Quantum Computing

Lecture 2: Phase shifts, rotations and state preparation

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Recall: our gate set so far

$$\{H,\, X,\, Y,\, Z\}$$

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Each gate acts on one qubit and is represented by a unitary matrix.

The phase-shift gate

$$P_\phi = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}, \qquad \phi \in \mathbb{R}.$$

Is this matrix a valid quantum operation?

Exercise: is the phase-shift gate unitary?

For any real angle ϕ, consider

$$P_\phi = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}.$$

1. Find the conjugate transpose $P_\phi^\dagger$.
2. Calculate $P_\phi^\dagger P_\phi$.
3. Is P_ϕ a valid unitary quantum gate for every $\phi \in \mathbb{R}$?

Use the criterion $U^\dagger U = I$.

Solution: the phase-shift gate is unitary

For real ϕ, complex conjugation gives $(e^{i\phi})^* = e^{-i\phi}$.

$$P_\phi^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi} \end{pmatrix}.$$

$$P_\phi^\dagger P_\phi = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi}e^{i\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

P_ϕ is unitary for every real ϕ.

Its inverse is $P_\phi^{-1} = P_\phi^\dagger = P_{-\phi}$; also $P_\phi P_\phi^\dagger = I$.

The phase-shift gate

$$P_\phi = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}, \qquad \phi \in \mathbb{R}$$

$$P_\phi |0\rangle = |0\rangle$$
$$P_\phi |1\rangle = e^{i\phi} |1\rangle$$

The Pauli Z gate: $\phi = \pi$.

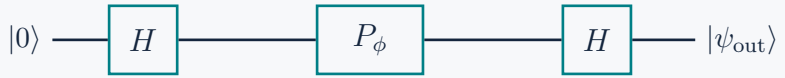
$$e^{i\pi} = \cos\pi + i \sin\pi = -1$$
$$P_\pi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$$

The T gate: $\phi = \pi/4$.

$$P_{\pi/4} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = T$$

A phase shift changes the relative phase of the amplitudes.

Exercise: phase shifts and interference



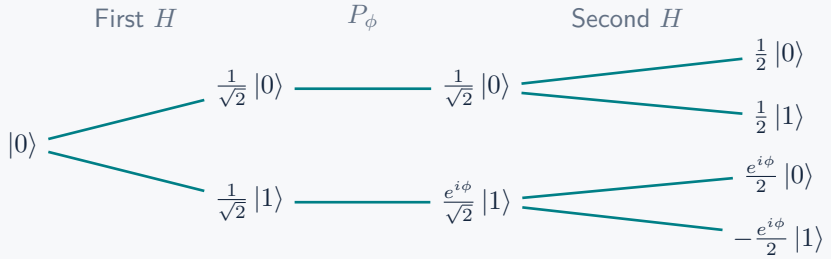
$$|\psi_{\text{out}}\rangle = HP_\phi H\,|0\rangle = \alpha_0\,|0\rangle + \alpha_1\,|1\rangle\,.$$

What are α_0 and α_1?

Then calculate the measurement probabilities.

$$P_0 = |\alpha_0|^2, \qquad P_1 = |\alpha_1|^2.$$

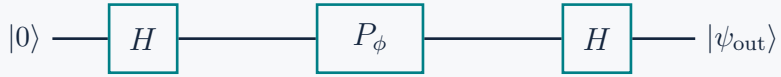
Following the amplitudes



$$|0\rangle \xrightarrow{H} \frac{|0\rangle + |1\rangle}{\sqrt{2}} \xrightarrow{P_\phi} \frac{|0\rangle + e^{i\phi}|1\rangle}{\sqrt{2}} \xrightarrow{H} \frac{1+e^{i\phi}}{2}|0\rangle + \frac{1-e^{i\phi}}{2}|1\rangle\,.$$

Add the contributions that end in the same basis state.

The output amplitudes



$$HP_\phi H\,|0\rangle = \frac{1}{2}(|0\rangle + |1\rangle) + \frac{e^{i\phi}}{2}(|0\rangle - |1\rangle)$$
$$= \left(\frac{1}{2} + \frac{1}{2}e^{i\phi}\right)|0\rangle + \left(\frac{1}{2} - \frac{1}{2}e^{i\phi}\right)|1\rangle\,.$$

$$\boxed{\alpha_0 = \frac{1 + e^{i\phi}}{2}}, \qquad \boxed{\alpha_1 = \frac{1 - e^{i\phi}}{2}}.$$

The measurement probabilities

$$P_0 = |\alpha_0|^2 = \alpha_0 \alpha_0^* = \left(\frac{1}{2} + \frac{1}{2}e^{i\phi}\right)\left(\frac{1}{2} + \frac{1}{2}e^{-i\phi}\right)$$
$$= \frac{1}{4} + \frac{1}{4}e^{-i\phi} + \frac{1}{4}e^{i\phi} + \frac{1}{4}$$
$$= \frac{1}{2} + \frac{1}{4}\left(e^{i\phi} + e^{-i\phi}\right) = \frac{1}{2} + \frac{1}{2}\cos\phi.$$

$$\boxed{P_0 = \cos^2(\phi/2)}, \qquad \boxed{P_1 = \frac{1 - \cos\phi}{2} = \sin^2(\phi/2)}.$$

The final Hadamard converts relative phase into measurable probabilities.

Statevector simulation versus hardware shots

PennyLane simulator

$$|\psi\rangle = U\,|000\rangle\,.$$

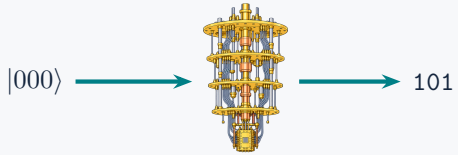
On `default.qubit`, `qml.state()` returns the evolved statevector.

Analytic mode (`shots=None`):

$$P(b) = |\alpha_b|^2.$$

No shot noise; finite numerical precision.
Simulators can also sample finite shots.

Quantum hardware



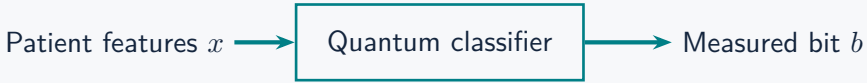
One shot: run and measure

Each shot returns a measured bitstring, not the statevector.

Same input and circuit; successive shots may give 101, 001, 101, ...

Statistics require repeated preparations and measurements.

A hypothetical HIV-test classifier



0: predicted negative 1: predicted positive.

$$|\psi(x)\rangle = \sqrt{0.2}\,|0\rangle + \sqrt{0.8}\,|1\rangle\,.$$

$$P(0 \mid x) = 0.2, \qquad P(1 \mid x) = 0.8.$$

One shot samples a class; it does not determine the score.

Repeat the **same input** for k shots; count m_1 ones:

$$\widehat{P}(1 \mid x) = \frac{m_1}{k}.$$

Use the estimated score for a decision: e.g. predict positive if $\widehat{P}(1 \mid x) > 1/2$.

For a known deterministic ideal output, one shot reads the prescribed bit.

Illustrative model, not a clinical test. A deterministic output can still be a wrong classification.

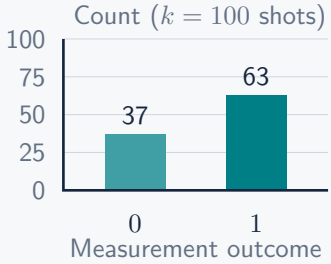
Estimating probabilities from repeated shots



Prepare afresh on every shot. Let m_j count outcome $j \in \{0, 1\}$.

$$m_0 + m_1 = k, \qquad \widehat{P}_0 = \frac{m_0}{k}, \quad \widehat{P}_1 = \frac{m_1}{k}.$$

Illustrative counts for 100 shots:
$\widehat{P}_0 = 37/100 = 0.37$, $\widehat{P}_1 = 63/100 = 0.63$.



Law of large numbers: $\widehat{P}_j \longrightarrow P_j$ as $k \to \infty$.

Exercise: is this matrix a valid quantum gate?

Let $\phi \in \mathbb{R}$. Consider

$$U = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}.$$

Does $UU^\dagger = I$?

Calculate the product and justify your answer.

Solution: checking unitarity

$$UU^\dagger = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix} \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}.$$

$$UU^\dagger = \begin{pmatrix} e^{-i\phi/2}e^{i\phi/2} & 0 \\ 0 & e^{i\phi/2}e^{-i\phi/2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Yes. This unitary is the rotation gate $R_z(\phi)$.

$$R_z(\phi)^{-1} = R_z(\phi)^\dagger = R_z(-\phi).$$

Exercise: acting on the basis states

$$R_z(\phi) = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}.$$

What does this gate do to $|0\rangle$ and $|1\rangle$?

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Give each answer as a vector and in Dirac notation.

Solution: rotating $|0\rangle$

$$R_z(\phi)\,|0\rangle = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\phi/2} \\ 0 \end{pmatrix}.$$

$$\boxed{R_z(\phi)\,|0\rangle = e^{-i\phi/2}\,|0\rangle}.$$

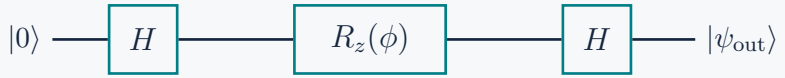
Solution: rotating $|1\rangle$

$$R_z(\phi)\,|1\rangle = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ e^{i\phi/2} \end{pmatrix}.$$

$$\boxed{R_z(\phi)\,|1\rangle = e^{i\phi/2}\,|1\rangle}.$$

Replacing the phase shift with R_z



$$|0\rangle \xrightarrow{H} \frac{|0\rangle + |1\rangle}{\sqrt{2}} \xrightarrow{R_z(\phi)} \frac{e^{-i\phi/2}|0\rangle + e^{i\phi/2}|1\rangle}{\sqrt{2}}$$

$$\xrightarrow{H} \frac{e^{-i\phi/2} + e^{i\phi/2}}{2}|0\rangle + \frac{e^{-i\phi/2} - e^{i\phi/2}}{2}|1\rangle$$

$$= \cos(\phi/2)|0\rangle - i\sin(\phi/2)|1\rangle .$$

For input $|0\rangle$: $P_0 = \cos^2(\phi/2)$ and $P_1 = \sin^2(\phi/2)$.

The R_x gate and an equivalent circuit

$$R_x(\phi) = \begin{pmatrix} \cos(\phi/2) & -i\sin(\phi/2) \\ -i\sin(\phi/2) & \cos(\phi/2) \end{pmatrix}.$$



$$HR_z(\phi)H = \frac{1}{2}\begin{pmatrix} e^{-i\phi/2} + e^{i\phi/2} & e^{-i\phi/2} - e^{i\phi/2} \\ e^{-i\phi/2} - e^{i\phi/2} & e^{-i\phi/2} + e^{i\phi/2} \end{pmatrix}$$
$$= R_x(\phi).$$

Exact equality for every input state; no extra global phase.

Our gate set now also includes $R_x(\phi)$.

Exercise: is this matrix a valid quantum gate?

Let $\theta \in \mathbb{R}$. Consider

$$U = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.$$

Does $UU^\dagger = I$?

Calculate the product and justify your answer.

Solution: checking unitarity

$$UU^\dagger = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.$$

Write $c = \cos(\theta/2)$ and $s = \sin(\theta/2)$.

$$UU^\dagger = \begin{pmatrix} c^2 + s^2 & cs - sc \\ sc - cs & s^2 + c^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Yes. This unitary is the rotation gate $R_y(\theta)$.

$$R_y(\theta)^{-1} = R_y(\theta)^\dagger = R_y(-\theta).$$

Exercise: acting on the basis states

$$R_y(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.$$

What does this gate do to $|0\rangle$ and $|1\rangle$?

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Give each answer as a vector and in Dirac notation.

Solution: rotating $|0\rangle$

$$R_y(\theta)\,|0\rangle = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix}.$$

$$\boxed{R_y(\theta)\,|0\rangle = \cos(\theta/2)\,|0\rangle + \sin(\theta/2)\,|1\rangle}.$$

Solution: rotating $|1\rangle$

$$R_y(\theta)\,|1\rangle = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}.$$

$$\boxed{R_y(\theta)\,|1\rangle = -\sin(\theta/2)\,|0\rangle + \cos(\theta/2)\,|1\rangle}.$$

The three Pauli rotation gates

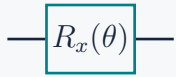
$$R_j(\theta) = e^{-i\theta\sigma_j/2} = \cos(\theta/2)I - i\sin(\theta/2)\sigma_j, \qquad j \in \{x, y, z\}.$$

Write $c = \cos(\theta/2)$ and $s = \sin(\theta/2)$; $\sigma_x = X$, $\sigma_y = Y$, $\sigma_z = Z$.

$R_x(\theta)$

$$\begin{pmatrix} c & -is \\ -is & c \end{pmatrix}$$

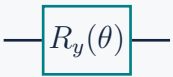
$R_x(\pi) = -iX$



$R_y(\theta)$

$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix}$$

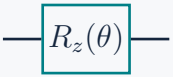
$R_y(\pi) = -iY$



$R_z(\theta)$

$$\begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

$R_z(\pi) = -iZ$



Our gate set so far

$$\{H,\, X,\, Y,\, Z,\, P_\phi,\, R_x(\theta),\, R_y(\theta),\, R_z(\theta)\}$$

Intermission

Next: preparing a chosen quantum state

Exercise: is this quantum state valid?

$$|\psi\rangle = \frac{3}{5}|0\rangle - \frac{4}{5}e^{i\pi/6}|1\rangle\,.$$

Is $|\psi\rangle$ normalized?

1. Identify the amplitudes α_0 and α_1.
2. Calculate $P_0 = |\alpha_0|^2$ and $P_1 = |\alpha_1|^2$.
3. Check whether $P_0 + P_1 = 1$.

Solution: checking the probabilities

$$|\psi\rangle = \frac{3}{5}|0\rangle - \frac{4}{5}e^{i\pi/6}|1\rangle\,.$$

$$P_0 = |\alpha_0|^2 = \left|\frac{3}{5}\right|^2 = \frac{9}{25}.$$

$$P_1 = |\alpha_1|^2 = \alpha_1\alpha_1^* = \left(-\frac{4}{5}e^{i\pi/6}\right)\left(-\frac{4}{5}e^{-i\pi/6}\right)$$

$$= \frac{16}{25}.$$

$P_0 + P_1 = \frac{9}{25} + \frac{16}{25} = 1$: the state is normalized.

Another check: the inner product

$$\langle\psi| = \frac{3}{5}\langle 0| - \frac{4}{5}e^{-i\pi/6}\langle 1|\,.$$

$$\begin{aligned}\langle\psi|\psi\rangle &= \left(\frac{3}{5}\langle 0| - \frac{4}{5}e^{-i\pi/6}\langle 1|\right)\left(\frac{3}{5}|0\rangle - \frac{4}{5}e^{i\pi/6}|1\rangle\right)\\ &= \frac{9}{25}\langle 0|0\rangle - \frac{12}{25}e^{i\pi/6}\langle 0|1\rangle\\ &\quad - \frac{12}{25}e^{-i\pi/6}\langle 1|0\rangle + \frac{16}{25}\langle 1|1\rangle\\ &= \frac{9}{25} + \frac{16}{25} = 1.\end{aligned}$$

Orthonormal basis: $\langle 0|0\rangle = \langle 1|1\rangle = 1$ and $\langle 0|1\rangle = \langle 1|0\rangle = 0$.

Exercise: prepare the state

$$|\psi\rangle = \frac{3}{5}|0\rangle - \frac{4}{5}e^{i\pi/6}|1\rangle\,.$$

Starting from $|0\rangle$, build a circuit that prepares $|\psi\rangle$.

Choose the gates, their order, and their parameters.

Available gates include $R_y(\theta)$ and P_ϕ.

State preparation: choosing the circuit

$$|\psi\rangle = \frac{3}{5}|0\rangle - \frac{4}{5}e^{i\pi/6}|1\rangle\,.$$



First set the magnitudes; then set the relative phase.

$$|\psi\rangle = P_\phi R_y(\theta)\,|0\rangle\,.$$

Step 1: set the magnitudes



$$|\psi_1\rangle = R_y(\theta)\,|0\rangle = \begin{pmatrix}\cos(\theta/2)\\ \sin(\theta/2)\end{pmatrix}$$

$$= \cos(\theta/2)\,|0\rangle + \sin(\theta/2)\,|1\rangle\,.$$

The rotation controls the probabilities of 0 and 1.

Step 2: set the relative phase

$$|\psi\rangle = P_\phi\,|\psi_1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix}.$$

$$\boxed{|\psi\rangle = \cos(\theta/2)\,|0\rangle + e^{i\phi}\sin(\theta/2)\,|1\rangle}.$$

The phase-shift gate leaves the magnitudes unchanged.

Solving for the parameters

Choose $\theta \in [0, \pi]$, so both sine and cosine below are nonnegative.

$$\cos^2(\theta/2) = \frac{9}{25} \quad \Longrightarrow \quad \theta = 2 \arccos \sqrt{\frac{9}{25}} = 2 \arccos \frac{3}{5}.$$

$$\sin^2(\theta/2) = 1 - \cos^2(\theta/2) = 1 - \frac{9}{25} = \frac{16}{25}.$$

$$e^{i\phi} = -e^{i\pi/6} \quad \Longrightarrow \quad \phi = \pi + \frac{\pi}{6} = \frac{7\pi}{6} \pmod{2\pi}.$$

$$e^{i(\pi+\pi/6)} = e^{i\pi} e^{i\pi/6} = -e^{i\pi/6}.$$

The completed preparation circuit



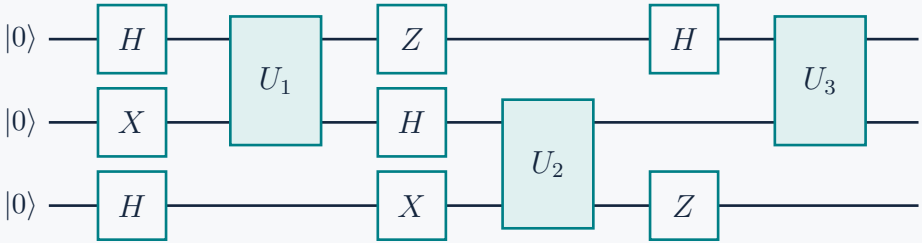
$$|\psi\rangle = \frac{3}{5}|0\rangle - \frac{4}{5}e^{i\pi/6}|1\rangle\,.$$

A computational-basis measurement gives

Outcome / post-measurement state	Probability
0 / $\|0\rangle$	$9/25$
1 / $\|1\rangle$	$16/25$

Beyond single-qubit gates

Computing with interacting qubits requires multi-qubit gates.



U_1, U_2, U_3 act jointly on two qubits.

Classical gates can combine inputs, e.g. $0 \oplus 1 = 1$. Quantum circuits also need gates that couple different wires.