Lecture 2
From bits to qubits:
Single-state quantum systems

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Information & Computation

How to represent, access and transform information in a computational device?

Two perspectives.

- Classical (either determinisitc or probabilistic)
- Quantum

Deterministic information

States as vectors

Bits, elements of the state space $S = \{0, 1\}$ represent binary information states

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Similarly, the result of rolling a dice can be expressed as an element of the state space $S = \{one, two, three, four, five, six\}$

$$= |four\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

When rows are labelled by all possible state identifiers, the presence of 1 in a specific cell indicates which state the vector represents.

Deterministic information

Operations as matrices

$$\boxed{x \;\mapsto\; f(x) \qquad \text{corresponds to} \qquad M|x\rangle = |f(x)\rangle}$$

$\boxed{I(x) = x}$ $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$ $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$

$\boxed{X(x) = \neg x}$ $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$ $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$

$\boxed{\underline{1}(x) = 1}$ $\begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$ $\begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$

$\boxed{\underline{0}(x) = 0}$ $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$ $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$

$$I|0\rangle = |0\rangle \qquad I|1\rangle = |1\rangle \qquad X|0\rangle = |1\rangle \qquad X|1\rangle = |0\rangle$$
$$\underline{1}|0\rangle = |1\rangle \qquad \underline{1}|1\rangle = |1\rangle \qquad \underline{0}|0\rangle = |0\rangle \qquad \underline{0}|1\rangle = |0\rangle$$

Deterministic information

A handy notation

$|x\rangle$ - column vector with 1 in the entry for x, and 0 otherwise

$\langle x|$ - column vector with 1 in the entry for x, and 0 otherwise

Product $|x\rangle\langle y|$ yields a square matrix with 1 in the entry (x, y), and 0 otherwise. which allows for a handy representation of the matrix corresponding to a function $f : S \longrightarrow S$:

$$\boxed{M \;=\; \textstyle\sum_{x \in S} \; |f(x)\rangle\langle x|}$$

Example ($\underline{1}$)

$$M \;=\; |\underline{1}(0)\rangle\langle 0| + |\underline{1}(1)\rangle\langle 1| \;=\; \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \;=\; \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Deterministic information

For every function, the corresponding matrix exists and is uniquely determined. Indeed

$$\begin{aligned} M|y\rangle &= \left(\sum_{x\in S} |f(x)\rangle\langle x|\right)|y\rangle \\ &= \sum_{x\in S} |f(x)\rangle\langle x||y\rangle \\ &= \sum_{x\in S} |f(x)\rangle\delta_{x,y} \\ &= |f(y)\rangle \end{aligned}$$

where $\delta_{x,y}$ is Dirac's impulse given by

$$\delta_{x,y} = \begin{cases} 1 & \Leftarrow x = y \\ 0 & \text{otherwise} \end{cases}$$

Probabilistic information

Often, knowledge about the actual state of a system is uncertain. For example, one may believe that a bit is in state 0 with probability $\frac{2}{3}$ and in state 1 with probability $\frac{1}{3}$. Such a probabilistic state is again expressed by a column vector, e.g.

$$|v\rangle = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$$

State: is a vector of probabilities whose entries are in $\mathcal{R}^+$

$$\begin{bmatrix} p_0 \cdots p_n \end{bmatrix}^T \text{ such that } \sum_i p_i = 1$$

expressing indeterminacy about the system's exact configuration.

Note that the system is always in some well-defined state, even if we do not know which.

Probabilistic information

When inspecting, or measuring, a system in a probabilistic state, one retrieves whatever classical state it is in. All uncertainty is resolved.

Expressing probabilistic states as linear combinations of classical ones

$$|v\rangle = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix} = \frac{3}{4}|0\rangle + \frac{1}{4}|1\rangle$$

makes clear that, once measured, the actual state is revealed with some probability given by its coefficient in the linear combination. Such coefficient for a state $|x\rangle$ is given by multiplying

$$\langle x||v\rangle \text{ usually written as } \langle x|v\rangle$$

Measuring entails a change of knowledge and, therefore, a change to another probabilistic state (typically, a trivial one). Note, however, that this means only that one's knowledge of the system has changed, not the system itself (which remains what it was).

Probabilistic operations

Operations as stochastic matrices

i.e., whose entries are in $\mathcal{R}^+$ and columns sum to 1. $M_{i,j}$ is the probability of evolution from state j to i.
They can be thought as

- as operations where randomness comes up during its execution

$$\begin{bmatrix} 1 & 1/3 \\ 0 & 2/3 \end{bmatrix}$$

  i.e., when the bit is in the classical state 0 nothing happens, otherwise it is flipped to 0 with $1/3$ probability.

- or, alternatively, as random choices of deterministic operations, e.g.

$$\begin{bmatrix} 1 & 1/3 \\ 0 & 2/3 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

  meaning that either I is applied with probability $2/3$ or $\underline{0}$ with probability probability $1/3$.

Quantum information

A quantum state is again expressed by a column vector whose indices label the system's classical states. However,

- Entries are complex numbers.
- The norm squared of the entries sum to 1.
  (Recall: $|c| = \sqrt{\overline{c}c}$ for a complex c)

This last condition is equivalent to require that the norm of a vector representing a quantum state

$$||v\rangle| = \sqrt{\sum_{k \in S} \overline{v_k} v_k}$$

is equal to 1, i.e. quantum states are represented by unital vectors.

Examples: $|0\rangle$, $|1\rangle$. $|+\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, $|-\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$, $|\psi\rangle = \begin{bmatrix} \frac{1+2i}{3} \\ -\frac{2}{3} \end{bmatrix}$

Measuring quantum information

Measurement acts as an interface between quantum and classical information: a way to extract the latter from the former, governed by the

Born rule

The probability of a classical state being retrieved is the norm squared of the corresponding entry in the state vector.

Example: $|v\rangle \;=\; \frac{1+2i}{3}|0\rangle - \frac{2}{3}|1\rangle$

- Probability of measuring $|0\rangle$ is $|\langle 0|v\rangle|^2 \;=\; |\frac{1+2i}{3}|^2 \;=\; \frac{5}{9}$
- Probability of measuring $|1\rangle$ is $|\langle 1|v\rangle|^2 \;=\; |-\frac{2}{3}|^2 \;=\; \frac{4}{9}$

Quantum operations

Operations as unitary matrices

$$UU^\dagger \;=\; UU^\dagger \;=\; I$$

i.e. $U^\dagger = U^{-1}$

where $U^\dagger$ is the conjugate trasnpose of U.

Property

Being unitary means that applying U to a quantum state $|v\rangle$,
i.e. multiplying matrix U by column vector $|v\rangle$,
preserves the norm of $|v\rangle$.

Quantum operations

Remark

We are becoming aware that the mathematical universe in which we can talk about quantum states is some sort of vector space over the field of complex numbers.
Note in particular that

- The dual of a matrix U, denoted by $U^\dagger$ is its conjugate trasnpose, i.e.
$$U^\dagger = \overline{U}^T$$
- Similarly, the dual of a quantum state $|v\rangle$ is the corresponding row vector with conjugate entries, represented by $\langle v|$, i.e.
$$|v\rangle^\dagger = \langle v|$$

(to be discussed later)

Examples of quantum operations

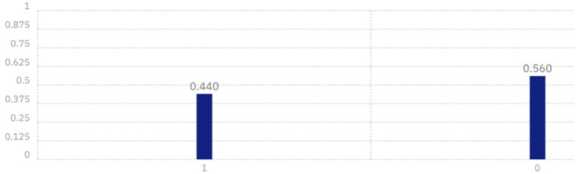
$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$$

Examples of quantum operations

Hadamard: a source of superposition

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$



$$H|0\rangle = |+\rangle = \overbrace{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}^{\text{superposition}}$$

$$H|1\rangle = |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Examples of quantum operations

Application: $H\left(\frac{1+2i}{3}|0\rangle - \frac{2}{3}|1\rangle\right)$

Through direct matrix multiplication:

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1+2i}{3} \\ -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{-1+2i}{3\sqrt{2}} \\ \frac{3+2i}{3\sqrt{2}} \end{bmatrix} = \frac{-1+2i}{3\sqrt{2}}|0\rangle + \frac{3+2i}{3\sqrt{2}}|1\rangle$$

Exploring linearity:

$$= \frac{1+2i}{3}H|0\rangle - \frac{2}{3}H|1\rangle = \frac{1+2i}{3}|+\rangle - \frac{2}{3}|-\rangle =$$
$$= \frac{1+2i}{3\sqrt{2}}|0\rangle + \frac{1+2i}{3\sqrt{2}}|1\rangle - \frac{2}{3\sqrt{2}}|0\rangle + \frac{2}{3\sqrt{2}}|1\rangle = \frac{-1+2i}{3\sqrt{2}}|0\rangle + \frac{3+2i}{3\sqrt{2}}|1\rangle$$

Examples of quantum operations

Remark

Observe that $|+\rangle$ and $|-\rangle$ are indistinguishable by observation (why?).

However, if measurement is preceeded by an application of H both states can be discriminated perfectly.

Actually, quantum states differing through a small change in the phase of the complex numbers entries may be indeed very different, even if not distinguished by measurement.

Examples of quantum operations

Phase shift

$$P_\phi = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}$$

$$\begin{aligned} P_\phi |0\rangle &= |0\rangle \\ P_\phi |1\rangle &= e^{i\phi}|1\rangle \end{aligned}$$

Qubit = a single-state quantum system

A qubit holds the information of both possible classical states:

A unit of information lives in a 2-dimensional complex vector space:

$$|v\rangle = \alpha|0\rangle + \beta|1\rangle$$

and thus possesses a continuum of possible values, so potentially, can store lots of classical data.

Qubit = a single-state quantum system

However, all this potential is hidden:

when observed $|v\rangle$ collapses into a classic state: $|0\rangle$, with probability $|\alpha|^2$, or $|1\rangle$, with probability $|\beta|^2$.

(Recall: $|\alpha| = \sqrt{\alpha\overline{\alpha}}$ for a complex α)

The outcome of an observation is probabilistic, which calls for a restriction to unit vectors, i.e. st

$$|\alpha|^2 + |\beta|^2 = 1$$

to represent quantum states.

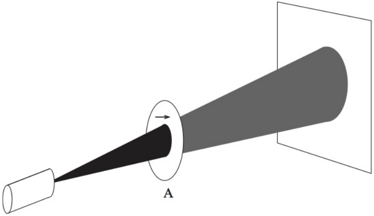
Quantum information is a different story

This quantum state is not a probabilistic mixture: it is not true that the state is really either $|0\rangle$ or $|1\rangle$ and we just do not happen to know which.

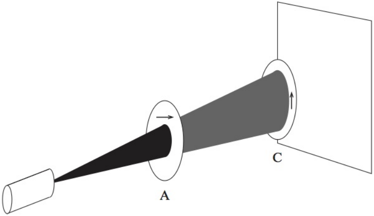
Amplitudes are not real numbers (e.g. probabilities) that can only increase when added, but complex so that they can cancel each other or lower their probability, thus capturing a fundamental quantum resource:

interference

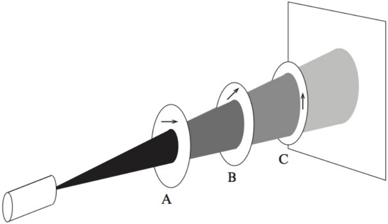
Quantum information: An experiment with a photon



$|0\rangle$ - horizontal polarization



$|1\rangle$ - vertical polarization



(from [Reifell & Polak, 2011])

Quantum information: An experiment with a photon

An explanation for a *single* photon experiment

- The photon's polarization state is modelled by a unit vector, for example the following linear combination of $|0\rangle$ and $|1\rangle$:

$$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

  which corresponds to a polarization of 45 degrees.
- On passing a polaroid the photon will be absorbed or leave the polaroid with its polarization aligned with the polaroid's axis.
- The probability to go through the polaroid is the square of the magnitude of the amplitude of its polarization in the direction of the polaroid's axis.

Quantum information: An experiment with a photon

- Polarization of polaroid B is $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$
- This vector, together with $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$ forms a basis for the 2-dimensional vector space
- of which $\{|0\rangle, |1\rangle\}$ is another one
  (the so-called computational basis).
- Expressing $|0\rangle$ in terms of $|+\rangle$ and $|-\rangle$ yields

$$|0\rangle = \frac{1}{\sqrt{2}}|-\rangle + \frac{1}{\sqrt{2}}|+\rangle$$

  which explains why a visible effect appears in the wall:
  the photon goes through C with 50% of probability (i.e. $|\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$).

A unit of quantum information

Photon's polarization states are represented as unit vectors in a 2-dimensional complex vector space, typically as a

non trivial linear combination $\equiv$ superposition of vectors in a basis

$$|v\rangle = \alpha|0\rangle + \beta|1\rangle$$

A basis provides an observation (or measurement) tool, e.g.

$$\bigcirc\frown\bigcirc = \{|0\rangle, |1\rangle\} \quad \text{or} \quad \bigcirc\frown\bigcirc = \{|-\rangle, |+\rangle\}$$

A unit of quantum information

Observation of a state

$$|v\rangle = \alpha|u\rangle + \beta|u'\rangle$$

transforms the state into one of the basis vectors in

$$\bigcirc\!\frown\!\bigcirc = \{|u\rangle, |u'\rangle\}$$

with a probability given by the norm square of its amplitude.

In other (the quantum mechanics) words:
measurement collapses $|v\rangle$ into a classic state
A subsequent measurement wrt the same basis returns $|u\rangle$ with probability 1.

Qubits

The space of possible polarization states of a photon is an example of a

$\boxed{\text{quantum bit (qubit)}}$

as does any quantum system (e.g. a electron spin or an atom) that can be modelled by a two-dimensional complex vector space.

- In practice it is not yet clear which two-state systems will be most suitable for physical realizations of qubits: it is likely that a variety of physical representation will be used.
- and they are fragile and unstable which entails the need for qubits' strong isolation, typically very hard to achieve.

Qubits

A qubit has ... a continuum of possible values

- potentially, it can store lots of classical data
- but the amount of information that can be extracted from a qubit by measurement is severely restricted: a single measurement yields at most a single classical bit of information;
- as measurement changes the state, one cannot make two measurements on the original state of a qubit.
- as an unknown quantum state cannot be cloned, it is not possible to measure a qubit's state in two ways, even indirectly by copying its state and measuring the copy.

Summing up

This lecture sought to present the quantum perspective on information and computation, emphasizing several formal similarities with the classical case. Indeed, their mathematical renderings are not so diverse.

However, in its foundations, namely in the way information is understood and (weird) effects of the physical reality are used as computational resources, both perspectives are remarkably different.

The previous lecture has already unveiled part of the game:

- If formally a quantum state distinguishes itself from a probabilistic one by replacing real coefficients by complex ones, how far-reaching is that move?
- Which physical reality does this move capture?

The *motto*

Information is a physical entity; Any computation is a physical process.

Thus, any abstract computational model has always to reflect what we know about reality, and quantum theory is our current best tool in this road.

Recall: The physical Church-Turing thesis

The class of functions computable in finite time by a physical device can be computed by a Turing machine.

which a statement about the physical universe, while the Church-Turing thesis — *The class of functions computable by a Turing machine corresponds exactly to the class of functions which can be described by an algorithm* — is a claim about what counts as an algorithm