Lecture 3
Hilbert spaces: A brief overview

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Recall: What is a qubit?

A qubit is a quantum system represented as a superposition, i.e. a linear combination of normalised and mutually orthogonal quantum states labelled by $|0\rangle$ and $|1\rangle$ with complex coeficients:

$$|v\rangle \;=\; \alpha|0\rangle + \beta|1\rangle \;=\; \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

When state $|v\rangle$ is measured (i.e. observed) one of the two basic states $|0\rangle, |1\rangle$ is returned with probability

$$|\alpha|^2 \quad \text{and} \quad |\beta|^2$$

respectively.

What is a qubit?

Being probabilities, the norm squared of the coefficients must satisfy

$$|\alpha|^2 + |\beta|^2 \ = \ 1$$

which enforces quantum states to be represented by unit vectors.

In practice, a qubit is a microscopic system, such as an atom, a nuclear spin, or a polarised photon.

The state space of a qubit

Global phases are useless

Unit vectors equivalent up to multiplication by a complex number of modulus one, i.e. a phase factor $e^{i\theta}$, represent the same state.

Let $|v\rangle = \alpha|u\rangle + \beta|u'\rangle$ and $e^{i\theta}|v\rangle = e^{i\theta}\alpha|u\rangle + e^{i\theta}\beta|u'\rangle$

$$|e^{i\theta}\alpha|^2 = (\overline{e^{i\theta}\alpha})(e^{i\theta}\alpha) = (e^{-i\theta}\overline{\alpha})(e^{i\theta}\alpha) = \overline{\alpha}\alpha = |\alpha|^2$$

and similarly for β.

As the probabilities $|\alpha|^2$ and $|\beta|^2$ are the only measurable quantities, the global phase has no physical meaning.

Representation redundancy

A qubit is not represented by a vector, but by the subspace it generates.

The state space of a qubit

Relative phase

It is a measure of the angle between the two complex numbers.
Thus, it cannot be discarded!

Those are different states

$$\frac{1}{\sqrt{2}}(|u\rangle + |u'\rangle) \quad \frac{1}{\sqrt{2}}(|u\rangle - |u'\rangle) \quad \frac{1}{\sqrt{2}}(e^{i\theta}|u\rangle + |u'\rangle)$$

...

Summing up

The State (or representation) Postulate

The state space of a quantum system is described by a unit vector in a Hilbert space up to a global phase

- In practice, with finite resources, one cannot distinguish between a continuous state space from a discrete one with arbitrarily small minimum spacing between adjacente locations.
- One may, then, restrict to finite-dimensional Hilbert spaces.

The underlying maths is that of Hilbert spaces.

A Language: Dirac's notation

Dirac's bra/ket notation provides a handy (easy to calculate with) way to represent elements of vector space over the complex field

$|u\rangle$ A ket stands for a vector in an Hilbert space V. In $\mathcal{C}^n$, a column vector of complex entries. The identity for $+$ (the zero vector) is just written 0.

$\langle u|$ A bra is a vector in the dual space V^*, i.e. scalar-valued linear maps $V \to \mathcal{C}$ — a row vector in $\mathcal{C}^n$.

There is a bijective correspondence between $|u\rangle$ and $\langle u|$

$$|u\rangle = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \Leftrightarrow \begin{bmatrix} \overline{u}_1 \cdots \overline{u}_n \end{bmatrix} = \langle u|$$

A Language: Dirac's notation

Multiplying a bra and ket gives a ...

$$\boxed{\text{a bracket:} \quad \langle u| \, |v\rangle \quad \rightsquigarrow \quad \langle u|v\rangle}$$

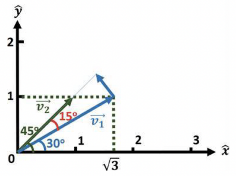
which provides the remaining link in our underlying mathematics:

the inner product

Notation: $\langle u|v\rangle \equiv \langle u, v\rangle \equiv (|u\rangle, |v\rangle)$

Recall: what is a inner product in $\mathbb{R}^2$?

From a linear algebra textbook: for $\vec{v_1} = \sqrt{3}\,\widehat{x} + \widehat{y}$ and $\vec{v_2} = \widehat{x} + \widehat{y}$, the inner (or dot, or scalar) product is



$$|\vec{v_1}||\vec{v_2}|\cos 15 \;\approx\; 2.732$$

i.e. the product of the (magnitudes of) $\vec{v_2}$ by $\vec{v_1}$ projected in the direction of $\vec{v_2}$.

The inner product measures how much the two vectors have in common after being scaled by their magnitudes.

Recall: what is a inner product in $\mathbb{R}^2$?

This definition is equivalent to another one which explicitly involves the vectors' coordinates:

$$\vec{v_1}^T \vec{v_2} \;=\; \begin{bmatrix} (v_1)_1 & (v_1)_2 & \ldots \end{bmatrix} \begin{bmatrix} (v_2)_1 \\ (v_2)_2 \\ \ldots \end{bmatrix}$$

Or, in Dirac's notation in $\mathbb{R}^2$,

$$\langle v_1, v_2 \rangle \;=\; \langle v_1 || v_2 \rangle \;=\; \langle v_1 | v_2 \rangle$$

Thus,

$$\langle v | v \rangle \;=\; v_1^2 + v_2^2 + \ldots$$

which entails positivity, i.e.

$$\langle v | v \rangle \geq 0 \quad \text{with} \quad \langle v | v \rangle = 0 \quad \text{iff} \quad | v \rangle = 0$$

The underlying maths: Hilbert spaces

Complex, inner-product vector space

A complex vector space with inner product

$$\langle - | - \rangle : V \times V \longrightarrow \mathbb{C}$$

such that

$$\begin{array}{ll} (1) & \langle v | \sum_i \lambda_i \cdot |w_i\rangle\rangle \; = \; \sum_i \lambda_i \langle v | w_i \rangle \\ (2) & \langle v | w \rangle = \overline{\langle w | v \rangle} \\ (3) & \langle v | v \rangle \geq 0 \;\; (\text{with equality iff } |v\rangle = 0) \end{array}$$

Note: $\langle - | - \rangle$ is conjugate linear in the first argument:

$$\langle \sum_i \lambda_i \cdot |w_i\rangle | v \rangle \; = \; \sum_i \overline{\lambda_i} \langle w_i | v \rangle$$

and linear in the second.

Inner product: examples

In $\mathcal{C}$

$$\langle a+bi | c+di \rangle = \overline{(a+bi)}(c+di)$$

In $\mathcal{C}^n$

$$\langle u|v\rangle = \left\langle \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right\rangle = \underbrace{\begin{bmatrix} \overline{u_1} & \overline{u_2} & \cdots & \overline{u_n} \end{bmatrix}}_{\langle u|} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{|v\rangle} = \sum_{i=1}^{n} \overline{u_i} v_i$$

An important note

Defining the dual vector in $\mathcal{C}^n$ in terms of the conjugate transpose preserves positivity of the internal product.

Indeed, in $\mathbb{R}^n$ positivity holds because the square of a real number is always positive, which is not the case for $\mathcal{C}$. However,

$$\overline{(a+bi)}(a+bi) \;=\; (a-bi)(a+bi) \;=\; a^2+b^2 \;\geq\; 0$$

Norms and orthogonality

Old friends

- $|v\rangle$ and $|w\rangle$ are orthogonal if $\langle v|w\rangle = 0$
- norm: $\||v\rangle| = \sqrt{\langle v|v\rangle}$, a nonnegative real number

This norm satisfies the triangle equalities $\||v\rangle + |w\rangle| \;\leq\; \||v\rangle| + \||w\rangle|$ due to the Cauchy-Schwarz inequality:

$$|\langle x|y\rangle| \;\leq\; \||x\rangle|\,\||y\rangle|$$

Norms and orthogonality

Recall

- normalization: $\frac{|v\rangle}{||v\rangle|}$
- $|v\rangle$ is a unit vector if $||v\rangle| = 1$
- A set of vectors $\{|i\rangle, |j\rangle, \cdots,\}$ is orthonormal if each $|i\rangle$ is a unit vector and

$$\langle i|j\rangle \;=\; \delta_{i,j} \;=\; \begin{cases} i = j & \Rightarrow 1 \\ \text{otherwise} & \Rightarrow 0 \end{cases}$$

Bases

Orthonormal basis

A orthonormal basis for a Hilbert space V of dimension n is a set $B = \{|i\rangle, \cdots\}$ of n linearly independent elements of V st

- $\langle i|j\rangle = \delta_{i,j}$ for all $|i\rangle, |j\rangle \in B$
- and B spans V, i.e. every $|v\rangle$ in V can be written as

$$|v\rangle \ = \ \sum_i \alpha_i |i\rangle \quad \text{for some } \alpha_i \in \mathcal{C}$$

Changing representations of a quantum state from one basis to another is a common technique in quantum algorithms.
(cf *why superposition can help you to date the girl/boy of your choice?)*

Example: The Hadamard basis

One of the infinitely many orthonormal bases for a space of dimension 2:

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$
$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

Check e. g.

$$\langle +|-\rangle \ = \ \frac{1}{2}\langle(|0\rangle + |1\rangle, |0\rangle - |1\rangle)\rangle \ = \ \frac{1}{2}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1 \\ -1\end{bmatrix} \ = \ 0$$

$$\||+\rangle| \ = \ \sqrt{\langle +|+\rangle} \ = \ \sqrt{\frac{1}{2}\langle|0\rangle + |1\rangle, |0\rangle + |1\rangle\rangle} \ = \ 1$$

When basis are around

If $|v\rangle$ is expressed wrt an orthonormal basis, i.e. $|v\rangle = \sum_i \alpha_i |i\rangle$, then the amplitude of $|v\rangle$ wrt $|i\rangle$ (i.e. the coefficient in $|i\rangle$ of vector $|v\rangle$ — its entry in line i) satisfies

$$\alpha_i = \langle i|v\rangle$$

because

$$\begin{aligned}
\langle i|v\rangle &= \langle i| \sum_j \alpha_j |j\rangle\rangle \\
&= \sum_j \alpha_j \langle i|j\rangle \\
&= \sum_j \alpha_j \delta_{i,j} \\
&= \alpha_i
\end{aligned}$$

Maps in a Hilbert space

Maps between Hilbert spaces are, of course,

linear transformations

typically represented by matrices whose entries are computed through the inner product:

$$A_{i,j} = \langle i|A|j\rangle$$

The dual space

V^*

If V is a Hilbert space, V^* is the space of linear maps from V to $\mathcal{C}$.

Elements of V^* are denoted by

$$\langle u| : V \longrightarrow \mathcal{C}$$

as discussed above, and defined through the inner product:

$$\langle u|(|v\rangle) = \langle u|v\rangle$$

In a matricial representation $\langle u|$ is the Hermitian conjugate, or conjugate transpose of $|u\rangle$,

i.e. the transpose of the vector formed by the complex conjugate of each element in $|u\rangle$.

The adjoint object

The bijective correspondence between V and V^* is esblished by $\dagger$:

$$(|u\rangle)^\dagger = \langle u| \quad \text{and} \quad (\langle u|)^\dagger = |u\rangle$$

which is antilinear:

$$\left(\sum_i \alpha_i |u_i\rangle\right)^\dagger = \sum_i \overline{\alpha_i}\langle u_i| \quad \text{and} \quad \left(\sum_i \alpha_i \langle u_i|\right)^\dagger = \sum_i \overline{\alpha_i}|u_i\rangle$$

The adjoint map

The adjoint $U^\dagger$ of a map $U : V \longrightarrow V$ is the unique map satisfying

$$(U^\dagger|w\rangle, |v\rangle) \;=\; (|w\rangle, U|v\rangle) \quad \text{or, in a simplified notation}$$
$$\langle U^\dagger w|v\rangle \;=\; \langle w|Uv\rangle$$

an equality that is often denoted by the expression

$$\langle u|U|v\rangle$$

Concretely,

$$\langle i|U^\dagger|j\rangle \;=\; \overline{\langle j|U|i\rangle}$$

i.e., the matrix representation of $U^\dagger$ is the conjugate transpose of U

Properties

- $(UV)^\dagger = V^\dagger U^\dagger$
- $U^{\dagger\dagger} \;=\; U$

A basis for V^*

If $\{|i\rangle, \cdots\}$ is an orthonormal basis for V, then

$$\{\langle i|, \cdots\}$$

is an orthonormal basis for V^*. Thus, a vector $\langle v|$ in the dual space is expressed wrt an orthonormal basis as

$$\langle v| \;=\; \sum_i \overline{\alpha_i}\langle i|$$

where

$$\overline{\alpha_i} \;=\; \langle v|i\rangle$$

Hilbert spaces

The complete picture

An Hilbert space is an inner-product space V st the metric defined by its norm turns V into a complete metric space, i.e.any Cauchy sequence

$$|v_1\rangle, |v_2\rangle, \cdots$$

$$\forall_{\epsilon>0}\ \exists_N\ \forall_{m,n>N}\ \ ||v_m - v_n\rangle| \leq \epsilon$$

converges
(i.e. there exists an element $|s\rangle$ in V st $\forall_{\epsilon>0}\ \exists_N\ \forall_{n>N}\ \ ||s - v_n\rangle| \leq \epsilon$)

The completeness condition is trivial in finite dimensional vector spaces