Lecture 3:
What's in a qubit?

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What is a qubit?

A qubit is a quantum system represented as a superposition, i.e. a linear combination of normalised and mutually orthogonal quantum states labelled by $|0\rangle$ and $|1\rangle$ with complex coeficients:

$$|v\rangle \; = \; \alpha|0\rangle + \beta|1\rangle \; = \; \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

When state $|v\rangle$ is measured (i.e. observed) one of the two basic states $|0\rangle, |1\rangle$ is returned with probability

$$|\alpha|^2 \quad \text{and} \quad |\beta|^2$$

respectively.

What is a qubit?

Being probabilities, the norm squared of coefficients must satisfy

$$|\alpha|^2 + |\beta|^2 = 1$$

which enforces quantum states to be represented by unit vectors.

In practice, a qubit is a microscopic system, such as an atom, a nuclear spin, or a polarised photon.

The state space of a qubit

Global phases are useless

Unit vectors equivalent up to multiplication by a complex number of modulus one, i.e. a phase factor $e^{i\theta}$, represent the same state.

Let $|v\rangle = \alpha|u\rangle + \beta|u'\rangle$ and $e^{i\theta}|v\rangle = e^{i\theta}\alpha|u\rangle + e^{i\theta}\beta|u'\rangle$

$$|e^{i\theta}\alpha|^2 = (\overline{e^{i\theta}\alpha})(e^{i\theta}\alpha) = (e^{-i\theta}\overline{\alpha})(e^{i\theta}\alpha) = \overline{\alpha}\alpha = |\alpha|^2$$

and similarly for β.

As the probabilities $|\alpha|^2$ and $|\beta|^2$ are the only measurable quantities, the global phase has no physical meaning.

Representation redundancy

qubit state space $\neq$ complex vector space used for representation

The state space of a qubit

Relative phase

It is a measure of the angle between the two complex numbers.
Thus, it cannot be discarded!

Those are different states

$$\frac{1}{\sqrt{2}}(|u\rangle + |u'\rangle) \quad \frac{1}{\sqrt{2}}(|u\rangle - |u'\rangle) \quad \frac{1}{\sqrt{2}}(e^{i\theta}|u\rangle + |u'\rangle)$$

...

Going general

Quantum computation explores the laws of quantum theory as computational resources.

Thus, the principles of the former are directly derived from the postulates of the latter.

- The representation postulate
- The evolution postulate
- The composition postulate
- The measurement postulate

Quantum states

The State (or representation) Postulate

The state space of a quantum system is described by a unit vector in a Hilbert space up to a global phase

- In practice, with finite resources, one cannot distinguish between a continuous state space from a discrete one with arbitrarily small minimum spacing between adjacente locations.
- One may, then, restrict to finite-dimensional Hilbert spaces.

The underlying maths is that of Hilbert spaces.

A Language: Dirac's notation

The starting point is a vector space over the complex field.

Dirac's bra/ket notation provides a handy (easy to calculate with) way to represent its elements and constructions

$|u\rangle$ A ket stands for a vector in an Hilbert space V. In $\mathcal{C}^n$, a column vector of complex entries. The identity for $+$ (the zero vector) is just written 0.

$\langle u|$ A bra is a vector in the dual space V^*, i.e. scalar-valued linear maps $V \to \mathcal{C}$ — a row vector in $\mathcal{C}^n$.

There is a bijective correspondence between $|u\rangle$ and $\langle u|$

$$|u\rangle = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \Leftrightarrow \begin{bmatrix} \overline{u}_1 \cdots \overline{u}_n \end{bmatrix} = \langle u|$$

A Language: Dirac's notation

The bijective correspondence between V and V^* is esblished by $\dagger$:

$$(|u\rangle)^\dagger = \langle u| \quad \text{and} \quad (\langle u|)^\dagger = |u\rangle$$

which is antilinear:

$$\left(\sum_i \alpha_1 |u_i\rangle\right)^\dagger = \sum_i \overline{\alpha_i} \langle u_i|; \quad \text{and} \quad \left(\sum_i \alpha_1 \langle u_i|\right)^\dagger = \sum_i \overline{\alpha_i} |u_i\rangle$$

A Language: Dirac's notation

Multiplying a bra and ket gives a ...

a bracket: $\langle u| \, |v\rangle$; $\rightsquigarrow$ $\langle u|v\rangle$

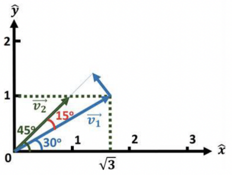
which provides the remaining link in our underlying mathematics:

the inner product

Notation: $\langle u|v\rangle \equiv \langle u, v\rangle \equiv (|u\rangle, |v\rangle)$

Recall: what is a inner product?

From a linear algebra textbook: for $\vec{v_1} = \sqrt{3}\,\widehat{x} + \widehat{y}$ and $\vec{v_2} = \widehat{x} + \widehat{y}$, the inner (or dot, or scalar) product is



$$\langle \vec{v_1}, \vec{v_2} \rangle \;=\; |\vec{v_1}||\vec{v_2}| \cos 15 \;\approx\; 2.732$$

i.e. the product of the (magnitudes of) $\vec{v_2}$ by $\vec{v_1}$ projected in the direction of $\vec{v_2}$.

The inner product measures how much the two vectors have in common after scaled by their magnitudes.

The underlying maths: Hilbert spaces

Complex, inner-product vector space

A complex vector space with inner product

$$\langle - | - \rangle : V \times V \longrightarrow \mathbb{C}$$

such that

$$\begin{aligned}
&(1) \quad \langle v | \sum_i \lambda_i \cdot | w_i \rangle \rangle \;=\; \sum_i \lambda_i \langle v | w_i \rangle \\
&(2) \quad \langle v | w \rangle = \overline{\langle w | v \rangle} \\
&(3) \quad \langle v | v \rangle \geq 0 \;\; (\text{with equality iff } | v \rangle = 0)
\end{aligned}$$

Note: $\langle - | - \rangle$ is conjugate linear in the first argument:

$$\langle \sum_i \lambda_i \cdot | w_i \rangle | v \rangle \;=\; \sum_i \overline{\lambda_i} \langle w_i | v \rangle$$

and linear in the second.

Inner product: examples

In $\mathcal{C}$

$$\langle a+bi|c+di\rangle = \overline{(a+bi)}(c+di)$$

In $\mathcal{C}^n$

$$\langle u|v\rangle = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \underbrace{\begin{bmatrix} \overline{u_1} & \overline{u_2} & \cdots & \overline{u_n} \end{bmatrix}}_{\langle u|} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{|v\rangle} = \sum_{i=1}^{n} \overline{u_i} v_i$$

where $\overline{c}$ is the complex conjugate of c

The dual space

V^*

If V is a Hilbert space, V^* is the space of linear maps from V to $\mathcal{C}$.

Elements of V^* are denoted by

$$\langle u| : V \longrightarrow \mathcal{C}$$

as discussed above, and defined through the inner product:

$$\langle u|(|v\rangle) = \langle u|v\rangle$$

In a matricial representation $\langle u|$ is the Hermitian conjugate, or conjugate transpose of $|u\rangle$,
i.e. the transpose of the vector formed by the complex conjugate of each element in $|u\rangle$.

Maps in a Hilbert space

Maps between Hilbert spaces are, of course, linear transformations which are typically represented by matrices whose entries are computed through the inner product:

$$A_{i,j} = \langle i|A|j\rangle$$

The adjoint map

The adjoint $U^\dagger$ of a map $U : V \longrightarrow V$ is the unique map satisfying

$$(U^\dagger|w\rangle, |v\rangle) = (|w\rangle, U|v\rangle) \quad \text{or, in a simplified notation}$$
$$\langle U^\dagger w|v\rangle = \langle w|Uv\rangle$$

an equality that is often denoted by the expression

$$\langle u|U|v\rangle$$

Concretely,

$$\langle i|U^\dagger|j\rangle = \overline{\langle j|U|i\rangle}$$

i.e., the matrix representation of $U^\dagger$ is the conjugate transpose of U

Properties

- $(UV)^\dagger = V^\dagger U^\dagger$
- $U^{\dagger\dagger} = U$

Other old friends: Norms and orthogonality

Old friends

- $|v\rangle$ and $|w\rangle$ are orthogonal if $\langle v|w\rangle = 0$
- norm: $\||v\rangle| = \sqrt{\langle v|v\rangle}$, a nonnegative real number

This norm satisfies $\||v\rangle + |w\rangle| \;\leq\; \||v\rangle| + \||w\rangle|$ due to the Cauchy-Schwarz inequality:

$$\langle x|y\rangle^2 \;\leq\; \langle x|x\rangle$$

Other old friends: Norms and orthogonality

Recall

- normalization: $\frac{|v\rangle}{||v\rangle|}$
- $|v\rangle$ is a unit vector if $||v\rangle| = 1$
- A set of vectors $\{|i\rangle, |j\rangle, \cdots,\}$ is orthonormal if each $|i\rangle$ is a unit vector and

$$\langle i|j\rangle \;=\; \delta_{i,j} \;=\; \begin{cases} i = j & \Rightarrow 1 \\ \text{otherwise} & \Rightarrow 0 \end{cases}$$

Other old friends: Bases

Orthonormal basis

A orthonormal basis for a Hilbert space V of dimension n is a set $B = \{|i\rangle\}$ of n linearly independent elements of V st

- $\langle i|j\rangle = \delta_{i,j}$ for all $|i\rangle, |j\rangle \in B$
- and B spans V, i.e. every $|v\rangle$ in V can be written as

$$|v\rangle \;=\; \sum_i \alpha_i |i\rangle \quad \text{for some } \alpha_i \in \mathcal{C}$$

Changing representations of a quantum state from one basis to another is a common technique in quantum algorithms.
(cf *why superposition can help you to date the girl/boy of your choice?)*

Example: The Hadamard basis

One of the infinitely many orthonormal bases for a space of dimension 2:

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$
$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

Check e. g.

$$\langle +|-\rangle = \frac{1}{2}(|0\rangle + |1\rangle, |0\rangle - |1\rangle) = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = 0$$

$$\||+\rangle| = \sqrt{\langle +|+\rangle} = \sqrt{\frac{1}{2}(|0\rangle + |1\rangle, |0\rangle + |1\rangle)} = \sqrt{\frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}} = 1$$

Old friends: Bases

If $|v\rangle$ is expressed wrt an orthonormal basis, i.e. $|v\rangle = \sum_i \alpha_i |i\rangle$, then the amplitude of $|v\rangle$ wrt $|i\rangle$ satisfies

$$\alpha_i = \langle i|v\rangle$$

because

$$\begin{aligned} \langle i|v\rangle &= \langle i| \sum_j \alpha_j j\rangle \\ &= \sum_j \alpha_j \langle i|j\rangle \\ &= \sum_j \alpha_j \delta_{i,j} \\ &= \alpha_i \end{aligned}$$

Hilbert spaces

The complete picture

An Hilbert space is an inner-product space V st the metric defined by its norm turns V into a complete metric space, i.e.any Cauchy sequence

$$|v_1\rangle, |v_2\rangle, \cdots$$

$$\forall_{\epsilon>0}\ \exists_N\ \forall_{m,n>N}\ \ ||v_m - v_n\rangle| \leq \epsilon$$

converges
(i.e. there exists an element $|s\rangle$ in V st $\forall_{\epsilon>0}\ \exists_N\ \forall_{n>N}\ \ ||s - v_n\rangle| \leq \epsilon$)

The completeness condition is trivial in finite dimensional vector spaces

The evolution postulate

If a quantum state is a ray (i.e., a unit vector in a Hilbert space V up to a global phase), its evolution is specified by a certain kind of linear operators $U : V \longrightarrow V$.

Linearity

$$U\left(\sum_j \alpha_j |v_j\rangle\right) = \sum_j \alpha_j U(|v_j\rangle)$$

Just by itself, linearity has an important consequence:

quantum states cannot be cloned

The no-cloning theorem

Linearity implies that quantum states cannot be cloned

Let $U(|a\rangle|0\rangle) = |a\rangle|a\rangle$ be a 2-qubit operator and $|c\rangle = \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle)$ for $|a\rangle$, $|b\rangle$ orthogonal.

If U is linear, then

$$U(\frac{1}{\sqrt{2}}(|a\rangle + |b\rangle)) = \frac{1}{\sqrt{2}}(U(|a\rangle|0\rangle) + U(|b\rangle|0\rangle)) = \frac{1}{\sqrt{2}}(|a\rangle|a\rangle + |b\rangle|b\rangle)$$

which is different from

$$U(|c\rangle|0\rangle) = |c\rangle|c\rangle = \frac{1}{\sqrt{2}}(|a\rangle|a\rangle + |a\rangle|b\rangle + |b\rangle|a\rangle + |b\rangle|b\rangle)$$

Computing with qubits

The evolution postulate

The evolution over time of the state of a closed quantum system is described by a unitary operator.

The evolution is linear

$$U\left(\sum_j \alpha_j |v_j\rangle\right) = \sum_j \alpha_j U(|v_j\rangle)$$

and preserves the normalization constraint

$$\text{If } \sum_j \alpha_j U(|v_j\rangle) = \sum_j \alpha'_j |v_j\rangle \text{ then } \sum_j |\alpha'_j|^2 = 1$$

Computing with qubits

Preservation of the normalization constraint means that unit length vectors (and thus orthogonal subspaces) are mapped by U to unit length vectors (and thus to orthogonal subspaces).

This entails a condition on valid quantum operators: they must preserve the inner product, i.e.

$$(U|v\rangle, U|w\rangle) = \langle v|U^\dagger U|w\rangle = \langle v|w\rangle$$

which is the case iff U is unitary, i.e. $U^\dagger = U^{-1}$, because

$$U^\dagger U = UU^\dagger = I$$

Unitarity

- Preserving the inner product means that a unitary operator maps orthonormal bases to orthonormal bases.
- Conversely, any operator with this property is unitary.
- If given in matrix form, being unitary means that the set of columns of its matrix representation are orthonormal (because the jth column is the image of $U|j\rangle$). Equivalently, rows are orthonormal (why?)

Unitarity

Unitarity is the only constraint on quantum operators: Any unitary matrix specifies a valid quantum operator.

This means that there are many non-trivial operators on a single qubit, in contrast with the classical case where the only non-trivial operation on a bit is complement.

Finally, because the inverse of a unitary matrix is also a unitary matrix, a quantum operator can always be inverted by another quantum operator

Unitary transformations are reversible

Recall the golden pattern

$$\underbrace{H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}}_{\text{Hadamard gate}} \quad \text{and} \quad \underbrace{P_\varphi = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{bmatrix}}_{\text{Phase shift gate}}$$

yielding

$$A = HP_\varphi H = e^{i\frac{\varphi}{2}} \begin{bmatrix} \cos\frac{\varphi}{2} & -i\sin\frac{\varphi}{2} \\ -i\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{bmatrix} = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix}$$

Recall the golden pattern

If the product $HP_\varphi H$ describes the action of the whole circuit, one may also step through its execution (ignoring the global phase), as follows

$$
\begin{aligned}
|0\rangle &\xrightarrow{H} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\
&\xrightarrow{P_\varphi} \frac{1}{\sqrt{2}}(|0\rangle + e^{i\varphi}|1\rangle) \\
&\xrightarrow{H} (\cos(\frac{\varphi}{2})|0\rangle - i\sin(\frac{\varphi}{2})|1\rangle)
\end{aligned}
$$

All the interference is controlled by the phase gate P_φ.

Phase gates: Three remarkable cases

Phase-flip $(\varphi = \pi)$ $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\frac{\pi}{4}$-phase $(\varphi = \frac{\pi}{2})$ $S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

$\frac{\pi}{8}$-phase $(\varphi = \frac{\pi}{4})$ $T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix}$

Note that

$$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{bmatrix} = \begin{bmatrix} e^{-i\frac{\varphi}{2}} & 0 \\ 0 & e^{i\frac{\varphi}{2}} \end{bmatrix}$$

why?

Pauli gates

$$\text{Identity} \quad I = |0\rangle\langle 0| + |1\rangle\langle 1| = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{Bit-flip} \quad X = |1\rangle\langle 0| + |0\rangle\langle 1| = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{Phase-flip} \quad Z = |0\rangle\langle 0| - |1\rangle\langle 1| = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = P_\pi$$

$$\text{BitPhase-flip} \quad Y = i(-|1\rangle\langle 0| + |0\rangle\langle 1|) = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

The X (bit-flip) gate

The $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ gate

$$X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$$

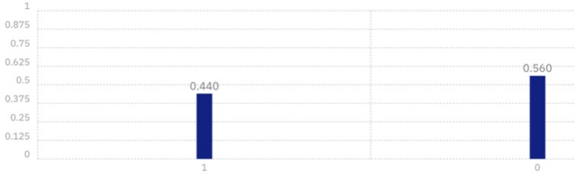
Pauli gates: Properties

Pauli gates

- are unitary and Hermitian ($G^\dagger = G$)
- square to the identity
- are anticommutative: $XY = -YX$, $XZ = -ZX$ and $YZ = -ZY$.

The Hadamard gate creates superpositions

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$



$$H|0\rangle = |+\rangle = \overbrace{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}^{\text{superposition}}$$

$$H|1\rangle = |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Pauli gates in the golden pattern

A few equalities

$$\begin{aligned} I &= HH \\ X &= HZH \\ Z &= HXH \\ -Y &= HYH \end{aligned}$$

States and gates

Quantum gates

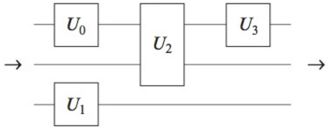
A gate is a transformation that acts on only a small number of qubits
Differently from the classical case, they do not necessarily correspond to physical objects

Is there a complete set?

In general no: there are uncountably many quantum transformations, and a finite set of generators can only generate countably many elements.

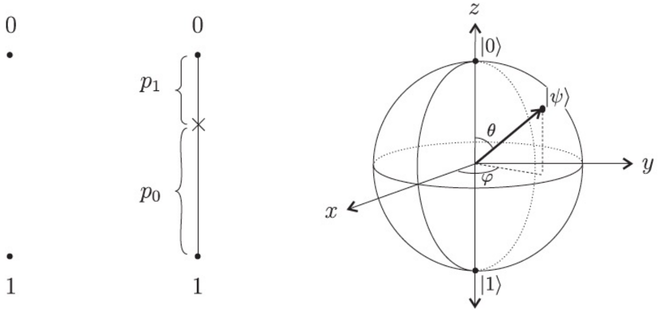
However, it is possible for finite sets of gates to generate arbitrarily close approximations to all unitary transformations.

Circuits



A handy representation for a single qubit

Deterministic, probabilistic and quantum bits



(from [Kaeys *et al*, 2007])

A handy representation for a single qubit

There is a simple way to visualise single-qubit state vectors. i.e.

$$|\phi\rangle \;=\; \alpha|0\rangle + \beta|1\rangle$$

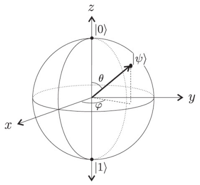
constrained by the relation

$$|\alpha|^2 + |\beta|^2 \;=\; 1$$

in terms of Euclidean vectors in three dimensions as

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

Inspecting the Bloch sphere



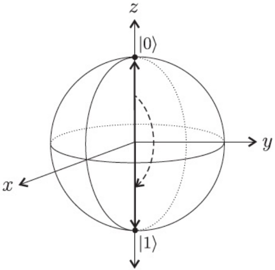
- Poles represent the classical bits. In general, orthogonal states correspond to antipodal points and every diameter to a basis for the single-qubit state space.
- Once measured a qubit collapses to one of the two poles. Which pole depends exactly on the arrow direction. The angle θ measures the colapsing probability: If the arrow points at the equator, there is 50-50 chance to collapse to any of the two poles.

Moreover, any unitary transformation on the state vector induces a rotation of the corresponding Bloch vector.

Inspecting the Bloch sphere: The X gate

The action of a 1-gate U on a quantum state $|\phi\rangle$ can be thought of as a rotation of the Bloch vector for $|\phi\rangle$ to the Bloch vector for $U|\phi\rangle$, eg.

Example: X



is a rotation about the x axis.

Inspecting the Bloch sphere: The phase shift gate P_ϕ

$$P_\phi = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}$$

$$\begin{aligned} P_\phi |0\rangle &= |0\rangle \\ P_\phi |1\rangle &= e^{i\phi}|1\rangle \end{aligned}$$

The gate acts by

$$\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle \mapsto \cos\frac{\theta}{2}|0\rangle + e^{i(\varphi+\phi)}\sin\frac{\theta}{2}|1\rangle$$

The azimuthal angle changes from φ to $\varphi + \phi$ and so the Bloch sphere is rotated anticlockwise by ϕ about the z-axis.

Note that rotating a vector wrt the z-axis does not affect which state the arrow will collapse to, when measured.

A parenthesis on the construction of the Bloch sphere

$$\left(\quad \ldots \right.$$

The Bloch sphere: Representing $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

- Express $|\psi\rangle$ in polar form

$$|\psi\rangle = \rho_1 e^{i\varphi_1}|0\rangle + \rho_2 e^{i\varphi_2}|1\rangle$$

- Eliminate one of the four real parameters multiplying by $e^{-i\varphi_1}$

$$|\psi\rangle = \rho_1|0\rangle + \rho_2 e^{i(\varphi_2-\varphi_1)}|1\rangle = \rho_1|0\rangle + \rho_2 e^{i\varphi}|1\rangle$$

making $\varphi = \varphi_2 - \varphi_1$,

which is possible because global phase factors are physically meaningless.

The Bloch sphere: Representing $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

- Switching back the coefficient of $|1\rangle$ to Cartesian coordinates

$$|\psi\rangle = \rho_1|0\rangle + (a + bi)|1\rangle$$

the normalization constraint

$$|\rho_1|^2 + |a + ib|^2 \;=\; |\rho_1|^2 + (a - ib)(a + ib) \;=\; |\rho_1|^2 + a^2 + b^2 \;=\; 1$$

yields the equation of a unit sphere in the real tridimensional space with Cartesian coordinates: (a, b, ρ_1).

The Bloch sphere: Representing $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

- The polar coordinates (ρ, θ, φ) of a point in the surface of a sphere relate to Cartesian ones (x, y, z) through the correspondence

$$
\begin{aligned}
x =& \rho \sin\theta \cos\varphi \\
y =& \rho \sin\theta \sin\varphi \\
z =& \rho \cos\theta
\end{aligned}
$$

- Recalling $\rho = 1$ (cf unit sphere),

$$
\begin{aligned}
|\psi\rangle &= \rho_1|0\rangle + (a + ib)|1\rangle \\
&= \cos\theta|0\rangle + \sin\theta(\cos\varphi + i\sin\varphi)|1\rangle \\
&= \cos\theta|0\rangle + e^{i\varphi}\sin\theta|1\rangle
\end{aligned}
$$

which, with two parameters, defines a point in the sphere's surface.

The Bloch sphere

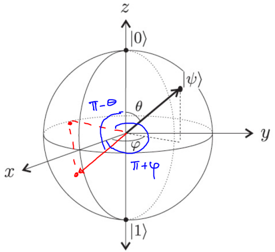
Actually, one may just focus on the upper hemisphere ($0 \leq \theta' \leq \frac{\pi}{2}$) as opposite points in the lower one differ only by a phase factor of -1, as suggested by

$$\begin{aligned} \theta' = 0 \;&\Rightarrow\; |\psi\rangle \;=\; \cos 0|0\rangle + e^{i\varphi}\sin 0|1\rangle \;=\; |0\rangle \\ \theta' = \frac{\pi}{2} \;&\Rightarrow\; |\psi\rangle \;=\; \cos\frac{\pi}{2}|0\rangle + e^{i\varphi}\sin\frac{\pi}{2}|1\rangle \;=\; e^{i\varphi}|1\rangle \;=\; |1\rangle \end{aligned}$$

Note that longitude (φ) is irrelevant in a pole!

The Bloch sphere

Indeed, let $|\psi'\rangle$ be the opposite point on the sphere with polar coordinates $(1, \pi - \theta, \varphi + \pi)$:



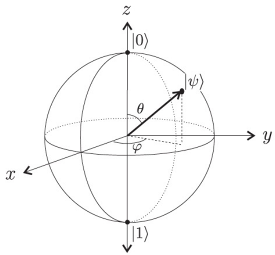
$$
\begin{aligned}
|\psi'\rangle &= \cos(\pi - \theta)|0\rangle + e^{i(\varphi+\pi)} \sin(\pi - \theta)|1\rangle \\
&= -\cos\theta|0\rangle + e^{i\varphi} e^{i\pi} \sin\theta|1\rangle \\
&= -\cos\theta|0\rangle + e^{i\varphi} \sin\theta|1\rangle \\
&= -|\psi\rangle
\end{aligned}
$$

The Bloch sphere

which leads to

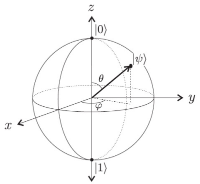
$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

where $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$



The map $\frac{\theta}{2} \mapsto \theta$ is one-to-one at any point but:
all points on the equator are mapped into a single point: the south pole.

The Bloch sphere



- The poles represent the classical bits. In general, orthogonal states correspond to antipodal points and every diameter to a basis for the single-qubit state space.
- Once measured a qubit collapses to one of the two poles. Which pole depends exactly on the arrow direction: The angle θ measures that probability: If the arrow points at the equator, there is 50-50 chance to collapse to any of the two poles.
- Rotating a vector wrt the z-axis results into a phase change (φ), and does not affect which state the arrow will collapse to, when measured.

End of parenthesis

...)