

Lecture 9 - The theory of quantum processes

Summary.

- (1) From pure quantum maps to quantum maps.
- (2) Causality.
- (3) Non-determinism and quantum processes.

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Introducing a discarding effect.

The key to provide a full picture of quantum processes is the introduction of a *discarding* effect providing a test which succeeds with certainty but does not reveal anything about (and, of course, does not depend on) the (normalised) state that gets discarded¹.

$$\begin{array}{c} \triangle \\ \text{?} \\ \downarrow \\ \triangle \\ \hat{\psi} \end{array} = \square$$

Exercise 1

Show that a discarding effect does not exist in the process theories of linear maps or pure quantum maps.

The discarding effect is defined as

$$\overline{\text{T}} := \begin{array}{c} \text{---} \\ \text{T} \\ \text{---} \end{array} := \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}$$

which satisfies the envisaged property:

$$\begin{array}{c} \overline{\text{T}} \\ \downarrow \\ \triangle \\ \hat{\psi} \end{array} = \square$$

because, noting that if $\hat{\psi}$ is normalised so is ψ ,

$$\begin{array}{c} \overline{\text{T}} \\ \downarrow \\ \triangle \\ \hat{\psi} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \downarrow \\ \triangle \\ \psi \end{array} = \begin{array}{c} \triangle \\ \psi \\ \downarrow \\ \triangle \\ \psi \end{array} = \square$$

¹The use of normalised states is essential here: suppose a discarding effect exists, thus yielding 1 when applied to an arbitrary state $\hat{\phi}$. Clearly the application to e.g. $5\hat{\phi}$ would return 5 rather than 1 ...

Note that this is the only possible definition. Indeed, suppose there was another effect \mathbf{d} sending all normalised pure quantum states to $\mathbf{1}$. As discussed in the previous lecture any orthonormal basis on a type \mathbf{A} (in the theory of linear maps) can be extended to a basis for $\mathbf{A} \otimes \mathbf{A}$ which is also a basis for $\hat{\mathbf{A}}$ in the theory of pure quantum maps. Let \mathbf{B} be corresponding normalised basis. Then applying any of the two candidates to be a discarding effect to all states in \mathbf{B} always yields $\mathbf{1}$, thus forcing them to coincide.

Exercise 2

Distinguish the discarding effect from cups in the theory of pure quantum maps.

Exercise 3

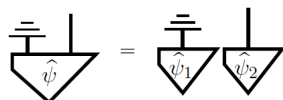
Characterise the discarding effect for types $\hat{\mathbf{A}} \otimes \hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$. Note that $\hat{\mathbf{C}}$ is type \mathbf{I} (the identity of \otimes) in the theory of pure quantum maps. Similarly, in the theory of linear maps, \mathbf{I} is \mathbb{C} (which can be regarded as the one-dimensional Hilbert space). The process we have been representing as $\mathbf{1}$, or depicting as the empty diagram, is, in any process theory, the identity on \mathbf{I} ($\text{id}_{\mathbf{I}}$).

Not only is the discard effect not a pure quantum effect, but, in general, reducing a pure state by discarding part of its output, i.e.

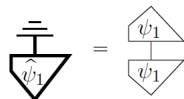


does not yield a pure quantum state. Actually, the reduced state is pure iff it is \otimes -separable. It is instructive to look at the proof of this claim.

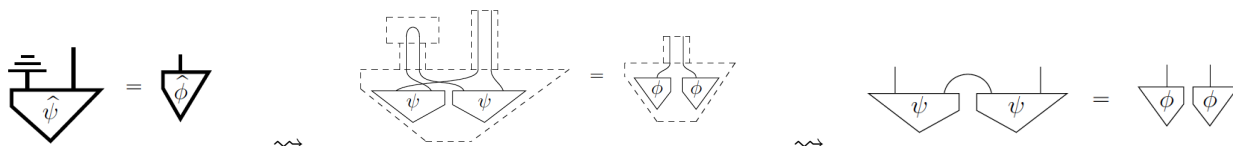
Consider, first, that the process is \otimes -separable, i.e.



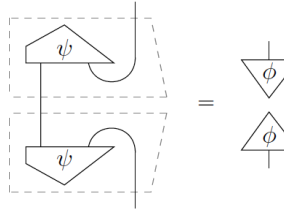
By construction,



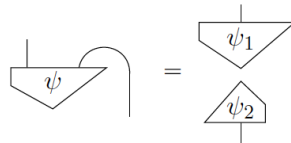
which, as a scalar in the theory of pure quantum maps, is a positive number. Thus, the reduced state is a pure quantum state. For the opposite direction, assume that the reduced state is equal to a pure state $\hat{\phi}$. Unfolding,



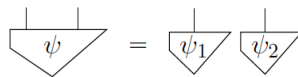
which is equivalent, by process-state duality, to



We may now resorting to a result discussed in the previous lecture stating that f is \otimes -separable iff $f^\dagger \cdot f$ is. Thus

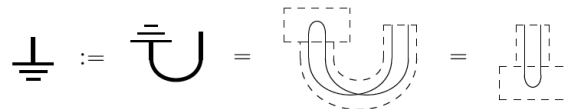


By process-state duality, ψ is \otimes -separable



The conclusion follows from doubling this last equation.

As discarding does not preserve pure quantum states (as just proved), one needs a more general notion of a state. Let us consider the adjoint of the discarding effect:

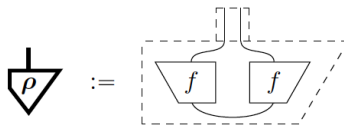


Its 'normalised' version

$$\frac{1}{D} \underline{\underline{\perp}}$$

is known as the *maximally mixed* (or *impure*) state, conveying a complete lack of knowledge about the system it stands for.

In general, quantum states are obtained through the composition of pure quantum maps and discarding. Their general form is



Verify this claim, recalling that ρ is \otimes -positive and, therefore, by the spectral theorem, there exists an orthonormal basis and positive scalars such that

$$\downarrow \rho := \sum_i r_i \downarrow i \downarrow i$$

This result indicates that as a causal state becomes more impure, the (squared-)norm will go lower and lower. The limit is the completely impure state, also called the *maximally mixed* state, which stands for a complete lack of knowledge about the system's actual state:

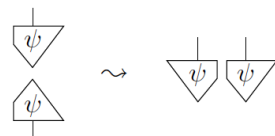
$$\frac{1}{D} \underline{\underline{\mathbb{1}}}$$

Clearly

$$\frac{1}{D} \underline{\underline{\mathbb{1}}} = \frac{1}{D^2} \text{[Diagram of a double state with a loop]} = \frac{1}{D}$$

Remark

The *doubling* procedure discussed in the previous lecture is closely related to the notion of a density operator $\tilde{\psi} = |\psi\rangle\langle\psi|$, which often in textbooks replaces vectors $|\psi\rangle$ as the standard notion of a pure quantum state. Actually, the density operator has the same data as a double state: one is obtained from the other by transposing the effect ψ into the conjugate of state ψ :



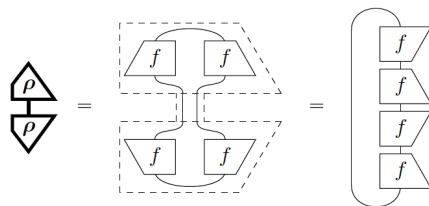
This density operator is a projector. Similarly, the density operator associated to a causal mixed state is a positive map with trace 1. Indeed, this representation is given by process-state duality

$$\downarrow \rho = \text{[Diagram of a state rho with two effects f]} \rightarrow \tilde{\rho} := \text{[Diagram of a state rho with a loop]} = \text{[Diagram of two effects f in series]}$$

Thus, discarding a state means taking its trace

$$\underline{\underline{\downarrow \rho}} = \text{[Diagram of a state rho with two effects f and a loop]} = \text{[Diagram of two effects f in series]}$$

and, similarly,



Therefore, the previous inequality becomes, in the density operator language, the well-known


$$\text{tr}(\tilde{\rho}^2) \leq \text{tr}(\tilde{\rho})^2$$

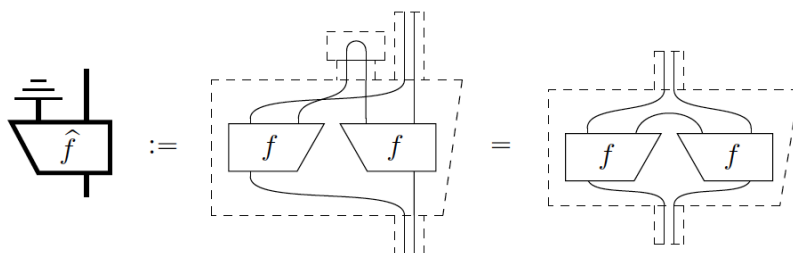
Quantum maps.

The theory of quantum maps is obtained from that of pure quantum maps by adding *discarding*. Clearly, this new theory admits string diagrams. It inherits from the pure case caps and cups, so it remains to show the existence of adjoints. The adjoint of a pure map is also a pure map, and the adjoint of discarding is


$$\left(\overline{\overline{\top}}\right)^\dagger = \overline{\overline{\cup}}$$

which composes a cup with discarding, making a quantum map again. Since adjoints need to preserve diagrams and all diagrams in quantum maps are made up of pure quantum maps and discarding, every quantum map has an adjoint.

This is the general form of a quantum map 



which means that quantum maps correspond to those linear maps f which are \otimes -positive. The pure quantum map \hat{f} above is known as the *purification* of the quantum map.

The result verified in Exercise 5 applied to a state 

provides a way to check if a quantum map is pure. Actually, it leads to

$$\left(\overline{\overline{\Phi}}\right)^\dagger \leq \left(\overline{\overline{\Phi}}\right)^2$$

with the equality holding iff ϕ is pure.

Causality also applies to maps: A quantum map is causal if

$$\begin{array}{c} \text{---} \\ \text{---} \\ \square \Phi \triangle \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array}$$

which may be read as *if the output of a process is discarded, it may as well have never happened.*

Exercise 6

Show that a causal map preserves causal states.

Causal effects are scarce: as shown in Exercise 3, discarding for type I amounts to doing nothing. Since effects have no outputs, causality reduces to

$$\begin{array}{c} \triangle \\ \square \rho \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array}$$

making discarding the only causal effect. Thus, there are no pure causal quantum effects. Moreover, any pure quantum map from a type \hat{A} to itself is causal iff it is unitary. The reason is that any isometry from \hat{A} to \hat{A} is unitary (a consequence of the dimension theorem), and one can prove that \hat{U} is causal iff it is an isometry; in pictures

$$\begin{array}{c} \text{---} \\ \text{---} \\ \square \hat{U} \triangle \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \quad \text{iff} \quad \begin{array}{c} \text{---} \\ \square \hat{U} \triangle \\ \square \hat{U} \triangle \\ \text{---} \\ \text{---} \end{array} = \text{---}$$

Indeed, unfolding the causality equation one gets

$$\begin{array}{c} \text{---} \\ \triangle U \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \triangle U \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array}$$

Thus,

$$\begin{array}{c} \text{---} \\ \square U \triangle \\ \square U \triangle \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \triangle U \\ \triangle U \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} = \text{---}$$

This result helps to build up intuition, as causality has probably a more direct physical interpretation than being unitary.

Mixing: An alternative view.

Impure quantum maps are obtained through discarding parts of a larger system. An alternative interpretation can be done as follows: First unfold the definition of the discarding effect and re-write the cap using explicit sums over a orthonormal basis:

$$\overline{\text{cap}} = \text{cap} = \sum_i \text{triangle}_i \text{triangle}_i = \sum_i \text{triangle}_i^{\text{cap}}$$

Then, any quantum map can be written as a sum of pure quantum maps

$$\text{box}_\Phi = \text{box}_{\widehat{f}}^{\overline{\text{cap}}} = \sum_i \text{box}_{\widehat{f}}^{\text{triangle}_i} = \sum_i \text{box}_{\widehat{f}_i} \quad \text{where} \quad \text{box}_{\widehat{f}_i} := \text{box}_{\widehat{f}}^{\text{triangle}_i}$$

which is known as a *Kraus decomposition* (which is not unique).

Conversely, any finite set of pure quantum maps is a quantum map

$$\sum_i \text{box}_{\widehat{f}_i} = \text{box}_{\widehat{f}}^{\overline{\text{cap}}} \quad \text{where} \quad \text{box}_f := \sum_i \text{triangle}_i \text{box}_{\widehat{f}_i}$$

Exercise 7

Prove that the theory of quantum maps is closed for sums (which is not the case for pure quantum maps).

Exercise 8

Show that the sum of causal quantum maps is not necessarily causal.

If one takes *mixtures* (i.e. *convex combinations*) instead of ordinary sums, causality is preserved. Indeed, a mixture of a family of causal quantum maps is a sum of the form

$$\sum_i p^i \text{box}_{\Phi_i}$$

where $\sum_i p^i = 1$. Thus,

$$\sum_i p^i \begin{array}{c} \overline{\overline{}} \\ \square \\ \Phi_i \\ \square \\ \overline{\overline{}} \end{array} = \sum_i p^i \overline{\overline{}} = \overline{\overline{}}$$

In a mixture, each p^i can be interpreted as the probability of the corresponding process happening. For example, the causal state

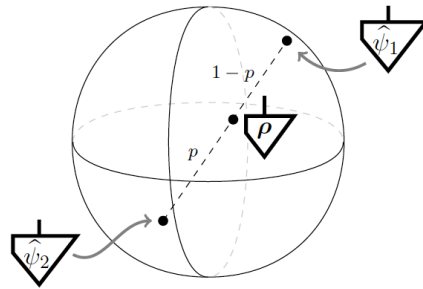
$$\begin{array}{c} \downarrow \\ \rho \end{array} = \sum_i p^i \begin{array}{c} \downarrow \\ \hat{\psi}_i \end{array}$$

can be regarded as a system which is in one of the pure states $\hat{\psi}_i$ with probability p^i . Actually, every causal quantum state can be expressed as a mixture of pure quantum states.

Exercise 9

Show that not every causal map can be expressed as a mixture of pure causal maps.

Geometrically, if pure causal states can be represented as points on the surface of the Bloch sphere, a mixed state appears as a point inside the corresponding ball:



where

$$\begin{array}{c} \downarrow \\ \rho \end{array} := p \begin{array}{c} \downarrow \\ \hat{\psi}_1 \end{array} + (1-p) \begin{array}{c} \downarrow \\ \hat{\psi}_2 \end{array}$$

Exercise 10

Show that, given a fixed orthonormal basis, probability distributions can be represented as causal quantum states of the form

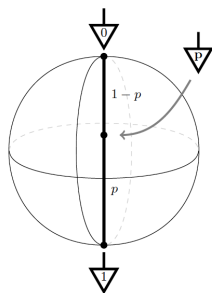
$$\begin{array}{c} \downarrow \\ \mathbf{p} \end{array} := \sum_i p^i \begin{array}{c} \downarrow \\ i \end{array}$$

The pure states correspond to point distributions.

In the two-dimensional case, probability distributions become

$$\downarrow_{\mathbf{p}} := p \downarrow_0 + (1-p) \downarrow_1$$

which are depicted as points in a line connecting two doubled basis states, each point corresponding to a number $p \in [0, 1]$.



There is, of course, a fundamental difference between probability distributions and quantum states. The former are uniquely decomposed into a probability distribution over point distributions (i.e. pure states), whereas the decomposition of the latter is, in general, not unique. A quantum state may decompose as many different mixtures of pure states. A typical example is the *maximally mixed state* introduced above which can be decomposed across any orthonormal basis:

$$\frac{1}{D} \underline{\underline{\mathbb{1}}} = \frac{1}{D} \cup = \frac{1}{D} \sum_i \downarrow_i \downarrow_i = \sum_i \frac{1}{D} \downarrow_i$$

Its name conveys the idea that it is equally distant from any pure state used in the decomposition. So, it can be thought of as pure noise, as it does not have any bias towards any meaningful data, i.e., any pure state.

Quantum processes.

Our journey to formalise quantum processes started from the theory of linear maps, to which some new ingredients were added along the way:

- *Doubling*, to capture probabilities as scalars, and get rid of global phases, leading to a theory of pure quantum maps.
- *Discarding*, to be able to ignore part of a system, thus capturing our lack of knowledge about its state. Such (impure) quantum maps can alternatively be described as probabilistic mixtures.

Quantum theory, however, deals with states and processes which are *non-deterministic* in a fundamental sense: such non-determinism cannot be accounted for solely based on lack of knowledge about the system at hand. Regardless of how perfect the current knowledge of the system is, non-deterministic processes will not have a fixed outcome until they occur².

On the other hand, quantum processes are supposed to be *causal*, which puts the theory out of conflict with other physical theories, namely special relativity, by forbidding *faster-than-light* signaling.

Exercise 11

Having proved in Exercise 3 that discarding a system of type $A \otimes B$ is the same as discarding individually subsystems A and B , and recalling that the only causal quantum effect is discarding itself, it is easy to conclude that all causal quantum effects are separable, i.e.

$$\begin{array}{c} \triangle \\ \rho \\ \hline \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}$$

Use this fact to show that the theory of causal quantum maps does not admit string diagrams.

This discussion motivates a more general definition: A *quantum process* is a collection of quantum maps

$$\left(\begin{array}{c} \text{---} \\ \text{---} \\ \Phi_i \\ \text{---} \\ \text{---} \end{array} \right)^i$$

each of which called a *branch* which together satisfy the following *causality* postulate:

$$\sum_i \begin{array}{c} \text{---} \\ \text{---} \\ \Phi_i \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}$$

A process is deterministic if this collection is singular. When acted by a quantum process one of the branches actually occurs and constitutes the outcome of the process.

A quantum process consisting of states

$$\left(\begin{array}{c} \text{---} \\ \text{---} \\ \rho_i \\ \text{---} \\ \text{---} \end{array} \right)^i$$

is a state in the theory. Its weight corresponds to its probability

$$P(\rho_i) := \begin{array}{c} \text{---} \\ \text{---} \\ \rho_i \\ \text{---} \\ \text{---} \end{array}$$

²cf, Einstein's famous aphorism expressing his skepticism wrt quantum mechanics — *God does not play dice*.

The causality requirement means that

$$\sum_i \overline{\overline{\rho_i}} = \square$$

Similarly, scalars in the theory of quantum processes are collections

$$\left(\diamond \mathbf{p}_i \right)^i \quad \text{such that} \quad \sum_i \diamond \mathbf{p}_i = \square$$

each of them thus forming a probability distribution.

Note that it is not possible to associate a fixed probability distribution to a general quantum process. Indeed, the probabilities will depend on the state to which the process is applied. Once applied, however, probabilities can be assigned and are, as usual, computed by the Born rule

$$P(\Phi_i | \rho) := \left. \begin{array}{c} \overline{\overline{\Phi_i}} \\ \hline \rho \end{array} \right\} \begin{array}{l} \text{effect} \\ \text{state} \end{array}$$

and satisfy

$$\sum_i \overline{\overline{\Phi_i}} \downarrow \rho = \overline{\overline{\rho}} = \square$$

as enforced by causality. For a deterministic process this last equation boils down to the definition of a causal quantum map:

$$\overline{\overline{\Phi}} \downarrow \rho = \overline{\overline{\rho}}$$

Sequential and parallel composition of quantum processes is defined to guarantee that any combination of valid branches can happen. Thus

$$\left(\begin{array}{c} \overline{\overline{\Psi_j}} \\ \hline \overline{\overline{\Phi_i}} \end{array} \right)^j_i := \left(\begin{array}{c} \overline{\overline{\Psi_j}} \\ \hline \overline{\overline{\Phi_i}} \end{array} \right)^{ij}$$

and

$$\left(\begin{array}{c} \overline{\overline{\Psi_j}} \\ \hline \overline{\overline{\Phi_i}} \end{array} \right)^j_i \left(\begin{array}{c} \overline{\overline{\Psi'_l}} \\ \hline \overline{\overline{\Phi'_k}} \end{array} \right)^l_k := \left(\begin{array}{cc} \overline{\overline{\Psi_j}} & \overline{\overline{\Psi'_l}} \\ \hline \overline{\overline{\Phi_i}} & \overline{\overline{\Phi'_k}} \end{array} \right)^{ijkl}$$

Exercise 12

Prove that causality is preserved by both sequential and parallel composition of quantum processes, which, therefore, can be organised into circuits.

Note that quantum processes admit string diagrams (once some extra notation replaces the family indices in the pictures ...). Moreover, any quantum map can be realized as a quantum process.