Revisiting Transition Systems — going probabilistic

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An alternative characterisation

Recall the definition of a LTS in Lecture 1. The isomorphism between relations $R \subseteq A \times B$ and functions $f : A \longrightarrow \mathcal{P}B$, given by

$$\langle a, b \rangle \in R \;\equiv\; b \in f\,a$$

supports an alternative, functional characterisation of LTS:

$$\langle S, N, \longrightarrow \rangle \;\equiv\; \alpha : S \longrightarrow \mathcal{P}(\mathbb{N} \times S)$$

given by

$$s \xrightarrow{a} s' \;\equiv\; \langle a, s' \rangle \in \alpha\, s$$

which allows us to easily draw a taxonomy of simple transition systems

A taxonomy of simple transition systems

$\alpha : S \longrightarrow \mathcal{P}(S)$	unlabelled TS
$\alpha : S \longrightarrow \mathbb{N} \times S + \mathbf{1}$	partial LTS (generative)
$\alpha : S \longrightarrow (S + \mathbf{1})^{\mathbb{N}}$	partial LTS (reactive)
$\alpha : S \longrightarrow \mathcal{P}(\mathbb{N} \times S)$	non deterministic LTS (generative)
$\alpha : S \longrightarrow \mathcal{P}(S)^{\mathbb{N}}$	non deterministic LTS (reactive)

Notation for sets

$A \times B$ Cartesian product

$A + B$ disjoint union

B^A function space

$\mathbf{1}$ Singular set: $\mathbf{1} \cong \{*\}$

What about bisimilarity?

For example:

Deterministic case

In a deterministic labelled transition system, two states are bisimilar iff they are trace equivalent, i.e.,

$$s \sim s' \;\Leftrightarrow\; \mathrm{Tr}(s) = \mathrm{Tr}(s')$$

Hint: define a relation R as

$$\langle x, y \rangle \in R \;\Leftrightarrow\; \mathrm{Tr}(x) = \mathrm{Tr}(y)$$

and show R is a bisimulation.

After thoughts

- The taxonomy is driven by the structure on the codomain of function α
- The definition of bisimulation follows, in every case, the same intuition

(... we are starting to think coalgebraically)

After thoughts

More consequences at the morphism level

A morphism $h : \langle S, \mathrm{next} \rangle \longrightarrow \langle S', \mathrm{next}' \rangle$ is a function $h : S \longrightarrow S'$ st the following diagram commutes

$$\begin{array}{ccc} S \times N & \xrightarrow{\ \mathrm{next}\ } & \mathcal{P}S \\ {\scriptstyle h \times id} \big\downarrow & & \big\downarrow {\scriptstyle \mathcal{P}h} \\ S' \times N & \xrightarrow{\ \mathrm{next}'\ } & \mathcal{P}S' \end{array}$$

i.e.,

$$\mathcal{P}h \cdot \mathrm{next} \;=\; \mathrm{next}' \cdot (h \times id)$$

or, going pointwise,

$$\{h\, x \mid x \in \mathrm{next}\ \langle s, a \rangle\} \;=\; \mathrm{next}'\ \langle h\, s, a \rangle$$

After thoughts

More consequences at the morphism level

A morphism $h : \langle S, \text{next}\rangle \longrightarrow \langle S', \text{next}'\rangle$

- preseves transitions:

$$s' \in \text{next}\ \langle s, a\rangle \Rightarrow h\ s' \in \text{next}'\ \langle h\ s, a\rangle$$

- reflects transitions:

$$r' \in \text{next}'\ \langle h\ s, a\rangle \Rightarrow \langle \exists\ s' \in S\ :\ s' \in \text{next}\ \langle s, a\rangle :\ r' = h\ s'\rangle$$

(why?)

After thoughts

- Both definitions coincide at the object level:

$$\langle s, a, s' \rangle \in T \quad \equiv \quad s' \in \mathsf{next}\ \langle s, a \rangle$$

- Wrt morphisms, the relational definition is more general, corresponding, in coalgebraic terms to

$$\mathcal{P}h \cdot \mathsf{next} \quad \subseteq \quad \mathsf{next}' \cdot (h \times id)$$

- More fundamentally: coalgebraic morphisms entail bisimulation: in particular, two states are bisimilar if connected by a morphism.

A zoo of transition systems

Simple transition systems can be extended with actions and suited to different sorts of behaviours (e.g. partial, non deterministic, etc).
... but the zoo is much broader, capturing

- probabilistic transitions (Prism)
- timed transitions (Uppaal, mCRL2)
- continuous evolutions (e.g. of physical processes) (KeYmaera)
- ... and several combinations thereof

(typical support tools are indicated in brown)

Bringing probabilities into the picture

Markov chains

$$\alpha : S \longrightarrow \mathcal{D}(S)$$

where $\mathcal{D}(S)$ is the set of all discrete probability distributions on set S

A Markov chain goes from a state s to a state s' with probability p if

$$\alpha\, s = \mu \;\text{ with }\; \mu\, s' = p > 0$$

Notation

$s \rightsquigarrow \mu$ and $s \overset{p}{\rightsquigarrow} s'$

Bringing probabilities into the picture

Recall

$\mu : S \longrightarrow [0, 1]$ is a discrete probability distribution

- if the support of μ, i.e. the set $\{s \in S \mid \mu\, s > 0\}$, is finite
- and $\sum_{s \in S} \mu\, s = 1$

Examples

Dirac distribution $\mu_s^1 = \{s \mapsto 1\}$

Product distribution $(\mu_1 \times \mu_2)\langle s, t\rangle = (\mu_1\, s) \cdot (\mu_2\, t)$

Bringing probabilities into the picture

Bisimilarity for Markov chains

An equivalence relation $R \subseteq S \times S$ is a bisimulation iff for all $\langle s, t \rangle \in R$

$$\text{if } s \rightsquigarrow \mu \text{ then there is a transition } t \rightsquigarrow \mu' \text{ such that } \mu \equiv_R \mu'$$

where $\mu \equiv_R \mu'$ iff $\mu[C] = \mu'[C]$ for all equivalence class C defined by relation R.

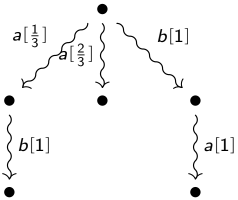
This means that the probability of getting from s or t to an element of C is the same

... of course, any two states in a Markov chain are bisimilar!
(hint: show that $S \times S$ is a bisimulation)

Reactive PTS

$$\alpha : S \longrightarrow (\mathcal{D}(S) + \mathbf{1})^N$$

- $s \overset{a}{\rightsquigarrow} \mu_a$ if $\alpha\, s\, a = \mu_a$
- $s \overset{a[p]}{\rightsquigarrow} s'$ if additionally s' in the support of μ and $\mu_a\, s' = p$
- $s \not\overset{a}{\rightsquigarrow}$ if $\alpha\, s\, a = *$
- Note the role of $\mathbf{1}$ (cf $\emptyset$ in the non deterministic LTS)



Reactive PTS

Bisimulation

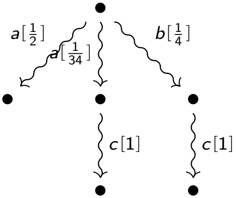
An equivalence relation $R \subseteq S \times S$ is a bisimulation iff for all $\langle s, t \rangle \in R$ and all $a \in N$

if $s \stackrel{a}{\rightsquigarrow} \mu$ then there is a distribution μ' with $t \stackrel{a}{\rightsquigarrow} \mu'$ such that $\mu \equiv_R \mu'$

Generative PTS

$$\alpha : S \longrightarrow \mathcal{D}(N \times S) + \mathbf{1}$$

- $s \overset{a}{\rightsquigarrow} \mu_a$ if $\alpha\, s = \mu$
- $s \overset{a[p]}{\rightsquigarrow} s'$ if additionally $\langle a, s' \rangle$ in the support of μ and $\mu\langle a, s' \rangle = p$
- $s \not\rightsquigarrow$ if $\alpha\, s = *$



Generative PTS

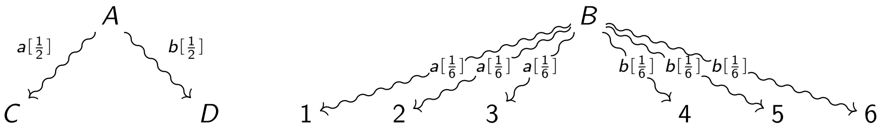
Bisimulation

An equivalence relation $R \subseteq S \times S$ is a bisimulation iff for all $\langle s, t \rangle \in R$

if $s \rightsquigarrow \mu$ then there is a distribution μ' with $t \rightsquigarrow \mu'$ such that $\mu \equiv_{R,A} \mu'$

Example

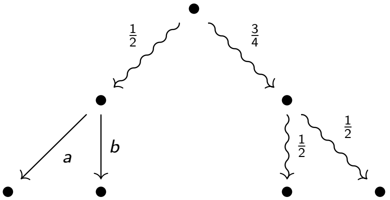
$R = \{\langle A, B\rangle, \langle C, 1\rangle, \langle C, 2\rangle, \langle C, 3\rangle, \langle D, 4\rangle, \langle D, 5\rangle, \langle D, 6\rangle\}$



A taxonomy of probabilistic transition systems

$\alpha : S \longrightarrow \mathcal{D}(S)$	simple PTS (Markov chain)
$\alpha : S \longrightarrow \mathcal{D}(N \times S) + \mathbf{1}$	generative PTS
$\alpha : S \longrightarrow (\mathcal{D}(S) + \mathbf{1})^N$	reactive PTS
$\alpha : S \longrightarrow \mathcal{D}(S) + (N \times S) + \mathbf{1}$	stratified PTS

Alternating PTS



Adding non determinism

$\alpha : S \longrightarrow \mathcal{P}(\mathcal{D}(N \times S))$	strict Segala PTS
$\alpha : S \longrightarrow \mathcal{P}(N \times \mathcal{D}(S))$	simple Segala PTS
$\alpha : S \longrightarrow \mathcal{P}(\mathcal{D}(\mathcal{P}(N \times S)))$	Pnueli-Zuck PTS

Transitions for simple and strict Segala PTS