

Lecture 4

Multi-state quantum systems: Composition and evolution

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What we have learned so far?

- We have introduced the notion of a **qubit**, the quantum analog to a bit, as the prototypical example of a **single** quantum state.
- A qubit is typically represented as a **superposition**, i.e., a linear combination of normalised and mutually orthogonal quantum states labelled by $|0\rangle$ and $|1\rangle$ with **complex** coefficients:

$$|v\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

- and we also learned that mathematically such states live in a **Hilbert space**, i.e., a complex vector space with **internal products**.
- Finally, we have seen that qubits evolve through the application of **unitary** operations, e.g., the Hadamard gate, and can also be **measured**.

The new questions

Questions

- How can bigger quantum states be formed from smaller ones?
- How do such systems evolve and be measured?

New states from older

States (classical, probabilistic, quantum) are represented by **vectors** in a specific **vector space**. Thus, putting two such states side by side

$$|v\rangle|u\rangle$$

corresponds to computing their **tensor product**: $|v\rangle \otimes |u\rangle$.

Example

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \otimes \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a \begin{bmatrix} d \\ e \\ f \end{bmatrix} \\ b \begin{bmatrix} d \\ e \\ f \end{bmatrix} \\ c \begin{bmatrix} d \\ e \\ f \end{bmatrix} \end{bmatrix} = \begin{bmatrix} ad \\ ae \\ af \\ bd \\ be \\ bf \\ cd \\ ce \\ cf \end{bmatrix}$$

Note

Some **convention** is needed to encode the joint classical states as row indices, typically the **lexicographic order**.

Tensor product $V \otimes W$

$\otimes$ is bilinear

- $(|u_1\rangle + |u_2\rangle) \otimes |z\rangle = |u_1\rangle \otimes |z\rangle + |u_2\rangle \otimes |z\rangle$
- $\alpha|u\rangle \otimes |z\rangle = |u\rangle \otimes \alpha|z\rangle = \alpha(|u\rangle \otimes |z\rangle)$
- and similarly in the second argument

Thus, scalars **float freely** over $\otimes$:

$$\alpha|u_1\rangle \otimes |u_2\rangle = |u_1\rangle \otimes \alpha|u_2\rangle = \alpha(|u_1\rangle \otimes |u_2\rangle)$$

Notation

$$|uv\rangle = |u\rangle|v\rangle = |u\rangle \otimes |v\rangle$$

New states from older

The Composition Postulate

The state space of a combined (quantum) system is the tensor product $V \otimes W$ of the state spaces V and W of its components.

Note

n m -dimensional vectors $\rightsquigarrow$ a vector in m^n -dimensional space

i.e., the state space of a quantum system grows exponentially with the number of particles.

Composing quantum states

Quantum states of multiple systems are represented by **column** vectors with **complex** entries and **norm equal to 1**, just as in the single-state case.

Example

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \\ -\frac{i}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} \text{ which corresponds to}$$

$$\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{6}}|01\rangle + \frac{i}{\sqrt{6}}|10\rangle + \frac{1}{\sqrt{6}}|11\rangle$$

Composing quantum states

Composing quantum states with $\otimes$ generates quantum states. The basic observation is that **norm** is **multiplicative** with respect to tensor:

$$\begin{aligned} \|u\rangle \otimes |v\rangle| &= \sqrt{\sum_{(a,b) \in A \times B} |\langle ab|u \otimes v\rangle|^2} \\ &= \sqrt{\sum_{a \in A} \sum_{b \in B} |\langle a|u\rangle \langle b|v\rangle|^2} \\ &= \sqrt{\sum_{a \in A} |\langle a|u\rangle|^2 \sum_{b \in B} |\langle b|v\rangle|^2} = \|u\| \|v\| \end{aligned}$$

Clearly, $\|u\rangle \otimes |v\rangle| = 1$, given that $\|u\rangle| = \|v\rangle| = 1$.

Composing quantum states

Example

The tensor product of $\alpha_1|0\rangle + \alpha_2|1\rangle$ and $\beta_1|0\rangle + \beta_2|1\rangle$ is

$$\alpha_1\beta_1|0\rangle \otimes |0\rangle + \alpha_1\beta_2|0\rangle \otimes |1\rangle + \alpha_2\beta_1|1\rangle \otimes |0\rangle + \alpha_2\beta_2|1\rangle \otimes |1\rangle$$

or, in a simplified notation,

$$\alpha_1\beta_1|00\rangle + \alpha_1\beta_2|01\rangle + \alpha_2\beta_1|10\rangle + \alpha_2\beta_2|11\rangle$$

Note

Notice the set $\{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$ is a **basis** for the space of two qubits.

Independent and entangled states

Tensoring quantum states yields **independent** multi-state systems.

However, most multi-state systems are **entangled**, which is itself a basic resource for quantum computation:

Examples

$$\begin{aligned}\text{CHZ} &= \frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle \\ W &= \frac{1}{\sqrt{3}}|001\rangle + \frac{1}{\sqrt{3}}|010\rangle + \frac{1}{\sqrt{3}}|100\rangle\end{aligned}$$

A note on representing multi-qubit states

Any unit vector in a 2^n Hilbert space represents a possible n -qubit state, up to

... a certain level of redundancy

- As before, vectors that differ only in a **global phase** represent the **same** quantum state
- but also the **same phase factor in different qubits** of a tensor product represent the **same** state:

$$|u\rangle \otimes (e^{i\phi}|v\rangle) = e^{i\phi}(|u\rangle \otimes |v\rangle) = (e^{i\phi}|u\rangle) \otimes |v\rangle$$

Actually, phase factors in qubits of a single term of a superposition can always be factored out into a coefficient for that term, i.e.
phase factors distribute over tensors

Bases

The computational basis for a vector space

$$\underbrace{V \otimes V \otimes \dots \otimes V}_n$$

corresponding to the composition of n qubits (each living in V) is the set

$$\begin{aligned} & \{ \underbrace{|0\rangle \dots |0\rangle |0\rangle}_n, \underbrace{|0\rangle \dots |0\rangle |1\rangle}_n, \underbrace{|0\rangle \dots |1\rangle |0\rangle}_n, \dots, \underbrace{|1\rangle \dots |1\rangle |1\rangle}_n \} \\ \stackrel{\text{abv}}{=} & \{ \underbrace{|0 \dots 00\rangle}_n, \underbrace{|0 \dots 01\rangle}_n, \underbrace{|0 \dots 10\rangle}_n, \dots, \underbrace{|1 \dots 11\rangle}_n \} \end{aligned}$$

which may be written in a compressed (decimal) way as

$$\{|0\rangle, |1\rangle, |2\rangle, |3\rangle, \dots, |2^n - 1\rangle\}$$

Bases

Example

The **computational basis** for a two qubit system is

$$\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$$

with

$$|0\rangle = |00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad |1\rangle = |01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad |2\rangle = |10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad |3\rangle = |11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Bases

There are of course other bases ... besides the **standard** one, e.g.

The Bell basis

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Compare with the Hadamard basis for the single qubit systems

Computing with quantum states

If a quantum state is a **ray** (i.e. a unit vector in a Hilbert space H up to a global phase), its evolution is specified a certain kind of **linear** operators

$$U : H \longrightarrow H$$

Linearity

$$U \left(\sum_j \alpha_j |v_j\rangle \right) = \sum_j \alpha_j U(|v_j\rangle)$$

just by itself has an important consequence:

unknown quantum states cannot be cloned

The no-cloning theorem

Linearity implies that unknown quantum states cannot be cloned

Let $U(|a\rangle|0\rangle) = |a\rangle|a\rangle$ be a 2-qubit operator and $|c\rangle = \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle)$ for $|a\rangle, |b\rangle$ orthogonal. Then,

$$\begin{aligned}U(|c\rangle|0\rangle) &= \frac{1}{\sqrt{2}}(U(|a\rangle|0\rangle) + U(|b\rangle|0\rangle)) \\&= \frac{1}{\sqrt{2}}(|a\rangle|a\rangle + |b\rangle|b\rangle) \\&\neq \frac{1}{\sqrt{2}}(|a\rangle|a\rangle + |a\rangle|b\rangle + |b\rangle|a\rangle + |b\rangle|b\rangle) \\&= |c\rangle|c\rangle \\&= U(|c\rangle|0\rangle)\end{aligned}$$

As already seen, $|x\rangle|y\rangle = |xy\rangle = |x\rangle \otimes |y\rangle$

The state evolution postulate

State Evolution Postulate

The evolution over time of the state of a closed quantum system is described by a unitary operator.

The evolution is **linear**

$$U \left(\sum_j \alpha_j |v_j\rangle \right) = \sum_j \alpha_j U(|v_j\rangle)$$

and preserves the **normalization constraint**

$$\text{If } \sum_j \alpha_j U(|v_j\rangle) = \sum_j \alpha'_j |v_j\rangle \text{ then } \sum_j |\alpha'_j|^2 = 1$$

Unitary operators

Preservation of the **normalization constraint** means that unit length vectors (and thus orthogonal subspaces) are mapped by U to unit length vectors (and thus to orthogonal subspaces).

This entails a condition on valid quantum operators: they must **preserve** the inner product, i.e.

$$(U|v\rangle, U|w\rangle) = \langle v|U^\dagger U|w\rangle = \langle v|w\rangle$$

which is the case iff U is **unitary**, i.e. $U^\dagger = U^{-1}$:

$$U^\dagger U = UU^\dagger = I$$

Unitarity

- Preserving the inner product means that a unitary operator maps **orthonormal bases** to **orthonormal bases**.
- Conversely, any operator with this property is unitary.
- If given in matrix form, being unitary means that the set of columns of its matrix representation are orthonormal (because the j th column is the image of $U|j\rangle$). Equivalently, rows are orthonormal (why?)

Unitarity

Unitarity is the **only** constraint on quantum operators: Any unitary matrix specifies a valid quantum operator.

This means that **there are many non-trivial operators on a single qubit**, in contrast with the **classical** case where the only non-trivial operation on a bit is **complement**.

Finally, because the **inverse** of a unitary matrix is also a unitary matrix, a quantum operator can always be inverted by another quantum operator

Unitary transformations are reversible

Partial measurements

Similarly to the probabilistic case, the quantum state corresponding to the aggregation of two systems takes the form of a **unit** vector

$$|u\rangle = \sum_{(a,b) \in A \times B} \alpha_{ab} |ab\rangle$$

When measured, the probability of obtaining (a, b) is given by

$$|\langle ab|u\rangle|^2 = |\alpha_{ab}|^2$$

If only the first system is measured, the probability of getting a is

$$\sum_{b \in B} |\langle ab|u\rangle|^2 = \sum_{b \in B} |\alpha_{ab}|^2$$

Question

What happens to the state of the second system?

Partial measurements

First observe that

$$|u\rangle = \sum_{a \in A} |a\rangle \otimes |u_a\rangle \quad \text{where} \quad |u_a\rangle = \sum_{b \in B} \alpha_{ab} |b\rangle$$

Then, the probability of obtaining a is given by

$$\sum_{b \in B} |\alpha_{ab}|^2 = \|u_a\|^2$$

and the quantum state becomes

$$|a\rangle \otimes \frac{|u_a\rangle}{\|u_a\|}$$

i.e., the state collapses consistently with the outcome a .

Note

There is, however, no analogous to the **marginal** probabilistic state:

$\sum_{(a,b) \in A \times B} \alpha_{ab} |a\rangle$ is not, in general, a quantum state.

Partial measurements

Example

Let

$$|u\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{6}}|01\rangle + \frac{i}{\sqrt{6}}|10\rangle + \frac{1}{\sqrt{6}}|11\rangle$$

Start rewriting $|u\rangle$ as

$$|u\rangle = |0\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{6}}|1\rangle \right) + |1\rangle \otimes \left(\frac{i}{\sqrt{6}}|0\rangle + \frac{1}{\sqrt{6}}|1\rangle \right)$$

The probability associated with outcome 0 is

$$\left| \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}} \right|^2 = \frac{2}{3}$$

and the resulting state

$$|0\rangle \otimes \frac{\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{6}}|1\rangle}{\sqrt{\frac{2}{3}}} = |0\rangle \otimes \left(\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle \right)$$