Exercises[1]

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Exercise 1

Explain what a monoidal category is and discuss in detail whether the category Rel of sets and relations is indeed monoidal. Characterize all the relevant defining elements. Does this category admit string diagrams? Justify.

Exercise 2

Explain what a monoidal category is and discuss in detail whether the category Pfn of sets and partial functions is indeed monoidal. Characterize all the relevant defining elements. Does this category admit string diagrams? Justify.

Exercise 3

Define in detail a category whose arrows are matrices over a field. Does it form a monoidal category? Does it admit string diagrams? Justify.

Exercise 4

Consider the categories Set, of sets and functions, Rel, of sets and relations, and Pfn, of of sets and partial functions. Discuss whether these categories admit the state-process duality. Justify.

Exercise 5

Give the diagrammatic equations of a process $\oplus$ taking two inputs and one output that express the algebraic properties of being

1. associative: $x \oplus (y \oplus z) \ = \ (x \oplus y) \oplus z$
2. commutative: $x \oplus y \ = \ y \oplus x$
3. having a unit, i.e. a process e (with no inputs) such that $x \oplus e \ = \ e \oplus x \ = \ x$
4. distributivity wrt another process $\odot$: $(x \odot y) \oplus z \ = \ (x \oplus z) \odot (y \oplus z)$

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[1] Pictures are taken from Coecke and Kissinger book, *Picturing Quantum processes*, CUP, 2017.

Exercise 6

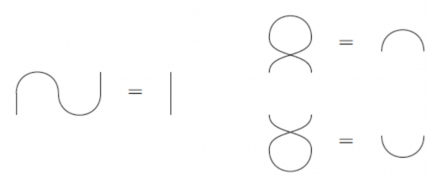
The process f^{-1} is said to be the *inverse* of a process f if, composing it with f on the right or on the left yields the processo identity. Prove that the following statements are equivalent, for any process f,

1. f is unitary.
2. $\mathsf{f}^\dagger$ is an isometry and admits an inverse.
3. f is an isometry and admits an inverse.

Exercise 7

The matrices for cups and caps in 2 dimensions are

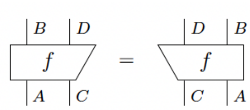
Show these definitions satisfy the *yanking* laws below:



Give the matrices for the cup and cap in 3 dimensions.

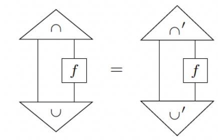
Exercise 8

The algebraic conjugate of a process f is defined as $\bar{\mathsf{f}} \mathrel{\widehat{=}} (\mathsf{f}^\mathsf{T})^\dagger = (\mathsf{f}^\dagger)^\mathsf{T}$, where f^T denotes the algebraic transpose of f. Show that all relations are algebraically self-conjugate. Identify the relations that are simply *self-conjugate*, i.e. that satisfy the following equation:



Exercise 9

Show that the trace of a process is independent of the particular choice of cup and cap, i.e. that if one has two cup/cap definitions, both satisfying the yanking laws, then



Exercise 10

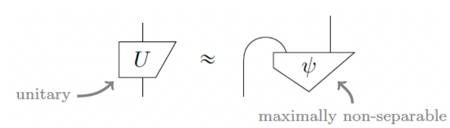
In the process theory of relations, a basis for a set A with n elements is composed by its singleton sets. Show that this is the only orthonormal basis of A. The orthonormality condition is actually not necessary for proving the uniqueness of the basis. Show that any basis (not necessarily orthonormal) of A must be the singleton basis.

Exercise 11

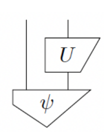
Explain, in your own words, the intuition underlying the notions of *positive* and $\otimes$-*positive* process. Show that the sequential composition of two $\otimes$-*positive* processes is still a $\otimes$-*positive* processe.

Exercise 12

A state is maximally non-separable if, up to a number, it corresponds to a unitary through the process-state duality, i.e.



Show that the application of a unitary to one side of a maximally non-separable state, i.e.

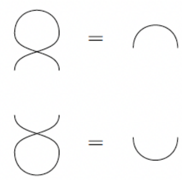


yields again a maximally non-separable state.

Exercise 13

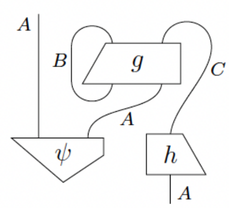
In the process theory corresponding to the matrix calculus discussed in the lectures, one may define

which satisfy the *yanking* laws. Show that this is indeed the case by verifying the following two equations:



Exercise 14

Compute the matrix corresponding to the following diagram, in the process theory corresponding to the matrix calculus discussed in the lectures.



Exercise 15

Show that a causal map preserves causal states.

Exercise 16

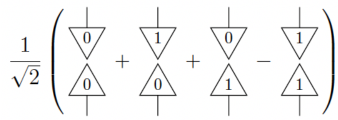
Consider the following definitions in the category of relations:

 $:: * \mapsto \{(a,a) \mid a \in A\}$  $:: \forall a \in A : (a,a) \mapsto *$

Prove they satisfy the three *yanking* equalities.

Exercise 17

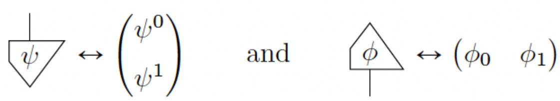
Show that the Hadamard gate can be written in matrix form with respect to the Z basis as



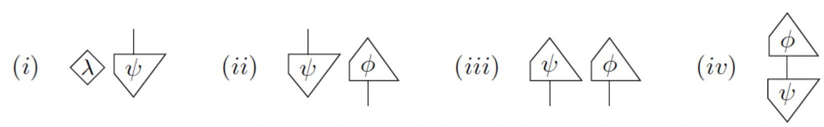
From this definition compute H as a matrix and compare with what you would obtain if considering the computational basis, X, instead. Justify.

Exercise 18

Consider the following 2-dimensional state and effect:

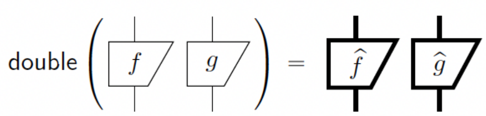


Write the matrices corresponding to the following processes, where λ is a number.



Exercise 19

Show that *doubling* preserves sequential and parallel composition of processes, i.e.



Exercise 20

Show that *doubling* preserves normalisation, i.e. a state ϕ is normalised if and only if its doubled state $\hat{\phi}$ is normalised. Discuss whether it preserves orthogonality as well.

LÓGICA QUÂNTICA - Módulo 2 (Exercícios)

Nr	Nome	1º Exercício	2º Exercício
PG 53729	Carolina Sobral	1	20
PG 53771	Diogo Ramos	2	19
PG 53821	Francisco José Silva Pinheiro	3	18
PG 53828	Gabriel Costa	4	17
PG 54024	Mafalda Pinto Couto	5	16
PG54111	Numa Nascimento Lima	6	15
PG 54147	Pedro Miguel Garrido Martins	7	14
PG 54160	Rafael Maria Igreja Gomes	8	13
PG 54267	Tomás Quintão de Araújo	9	12
PG 54274	Vítor Mendes	10	11