A Zoo of Monads

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Monad of Exceptions

Useful for raising exceptions and keeping track of them along computations

$$\frac{}{\Gamma \vdash_{\mathsf{c}} e : \mathbb{A}}$$

Definition

Set-constructor: $X \mapsto X + 1$

Unit: $\eta : X \to X + 1, \quad x \mapsto i_1(x)$

Kleisli lifting:

$$\frac{f : X \to Y + 1}{f^{\star} : X + 1 \to Y + 1, \quad f^{\star} = [f, i_2]}$$

Monad of Durations

Useful for keeping track of execution times

$$\frac{\Gamma \vdash_{\mathsf{c}} M : \mathbb{A}}{\Gamma \vdash_{\mathsf{c}} \mathrm{wait}_{\mathrm{n}}(M) : \mathbb{A}} \qquad \mathrm{wait}_{\mathrm{n}}(\mathrm{wait}_{\mathrm{m}}(M)) = \mathrm{wait}_{\mathrm{n+m}}(M)$$

Definition

Set-constructor: $X \mapsto \mathbb{N} \times X$

Unit: $\eta : X \to \mathbb{N} \times X, \quad x \mapsto (0, x)$

Kleisli lifting:

$$\frac{f : X \to \mathbb{N} \times Y}{f^{\star}(n, x) = (n + m, y) \text{ where } f(x) = (m, y)}$$

Monad of Boolean inputs

Useful for modelling computations depending on Boolean inputs

$$\frac{\Gamma \vdash_{\mathsf{c}} M : \mathbb{A} \quad \Gamma \vdash_{\mathsf{c}} N : \mathbb{A}}{\Gamma \vdash_{\mathsf{c}} \mathrm{read}(M, N) : \mathbb{A}}$$

Definition

Set-constructor: $X \mapsto \mathrm{LTree}(X)$

Unit: $\eta : X \to \mathrm{LTree}(X), \quad x \mapsto \mathrm{Leaf}(x)$

Kleisli lifting:

$$\frac{f : X \to \mathrm{LTree}(Y)}{f^\star(\mathit{Leaf}\ x) = f(x), \quad f^\star(\mathit{Fork}(x, y)) = \mathit{Fork}(f^\star(x), f^\star(y))}$$

Monad of Boolean Messages

Useful for keeping track of messages produced by computations

$$\frac{\Gamma \vdash_{\mathsf{c}} M : \mathbb{A}}{\Gamma \vdash_{\mathsf{c}} \mathrm{write}_{\mathrm{tt}}(M) : \mathbb{A}} \qquad \frac{\Gamma \vdash_{\mathsf{c}} M : \mathbb{A}}{\Gamma \vdash_{\mathsf{c}} \mathrm{write}_{\mathrm{ff}}(M) : \mathbb{A}}$$

Definition

Set-constructor: $X \mapsto [Bool] \times X$

Unit: $\eta : X \to [Bool] \times X, \quad x \mapsto ([], x)$

Kleisli lifting:

$$\frac{f : X \to [Bool] \times Y}{f^{\star}(l, x) = (l +\!\!+ m, y) \text{ where } f(x) = (m, y)}$$

Boolean inputs + Boolean messages

It is possible to combine the two previous monads into a new monad

The new monad serves as a model of communication between computations (recall process algebra)

Nondeterministic choice

Useful for modelling nondeterministic computation

$$\frac{\Gamma \vdash_{\mathsf{c}} M : \mathbb{A} \quad \Gamma \vdash_{\mathsf{c}} N : \mathbb{A}}{\Gamma \vdash_{\mathsf{c}} \mathrm{choice}(M, N) : \mathbb{A}} \qquad \mathrm{choice}(M, M) = M$$

$$\mathrm{choice}(M, N) = \mathrm{choice}(N, M), \quad \text{and associativity}$$

Definition

Set-constructor: $X \mapsto \mathrm{P}(X)$

Unit: $\eta : X \to \mathrm{P}(X), \quad x \mapsto \{x\}$

Kleisli lifting:

$$\frac{f : X \to \mathrm{P}(Y)}{f^{\star}(A) = \cup_{a \in A} f(a)}$$

Boolean inputs + Boolean messages + Nondeterminism

It is possible to combine the three previous monads into a new monad

The new monad handles communication and concurrency

It serves as a basis for process algebra

Monad of Cyber-Physical Computation

Useful for modelling interactions with physical processes

$$\frac{\Gamma \vdash_{\mathsf{c}} M : \mathbb{A}}{\Gamma \vdash_{\mathsf{c}} \mathrm{run}(\mathrm{diff.\ eq\ for\ n}; M) : \mathbb{A}}$$

Definition

Set-constructor: $X \mapsto \left(\sum_{r\in[0,\infty)} \mathbb{R}^{n^{[0,r)}}\right) \times X$

Unit: $\eta : X \to \left(\sum_{r\in[0,\infty)} \mathbb{R}^{n^{[0,r)}}\right) \times X, \quad x \mapsto (!, x)$ where $! \in \mathbb{R}^{n^{[0,0)}}$

Kleisli lifting:

$$\frac{f : X \to \left(\sum_{r\in[0,\infty)} \mathbb{R}^{n^{[0,r)}}\right) \times Y}{f^{\star}(l, x) = (l + m, y) \text{ where } f(x) = (m, y)}$$

Monad of internal Boolean memory

Useful for manipulating internal memory

$$\frac{\Gamma \vdash_{\mathsf{c}} M : \mathbb{A} \quad \Gamma \vdash_{\mathsf{c}} N : \mathbb{A}}{\Gamma \vdash_{\mathsf{c}} \mathrm{lookup}(M, N) : \mathbb{A}}$$

$$\frac{\Gamma \vdash_{\mathsf{c}} M : \mathbb{A}}{\Gamma \vdash_{\mathsf{c}} \mathrm{write}_{\mathrm{tt}}(M) : \mathbb{A}} \qquad \frac{\Gamma \vdash_{\mathsf{c}} M : \mathbb{A}}{\Gamma \vdash_{\mathsf{c}} \mathrm{write}_{\mathrm{ff}}(M) : \mathbb{A}}$$

Several equations expressing the interaction between computations and internal memory, *e.g.*

$$\begin{aligned} \mathrm{lookup}(M, M) &= M \\ \mathrm{write}_{\mathrm{tt}}(\mathrm{lookup}(M, N)) &= M \\ &\dots \end{aligned}$$

Monad of internal Boolean memory

Useful for manipulating internal memory

$$\frac{\Gamma \vdash_{\mathsf{c}} M : \mathbb{A} \quad \Gamma \vdash_{\mathsf{c}} N : \mathbb{A}}{\Gamma \vdash_{\mathsf{c}} \mathrm{lookup}(M, N) : \mathbb{A}}$$

$$\frac{\Gamma \vdash_{\mathsf{c}} M : \mathbb{A}}{\Gamma \vdash_{\mathsf{c}} \mathrm{write}_{\mathrm{tt}}(M) : \mathbb{A}} \qquad \frac{\Gamma \vdash_{\mathsf{c}} M : \mathbb{A}}{\Gamma \vdash_{\mathsf{c}} \mathrm{write}_{\mathrm{ff}}(M) : \mathbb{A}}$$

Definition

Set-constructor: $X \mapsto (Bool \times X)^{Bool}$

Unit: $\eta : X \to (Bool \times X)^{Bool}, \quad x \mapsto (b \mapsto (b, x))$

Kleisli lifting:

$$\frac{f : X \to (Bool \times Y)^{Bool}}{f^{\star}(b \mapsto (b', x)) = b \mapsto (b'', y) \text{ where } f(x) = b' \mapsto (b'', y)}$$

Other monads

See other monads in:

`https://wiki.haskell.org/All_About_Monads#A_Catalog_of_Standard_Monads`