Lecture 3: Universal Properties

Summary.
(1) Universal properties: concept, examples and ubiquity.

(2) Initial and final objects in a category.

(3) Universal characterisation of Cartesian product in Set. The categorial product construction.

(4) Universal properties 'come in pairs': the coproduct construction. Properties of products and coproducts.

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Opening.

If there is a 'main topic' in category theory, this is certainly the study of *universal* properties. Roughly speaking, an entity ϵ is universal among a family of 'similar' entities if it is the case that every other entity in the family can be *reduced* or *traced back* to ϵ. For example, an object T is said to be *final* in a category C if, from every other object X in C, there exists a unique arrow $!_X$ to T. Therefore, there is a canonical, *unique* way to relate every object in C to T — *finality* is thus an universal property.

A nice thing about universal properties is the fact they always 'come in pairs': the *dual* of an universal is still an universal. Dualising finality, we arrive at *initiality*: an object is *initial* in C if there is one and only one arrow in C from it to any other object in the category.

Universal properties, like finality or initiality, can be recognised, usually under a different terminology, in many branches of Mathematics. Moreover, they happen to play a major role in the structure of 'mathematical spaces'. Therefore, category theory provides a setting for studying abstractly such 'spaces' and their relationships.

Let us consider an illustrative example (adapted from [2]). The study of bilinear (i.e. linear in both arguments) maps out of two vector spaces U and V can be reduced to the study of linear maps because there is a *universal* bilinear map $\epsilon : U \times V \longrightarrow T$ through which all the others factor, i.e. for all $f : U \times V \longrightarrow X$, there exists one and only one linear map $\overline{f} : T \longrightarrow X$ such that $f = \overline{f} \cdot \epsilon$. Look for a moment how uniqueness is proved. Suppose both ϵ and $\epsilon' : U \times V \longrightarrow T'$ satisfy the universal property above. Thus, we obtain linear maps $\overline{\epsilon}$ and $\overline{\epsilon'}$, such that

$$\epsilon' = \overline{\epsilon'} \cdot \epsilon \quad \text{and} \quad \epsilon = \overline{\epsilon} \cdot \epsilon'$$

because, respectively, ϵ and ϵ' are universal by assumption. Clearly, $\overline{\epsilon} \cdot \overline{\epsilon'} \cdot \epsilon = \overline{\epsilon} \cdot \epsilon' = \epsilon$ as depicted in the following diagram:

$$\begin{tikzcd}
& U \times W \arrow[dl, "\epsilon"'] \arrow[d, "\epsilon'"] \arrow[dr, "\epsilon"] & \\
T \arrow[r, "\overline{\epsilon'}"] & T' \arrow[r, "\overline{\epsilon}"] & T
\end{tikzcd}$$

However, $\mathrm{id}_T \cdot \epsilon = \epsilon$, which entails $\overline{\epsilon} \cdot \overline{\epsilon'} = \mathrm{id}_T$ by the uniqueness of ϵ. A similar argument, relying on the universality of ϵ', yields $\overline{\epsilon'} \cdot \overline{\epsilon} = \mathrm{id}_{T'}$. Thus, $\overline{\epsilon}$ is an isomorphism witnessing $T \cong T'$.

Vector space T is the *tensor* product of U and V, often written as $U \otimes V$; and what the universal property tells is that it is essentially unique. The way it is constructed is, to a large extent, irrelevant: the universal property is enough.

Exercise 1

Characterise the initial and final objects in a preorder regarded as a category. Give an example of a preorder in which such objects do not exist.

Exercise 2

Show that any singleton set is both initial and final in $\mathsf{Set}_\perp$ (and, therefore, called a *zero* object). Can you think of another familiar category with a zero object?

Exercise 3

Let Rng be the category of rings and consider $Z = \langle \mathbb{Z}, +, 0, -, \cdot, 1 \rangle$ the ring of integer numbers. Show that there is a unique ring homomorphism h from Z to any other ring $\langle S, +', 0', -', \cdot', 1' \rangle$ given by

$$h(n) \mathrel{\widehat{=}} \begin{cases} 0' & \Leftarrow \; n = 0 \\ -'h(-n) & \Leftarrow \; n < 0 \\ \underbrace{1' +' 1' +' \cdots +' 1'}_{n} & \Leftarrow \; n > 0 \end{cases}$$

Exercise 4

Based on the previous exercise, conclude that Z is the initial object in Rng, showing that any other ring satisfying the universal property is isomorphic to Z.

Exercise 5

Show that any map from a final object in a category to an initial one is an isomorphism.

Exercise 6

Coalgebras are a generic way represent transition systems. Formally, a *coalgebra* for a functor $F : C \longrightarrow C$, thought of as the type of the allowed transitions, is an object U, called its carrier, or state space, and an arrow $c : U \longrightarrow T(U)$ of C. A morphism between coalgebras c and c' is an arrow $h : U \longrightarrow V$ in C making the following diagram comute:

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "c"'] & V \arrow[d, "c'"] \\ F(U) \arrow[r, "F(h)"'] & F(V) \end{tikzcd}$$

1. Instantiate the definition for $C = Set$ and $F(X) = \mathcal{P}(L \times X)$, where $\mathcal{P}$ is the finite powerfunctor and L an arbitrary set (of labels, say). What sort of transition systems correspond to this type of coalgebras?
2. Show that coalgebras and their morphisms form a category.
3. Prove that, if coalgebra $(W, \omega : W \longrightarrow F(W))$ is final in the category of F-coalgebras, ω is an isomorphism.

Exercise 7

Dualise the definition of a coalgebra given above to arrive to the dual concept of a F-*algebra*, $(A, a : F(A) \longrightarrow A)$. Show that an initial algebra in the corresponding category is also an isomorphism — notice the proof strucuture is exactly the same used in the last question of the previous exercise.

Exercise 8

Characterise product and coproduct in a poset regarded as a category. Do the same for the category Pos whose objects are posets and arrows are monotone functions.

Exercise 9

Characterise product and coproduct in a discrete category.

Exercise 10

Resorting to the corresponding universal property, show that the product (respectively, coproduct) construction in a category is functorial. Show, in particular that, given two arrows $f : A \longrightarrow B$ and $g : C \longrightarrow D$, $f \times g : A \times C \longrightarrow B \times D \;=\; \langle f \cdot \pi_1, g \cdot \pi_2 \rangle$. What about $f + g$?

Exercise 11

Derive, from the universal property of products, the equality $\langle f, g \rangle \cdot h \;=\; \langle f \cdot h, g \cdot h \rangle$, for f, g and h suitably typed, and $\langle id_A, id_B \rangle = id_{A \times B}$. These results are known in classical program calculi [1], as the product *fusion* and *reflection* laws, respectively.

Exercise 12

A coproduct in Rel is given by disjoint union, with the universal arrow in the diagram below defined as

$$[R, S] \;\widehat{=}\; R \cdot \iota_1^\circ \cup S \cdot \iota_2^\circ$$

$$\begin{tikzcd} A \arrow[r, "\iota_1"] \arrow[dr, "R"'] & A + B \arrow[d, "{[R,S]}"] & B \arrow[l, "\iota_2"'] \arrow[dl, "S"] \\ & C & \end{tikzcd}$$

Define product in Rel by dualising this construction. Recall that Rel is a self-dual category.

Exercise 13

The product of two vector spaces U, V over a field K, in $Vect_K$ usually represented as $U \oplus V$, is given by $U \times V = \{(u, v) \mid u \in U, v \in V\}$ made into a vector space by defining addition and scalar multiplication as follows:

$$(x, y) + (x', y') \;=\; (x + x', y + y') \;\text{ and }\; k\,(x, y) \;=\; (k\,x, k\,y)$$

Projections and the universal arrow are as in Set but required to be linear. Show such is the case indeed.

Exercise 14

The Cartesian product $U \oplus V$ of two vector spaces U, V over a field K, in $Vect_K$, is simultaneously their product (as discussed in the previous exercise) and coproduct. Define the embeddings $\iota_1 : U \longrightarrow U \oplus V$ and $\iota_2 : V \longrightarrow U \oplus V$ as

$$\iota_1(x) \;=\; (x, 0_V) \;\text{ and }\; \iota_2(y) \;=\; (0_U, y)$$

where $0_U, 0_V$ are the additive identities in U and V, respectively. For $f : U \longrightarrow Z$ and $g : V \longrightarrow Z$, define the universal arrow $[f, g] : U \oplus V \longrightarrow Z$ by

$$[f, g](x, y) \;=\; f(x) + g(y)$$

and prove that the relevant arrows are linear and this construction defines indeed a coproduct.

References

[1] R. Bird and O. Moor. *The Algebra of Programming*. Series in Computer Science. Prentice-Hall International, 1997.

[2] T. Leinster. *Basic Category Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2014.