Paraconsistency for the Working Software Engineer

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MEF
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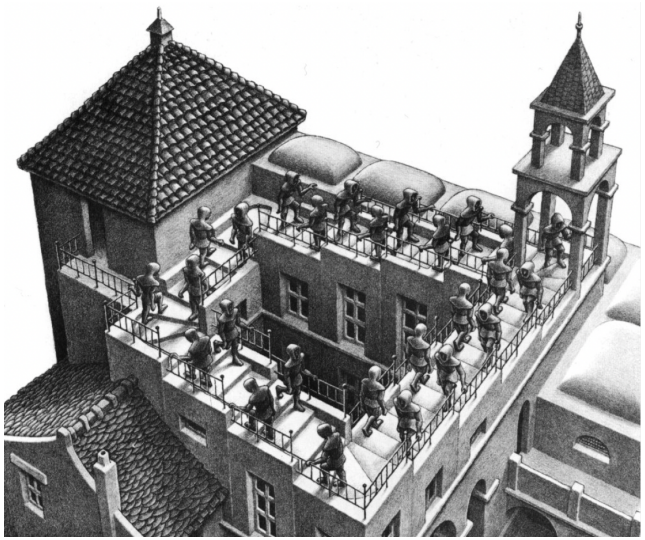
What's up on the table?

The need to articulate different perspectives on complex information/requirements:

local consistency

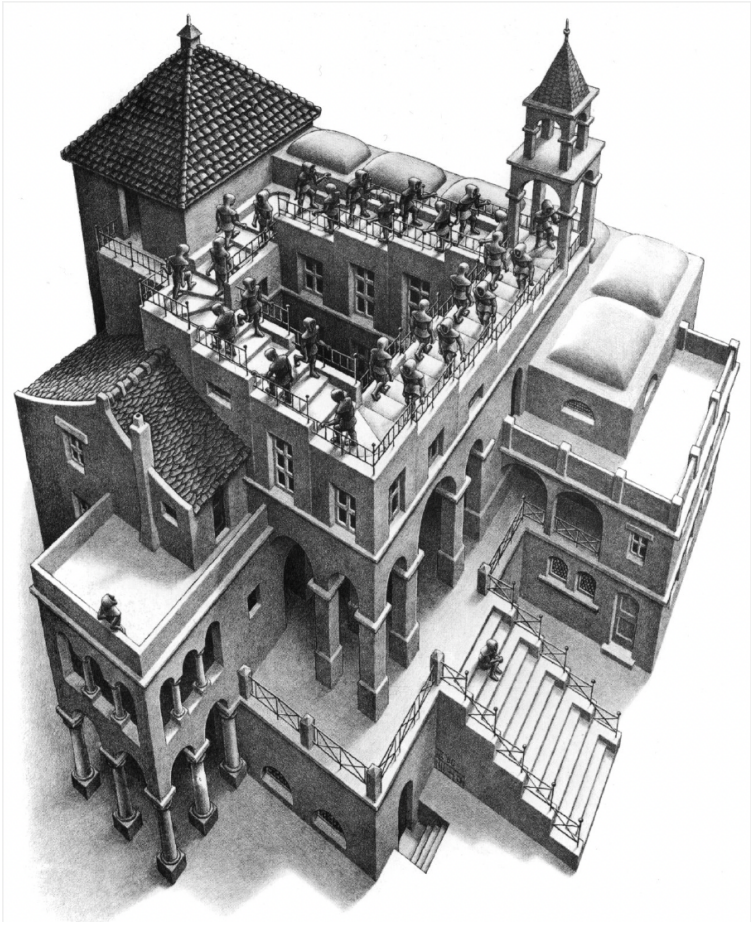


global inconsistency



... and why we should care?

Information (reality?) is not what we thought it was ...



M. C. Escher, *Ascending and descending*

Vagueness and contradiction

In particular, modelling contexts in which bivalent, or even probabilistic reasoning is not enough, entail the need for capturing both

- lack of information (vagueness or uncertainty)
- excess of information (potential inconsistency)

Vagueness and contradiction

Vagueness is addressed in fuzzy logics,

but potentially contradictory information arises in a number of scenarios. (e.g. knowledge representation, data integration, etc.)

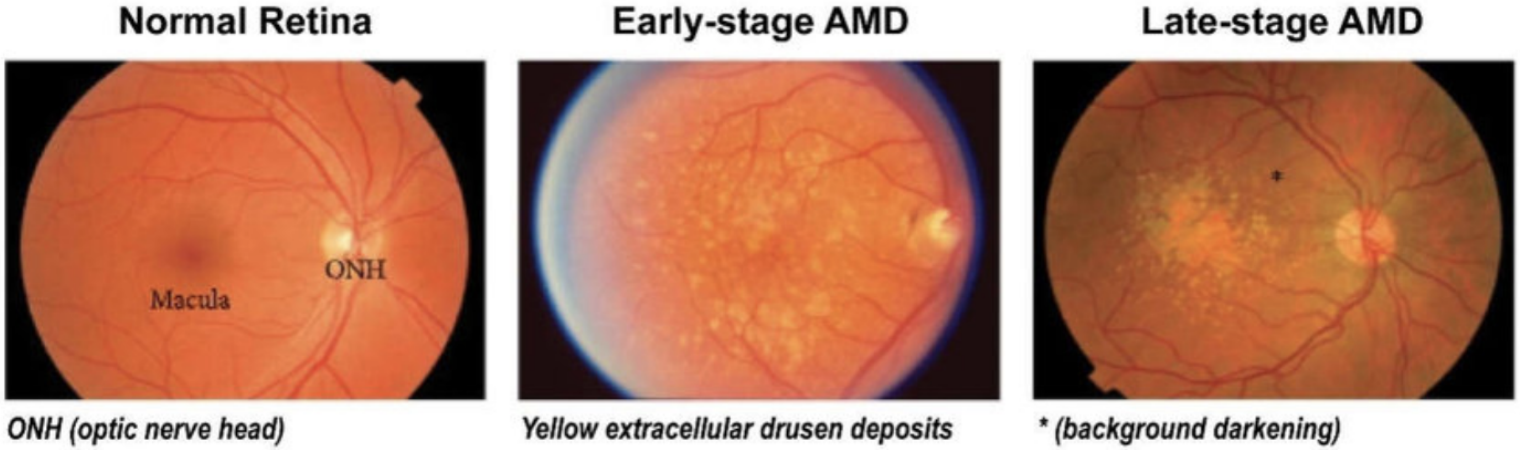
Two real-world scenarios

- Inference over huge data sets for Age-related Macular Degeneration (AMD) diagnosis
- Identification of success factors in inter-agency partnerships

Age-related macular degeneration (AMD) diagnosis

A disease of the macula, leading cause of vision loss in people over 55.

- multifactorial disease, with a complex pathophysiology, for which the onset and progression are different, with different risk factors, environmental and genetic, contributing.
- progression not well understood (e.g. different rates/patterns for similar patients and even between the two eyes of the same patient).



Age-related macular degeneration (AMD) diagnosis

- Highly heterogeneous data collected in large epidemiological studies over long time spans
- Crucial role played by expert assessment: the evidence level assigned to each data factor (e.g. an image) as an enabler for a specific future development of the disease may vary from an expert to another, leading to complex data consolidation processes.
- Thus, potentially contradicting medical judgments cannot be swept under the carpet.

A recently funded proposal supported by a leading eye lab to develop an inference framework able to reason about contradictory, or even inconsistent data in a sound and effective way.

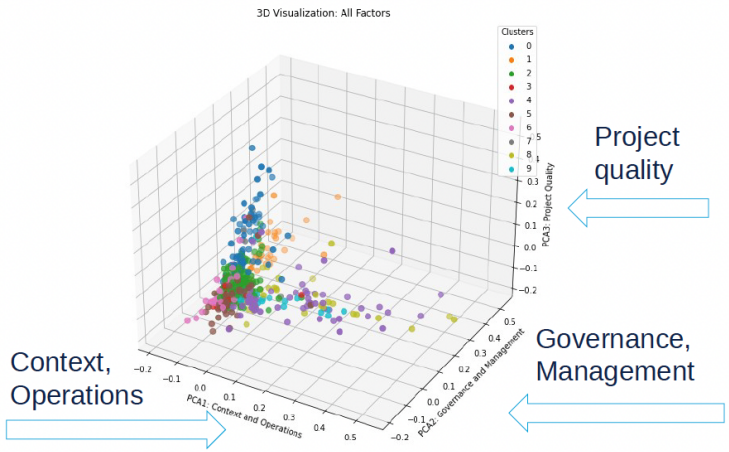


International inter-agency partnerships

19% - 25% international partnerships are unsuccessful, translating into diminished cooperation impact and public expenditure inefficiency.

- Data collected through text mining over a huge documental fund, resorting to different techniques (XAI, neural nets, etc)
- lead to the identification of 735 classes with opposite-polarity weights from different machine learning models

Cluster ID	Number of Factors	Topics
0	67	Quality-related issues
1	32	Governance and Management frameworks
2	343	Bulky cluster (sub-clusters)
3	46	Mutual commitment
4	61	Country-level capacity
5	42	Staff
6	23	Previous collaboration
7	23	Cost-benefit, cost coverage
8	40	Country-level strategy
9	33	Continued relevance in context



Success factors identification in inter-agency partnerships

factor ID		factor text	number of semantically similar corpus sentences	
753-factor list	24-factor list		positive polarity	negative polarity
18	0	mechanisms for expression of stakeholder viewpoint on the partnership	45	43
51	1	gender-sensitive equity indicators measuring reach and benefit for disadvantaged groups (PPP)	54	52
83	2	capacity building	28	37
152	3	local borrower performance history	44	50
157	4	staff requirement and capacity assessment	48	52
162	5	clear, result-oriented, competitive procurement processes, bidding procedures and contracts	OOR	28
192	6	partner performance assessment in project completion reports or evaluation reports	84	81
219	7	project processing and implementation capacity	31	OOR
228	8	risk identification, risk management framework at the strategic and project level from the outset	OOR	40
263	9	stakeholder involvement	68	72
300	10	capacity building enabling full participation	40	47
306	11	focus on outcomes in the partnership strategy	29	OOR
308	12	specific guidance on approaches and outcomes in the partnership strategy	33	35
313	13	cost-benefit analysis	44	32
379	14	early-on attention to sustainability of project benefits	57	72
440	15	donors supportive of the aims and operations of the partnership in the local context	28	40
463	16	capacity building in local partners	OOR	30
468	17	quality of project monitoring and evaluation systems design	61	54
550	18	lack of harmonization of procurement and disbursement procedures	59	69
557	19	outdated procurement procedures	45	58
559	20	rigid and diverging procedures for procurement and disbursement	27	OOR
602	21	complexity of procurement procedures	OOR	28
623	22	lack of stakeholder involvement during initial processes	31	OOR
656	23	involvement of stakeholder country in the design of the partnership or of its programmes, for ownership	32	34

Joint project to design a inference engine to predict partnership performance from ex-ante documents

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Summing up: Data is a mined field

- Not only the values and structure of data changes but also the logic under which this information needs to be understood changes as well
- Informational states may exhibit
  - potentially inconsistent (or partially consistent) data,
  - reflecting the diversity of judgements (e.g. from different domain experts).

Summing up: Data is a mined field

- Moreover, they may be linked
  - positively (witnessing e.g. the existence of a relationship) and
  - negatively (recording whatever prevents such a relationaship)
- Finally, the weights of such transitions are, in most cases, non-complementary, opening an inference arena encompassing both classical, vague, and even (controlled forms of) inconsistent reasoning.

Paraconsistent Logic

Originally developed in Latin America in 50's, mainly by F. Asenjo and Newton da Costa, accommodates inconsistency in a controlled way, treating inconsistent information as potentially informative.

Paraconsistency is the study of logical systems in which the presence of a contradiction does not imply triviality, therefore avoiding the principle of explosion according to which any statement can be proven from a contradiction, i.e. it is not always the case that

$$\varphi, \neg\varphi \models \psi$$

... therefore separating

contradiction from deductive triviality
inconsistency from contradiction
consistency from absence of contradiction

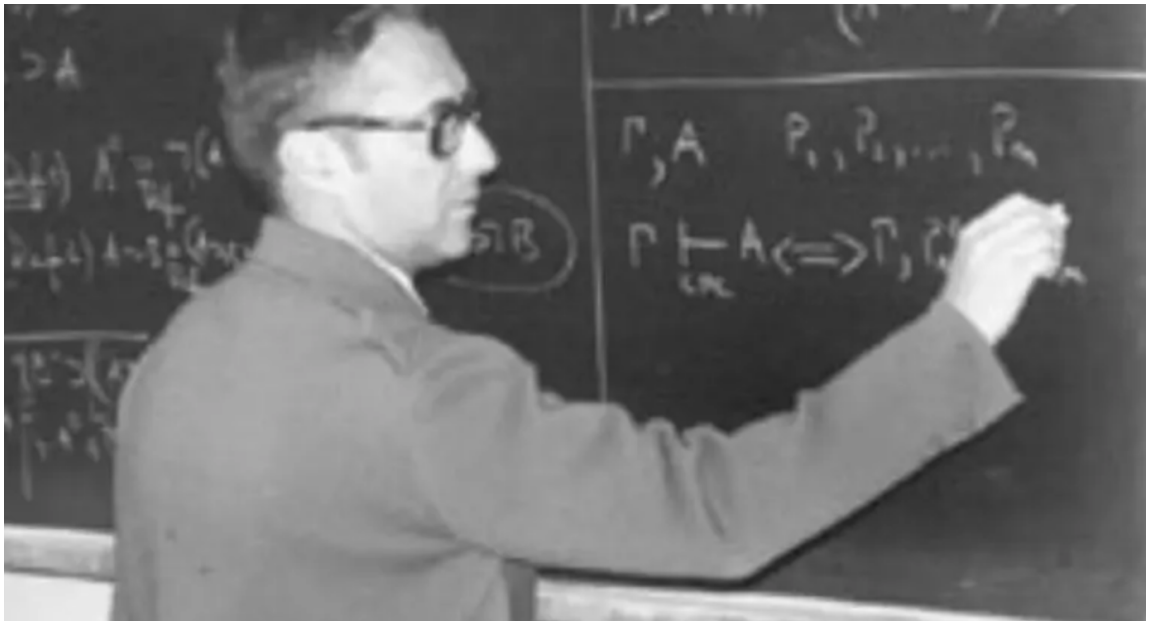
A brief history of an idea

- 1910 J. Łukasiewicz and N.Vasiliev: denying the law of noncontradiction would lead to still meaningful, although non-Aristotelian logics.
- S. Jaśkowski's study of empirical theories including contradictory assumptions.
- 1958: Newton da Costa (1929 - 2024) seminal paper .

Nota sôbre o conceito de contradição.
Anuário da Sociedade Paranaense de Matemática, 2 (1), 1958.

paved the way to the remarkable influence of the Brazilian School

A brief history of an idea



"I decided to do it the other way round: mathematics with contradictions. Existence in mathematics means anything but the absence of contradiction. Contradictions begin to appear at the edges of mathematics. There are always problems."

[Newton da Costa, *available from youtube*, 2019]

A brief history of an idea

- A main development in the Brazilian School led to the logic of formal inconsistency (LFI) in which the consistency of a sentence is made explicit by a unary operator
(W. Carnielli, M. E. Coniglio, J. Marcos)
- 1990: *Mathematical Reviews* added a new entry 03B53 entitled Paraconsistent Logic later expanded to Logics admitting inconsistency
- ... and applications pop out: from Philosophy of Science to Mathematics, from Economics, to Quantum Mechanics.
- ... and in Computer Science: AI, databases, semantics of concurrency, quantum computation ...

Our approach

Bringing paraconsistency to software engineering

- Transition systems with both positive and negative accessibility relations, with non complementary weights:
  - one weighting the possibility of a transition to be present,
  - the other weighting the possibility of being absent
- used in
  - software modelling (as a semantic structure)
  - software analysis/reasoning
    (as a Kripke frames for a modal logic)

A parenthesis on transition systems and their logics

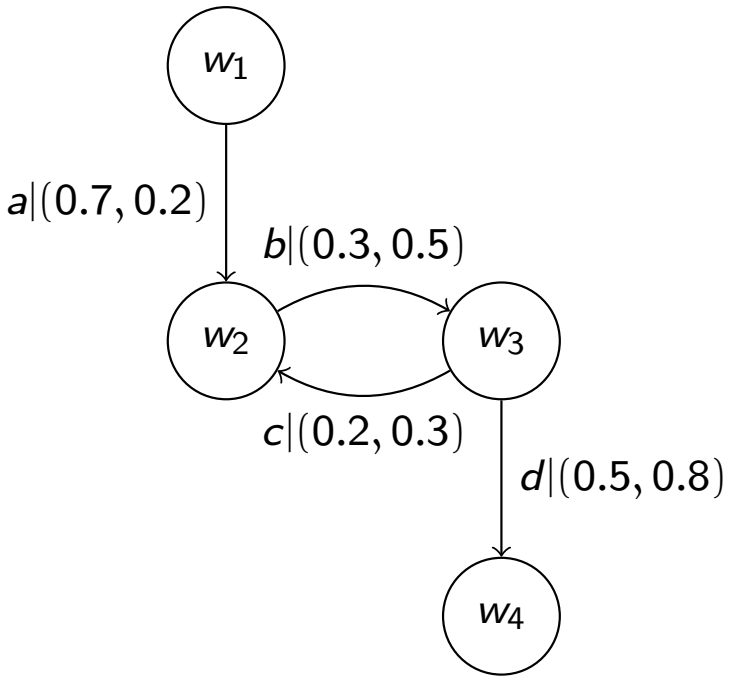
(...

End of parenthesis

...)

Anticipating an example

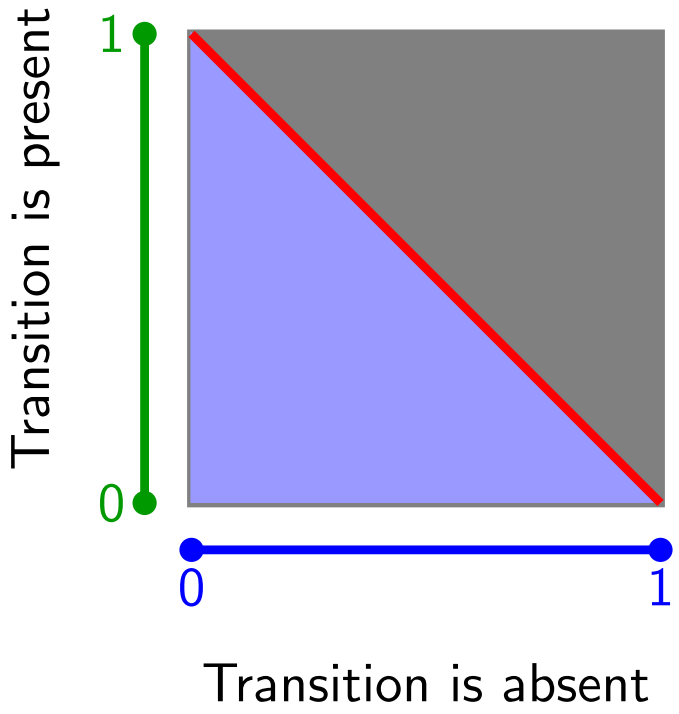
Paraconsistent transition Systems



Questions:

- How is information weighted?
- How to consider positive and negative weights simultaneously?

The information structure



How is information weighted?

Underlying semantic structure

- A residuated lattice, i.e. a bounded lattice $\boldsymbol{A} = \langle A, \sqcap, \sqcup, 1, 0 \rangle$ equipped with a monoid $\langle A, \odot, e \rangle$ such that $\odot$ has a right adjoint,

$$a \odot b \leqslant c \;\Leftrightarrow\; b \leqslant a \to c$$

- st the monoidal operation $\odot$ coincides with meet $\sqcap$
- plus a prelinearity condition:

$$(a \to b) \sqcup (b \to a) = 1$$

(iMTL-algebra, after *integral monoidal t-norm based logic*)

Some examples

$3 = \langle \{\bot, u, \top\}, \wedge_3, \vee_3, \top, \bot, \rightarrow_3 \rangle$

$\vee_3$	$\bot$	u	$\top$
$\bot$	$\bot$	u	$\top$
u	u	u	$\top$
$\top$	$\top$	$\top$	$\top$

$\wedge_3$	$\bot$	u	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
u	$\bot$	u	u
$\top$	$\bot$	u	$\top$

$\rightarrow_3$	$\bot$	u	$\top$
$\bot$	$\top$	$\top$	$\top$
u	$\bot$	$\top$	$\top$
$\top$	$\bot$	u	$\top$

$\ddot{\mathbf{G}} = \langle 0..1, min, max, 0, 1, \rightarrow \rangle$ (Gödel)

$$a \rightarrow b = \begin{cases} 1, & \text{if } a \leqslant b \\ b, & \text{otherwise} \end{cases}$$

How is information weighted?

... endowed with a metric $d : A \times A \longrightarrow \mathbb{R}^+$

Examples

2 and 3

2

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

3

d	$\bot$	u	$\top$
$\bot$	0	1	2
u	1	0	1
$\top$	2	1	0

$\ddot{\mathbf{G}}$ (Gödel)

$$\ddot{\mathbf{G}} = \langle [0,1], min, max, 0, 1, \rightarrow, d \rangle$$

where $d(x,y) = y - x$

Computing with pairs of weights

A twisted structure, obtained through the direct product of $\boldsymbol{A}$ and its order dual:

$$\boldsymbol{A}^2 = \langle A \times A, \sqcap, \sqcup, \Rightarrow, /\!\!/ \rangle$$

- $(a, b) \sqcap (c, d) = (a \sqcap c, b \sqcup d)$
- $(a, b) \sqcup (c, d) = (a \sqcup c, b \sqcap d)$
- $(a, b) \Rightarrow (c, d) = ((a \rightarrow c) \sqcap (d \rightarrow b), a \sqcap d)$
- $/\!\!/(a, b) = (b, a)$

with

$$(a, b) \preccurlyeq (c, d) \text{ iff } a \leqslant c \text{ and } b \geqslant d$$

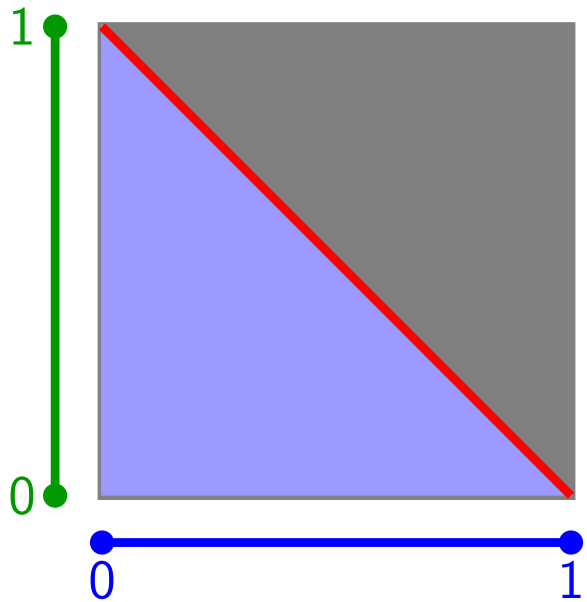
... and a metric $D((a, b), (c, d)) = \sqrt{d(a, c)^2 + d(b, d)^2}$

Computing with pairs of weights

(paraconsistent): $\Delta_P = \{(a,b) | D((a,b),(1,1)) \leqslant D((a,b),(0,0))\}$

(consistent): $\Delta_C = \{(a,b) | D((a,b),(0,0)) \leqslant D((a,b),(1,1))\}$

(strictly consistent): $\Delta = \Delta_P \cap \Delta_C$



Computing with pairs of weights

Alert: Adjunction $\sqcap \dashv \rightarrow$ does not lift

i.e. $(a,b) \sqcap\!\!\sqcap (c,d) \preccurlyeq (e,f)$ iff $(c,d) \preccurlyeq (a,b) \Rightarrow (e,f)$ fails:

e.g. for $(a,b) = (0.8, 0.4)$, $(c,d) = (0.5, 0.2)$ and $(e,f) = (0.6, 0.3)$,

$$(a,b) \sqcap\!\!\sqcap (c,d) = (min\{0.8, 0.5\}, max\{0.4, 0.2\}) = (0.5, 0.4)$$
$$(a,b) \Rightarrow (e,f) = (min(0.8 \rightarrow 0.6, 0.3 \rightarrow 0.4), min\{0.8, 0.3\}) = (0.6, 0.3)$$

Thus,

$$(0.5, 0.4) \preccurlyeq (0.6, 0.3) \text{ but } (0.5, 0.2) \not\preccurlyeq (0.6, 0.3)$$

but is recovered replacing $\sqcap\!\!\sqcap$ by

$$(a,b) \otimes (c,d) = (a \sqcap c, (a \rightarrow d) \sqcap (c \rightarrow b))$$

$$\boxed{(a,b) \otimes (c,d) \preccurlyeq (e,f) \text{ iff } (c,d) \preccurlyeq (a,b) \Rightarrow (e,f)}$$

Some properties

- The twist construction brings an involution
- and keeps the complete, universally distributive lattice structure, as the product of two such lattices

Some properties

$$/\!\!/ \left(/\!\!/(a,b) \barwedge /\!\!/(c,d)\right) = (a,b) \barvee (c,d)$$

$$/\!\!/ \left(\overline{\bigwedge_{i=1}^{n}}(a_i,b_i)\right) = \underline{\bigvee_{i=1}^{n}} /\!\!/(a_i,b_i)$$

$$/\!\!/ \left((a,b) \Rightarrow /\!\!/(c,d)\right) = (a,b) \otimes (c,d)$$

Some properties

...

if $(a', b') \preccurlyeq (a, b)$ and $(c, d) \preccurlyeq (c', d')$ then $(a, b)[\Rightarrow | \otimes](c, d) \preccurlyeq (a', b') \Rightarrow (c', d')$

$$(a, b) \otimes \Big((c, d) \sqcup (e, f) \Big) = \Big((a, b) \otimes (c, d) \Big) \sqcup \Big((a, b) \otimes (e, f) \Big)$$

$$(a, b) \otimes \Big((c, d) \sqcap (e, f) \Big) \preccurlyeq \Big((a, b) \otimes (c, d) \Big) \sqcap \Big((a, b) \otimes (e, f) \Big)$$

$$(a, b) \otimes \Big((c, d) \Rightarrow (e, f) \Big) \preccurlyeq (c, d) \Rightarrow \Big((a, b) \otimes (e, f) \Big)$$

$$(a, b) \Rightarrow \Big((c, d) \Rightarrow (e, f) \Big) \preccurlyeq \Big((a, b) \otimes (c, d) \Big) \Rightarrow (e, f)$$

$$(a, b) \otimes \Big((c, d) \Rightarrow (e, f) \Big) \preccurlyeq \Big((a, b) \Rightarrow (c, d) \Big) \Rightarrow (e, f)$$

$$(a, b) \Rightarrow \Big((c, d) \Rightarrow (e, f) \Big) \preccurlyeq (c, d) \Rightarrow \Big((a, b) \Rightarrow (e, f) \Big)$$

...

Paraconsistent Labelled Transition Systems

The approach

- Introduce labels from a set of identifiers Act, and an initial state
- Define morphism to organise PLTS over $\boldsymbol{A}$ into a category
- Derive an algebra, to get new PLTS from old, from the underlying categorical structure
  (... possibly leading to a language and a dynamic logic)
- Develop a multimodal logic (*à la* Hennessy-Milner) for PLTS

Paraconsistent Labelled Transition Systems

A PLTS over a iMTL-algebra $\boldsymbol{A}$, and a set of atomic actions Act is a structure

$$T = \langle W, i, R \rangle$$

where,

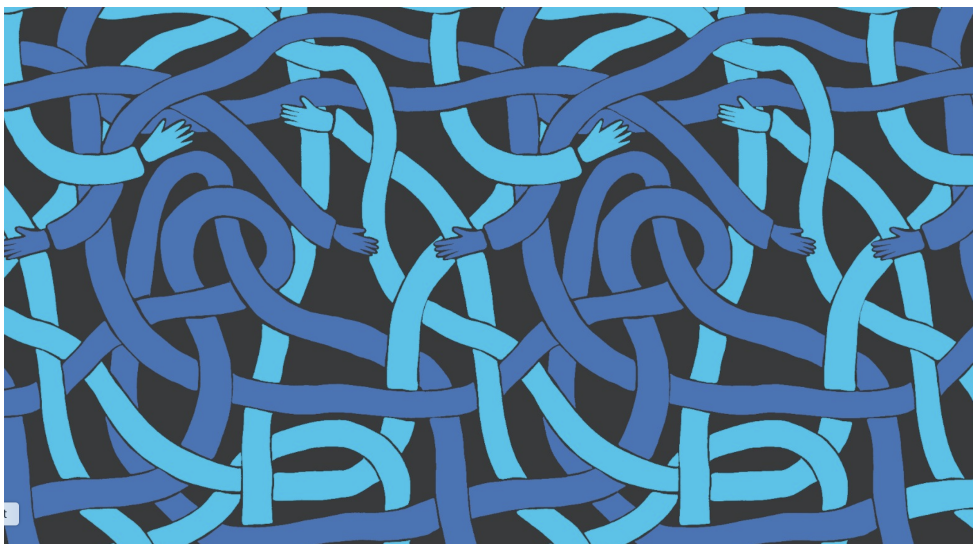
- W is a non-empty set of states
- $i \in W$ is the initial state
- $R = (R_a \colon W \times W \to A \times A)_{a \in Act}$ is an Act-indexed family of functions

$$R_a(w_1, w_2) = (\alpha, \beta)$$

with α weighting evidence of the transition through a and β its absence.

Where can we go from here?

- Modelling (Programming): requires a notion of morphism to discuss compositionality
- Reasoning (Verification): requires extension to a Kripke structure, a notion of (bi)simulation and a logic



Morphism

$(h, \lambda) : T \to T'$

- $h : W \to W'$
- $\lambda : Act \to_{\perp} Act'$

such that

- $h(i) = i'$ and
- for any $a \in Act$,

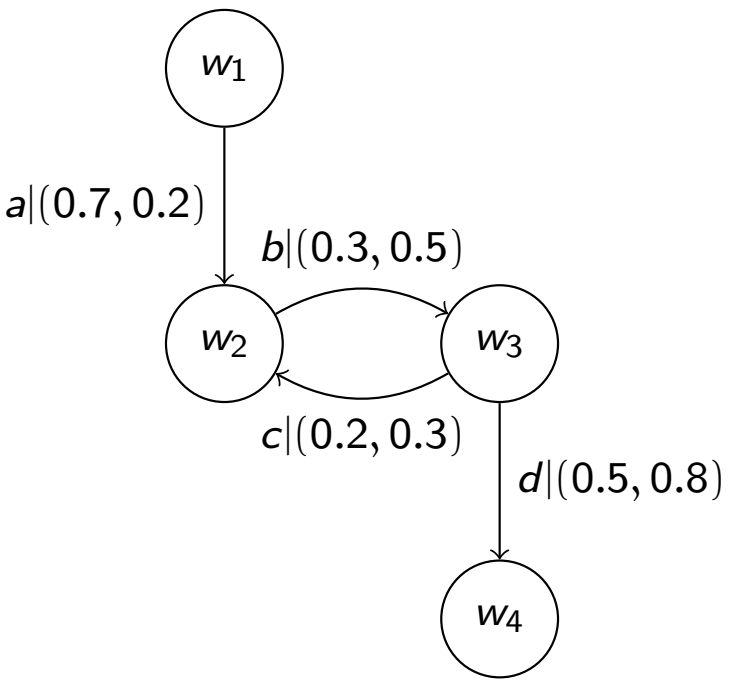
$$R_a(w, w') \preccurlyeq R'^{\perp}_{\lambda(a)}(h(w), h(w'))$$

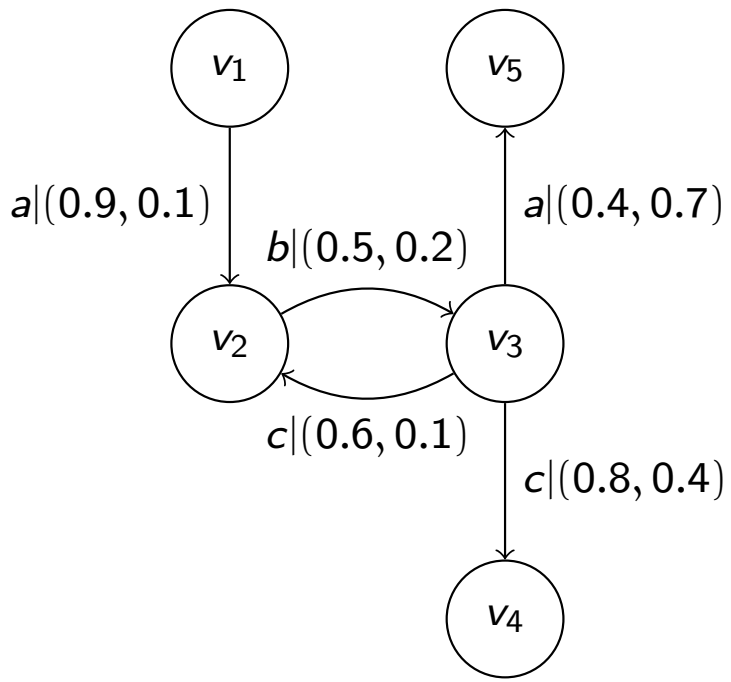
where $R^{\perp} = R \cup R_{\perp}$ with $R_{\perp}(w, w) = (1, 0)$ for any state $w \in W$

Morphism

Example

(h, id) with $h = \{w_1 \mapsto v_1, w_2 \mapsto v_2, w_3 \mapsto v_3, w_4 \mapsto v_4\}$





Kripke Structures & (Bi)simulation

A PLTS $T = \langle W, i, R \rangle$ over a iMTL-algebra $\boldsymbol{A}$, and a set of atomic actions Act, generates a Kripke structure

$$\langle W, R, V \rangle$$

where

$$V : W \times Prop \to A \times A$$

is a valuation function over a set *Prop* of proposition symbols.

Kripke Structures & (Bi)simulation

The crisp case

$S \subseteq W \times W'$ is a simulation if, for all $p \in Prop$, $a \in Act$,

if wSw' then

- $V(w,p) \preccurlyeq V'(w',p)$
- If $R_a(w,v) = (\alpha,\beta)$ then,

$$\exists_{w' \in W'}.\ R_a(w',v') = (\delta,\gamma) \text{ and } w'Sv' \text{ and } (\alpha,\beta) \preccurlyeq (\delta,\gamma)$$

Kripke Structures & (Bi)simulation

The crisp case

$S \subseteq W \times W'$ is a simulation if, for all $p \in Prop$, $a \in Act$, if wSw' then

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- If $R_a(w,v) = (\alpha,\beta)$ then,

$$\exists_{w' \in W'}.\ R_a(w',v') = (\delta,\gamma) \text{ and } w'Sv' \text{ and } (\alpha,\beta) \preccurlyeq (\delta,\gamma)$$

The graded case

$G : W \times W' \to A \times A$ is a graded simulation if, for all $p \in Prop$, $a \in Act$,

- $G(w,w') \preccurlyeq (V(w,p) \Rightarrow V'(w',p))$
- $\exists_{v' \in W'}\ (G(w,w') \otimes R_a(w,v)) \preccurlyeq (R'_a(w',v') \otimes G(v,v'))$

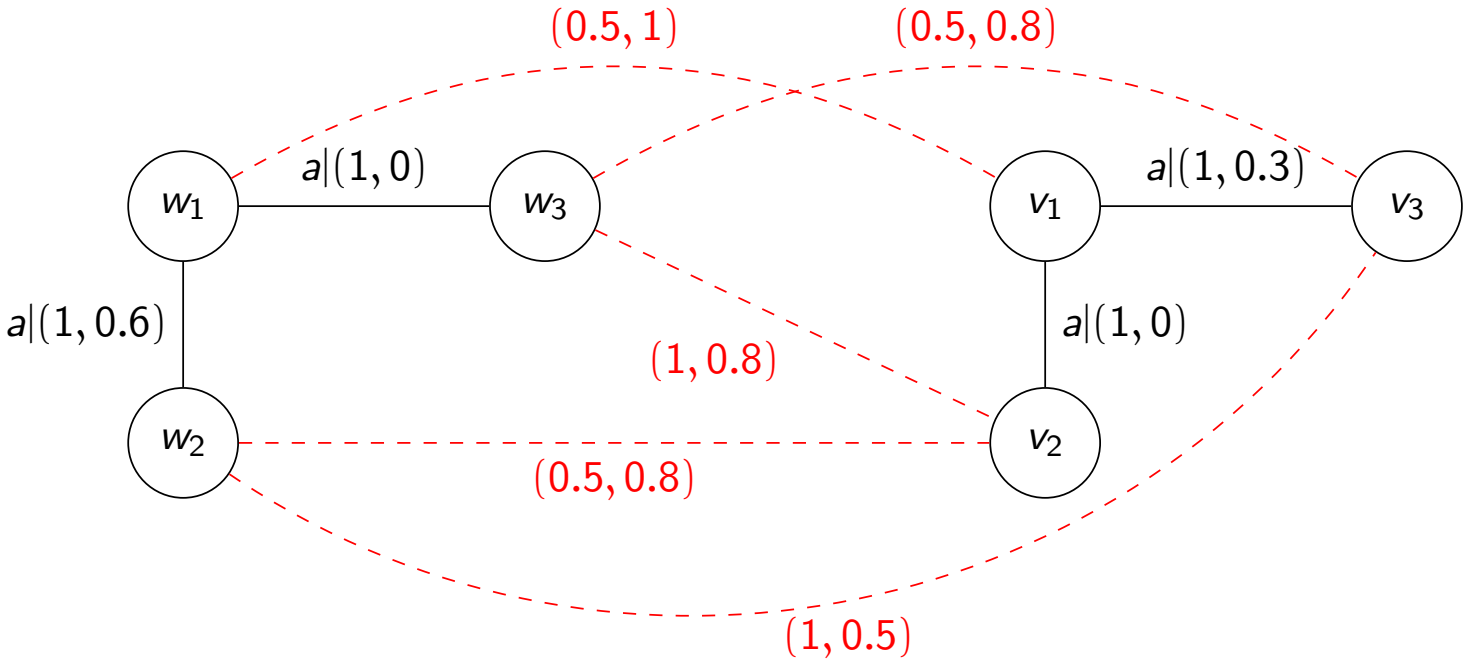
Kripke Structures & (Bi)simulation

Graded bisimulation

$G : W \times W' \to A \times A$ is a graded bisimulation if, for all $p \in Prop$, $a \in Act$,

- $G(w, w') \preccurlyeq (V(w, p) \Leftrightarrow V'(w', p))$
- $\exists_{v' \in W'} \ (G(w, w') \otimes R_a(w, v)) \preccurlyeq (R'_a(w', v') \otimes G(v, v'))$
- $\exists_{v \in W} \ (G(w, w') \otimes R'_a(w', v')) \preccurlyeq (R_a(w, v) \otimes G(v, v'))$

Kripke Structures & (Bi)simulation



where
$V(w_1, p) = V'(v_1, p) = (1, 1)$
$V(w_2, p) = V'(v_3, p) = (1, 0.5)$
$V(w_3, p) = V'(v_2, p) = (0.8, 0.8)$

Kripke Structures & (Bi)simulation

Alert: Crisp (bi)simulations are not necessarily graded

i.e. a crisp (bi)simulation S st wSw' does not entail the existence of a graded one G st $G(w, w') = (1, 0)$.

- Conditions enforcing coincidence are identified for both simulation and bisimulation
- This helps to explain some weird behaviour of graded behavioural relations.

Kripke Structures & (Bi)simulation

Example



with $V(w,p) = V'(w',p) = (0.2, 0.2)$.

Relation $G(w,w') = (1,0)$ is not a graded simulation:

$$(1,0) \not\leq (0.2,0.2) \Rightarrow (0.2,0.2)$$
$$\not\leq (1,0.2)$$

Paraconsistency for the working software engineer

- Modelling (Programming): New PLTS from old

  .. exploring the underlying category $Pt_{\boldsymbol{A}}$ of PLTS and their morphisms

- Reasoning (Verification): A multimodal logic $PML(\boldsymbol{A})$

  ... action-indexed modalities interpreted over paraconsistent Kripke structures

New PLTS from old

Restriction and relabeling

Let T be a PLTS. Then,

- Restriction: for $\lambda : Act' \to Act$ an inclusion, $T \upharpoonright \lambda$ is a Cartesian lifting in $Pt_{\boldsymbol{A}}$, i.e., any other morphism whose action component is λ factors through the restriction morphism from $T \upharpoonright \lambda$ to T.
- Relabelling: for $\lambda : Act' \to Act$ total, $T\{\lambda\}$ is a co-Cartesian lifting in $Pt_{\boldsymbol{A}}$

New PLTS from old

New PLTS from old: Parallel Composition

$T_1 \times T_2$ is the categorical product in $Pt_{\boldsymbol{A}}$:

$$T_1 \times T_2 = \langle W_1 \times W_2, (i_1, i_2), R \rangle$$

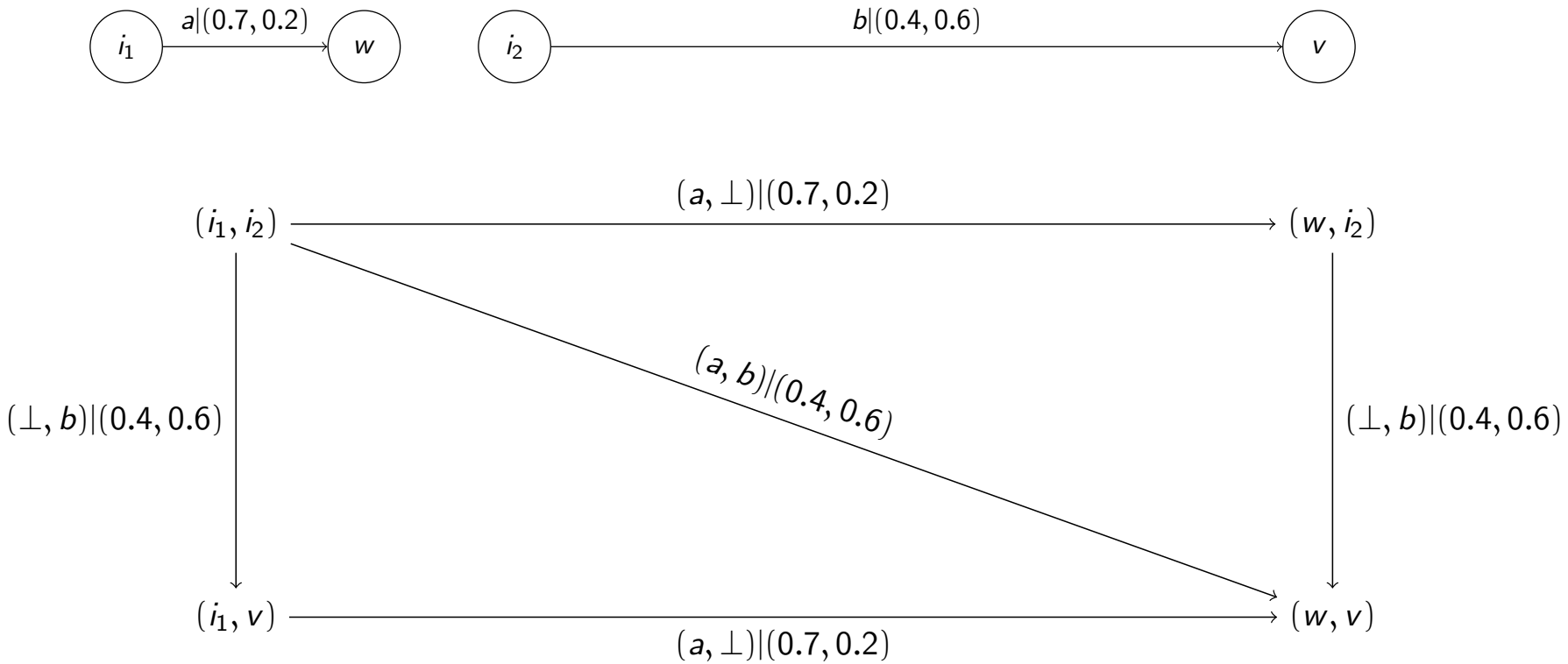
over

$$Act_1 \times_\perp Act_2 = \{(a, \perp) \mid a \in Act_1\} \cup \{(\perp, b) \mid b \in Act_2\} \cup \{(a, b) \mid a \in Act_1, b \in Act_2\}$$

such that

$$\begin{aligned} R_{(a,b)}((w_1, w_2), (v_1, v_2)) = (\alpha, \beta) \text{ iff} \\ (R_1)^\perp_a(w_1, v_1) = (\alpha_1, \beta_1) \text{ and} \\ (R_2)^\perp_b(w_2, v_2) = (\alpha_2, \beta_2) \text{ and} \\ (\alpha, \beta) = (\alpha_1, \alpha_2) \sqcap (\beta_1, \beta_2) \end{aligned}$$

Example


i_1
$a|(0.7, 0.2)$
w
i_2
$b|(0.4, 0.6)$
v
(i_1, i_2)
$(a, \bot)|(0.7, 0.2)$
(w, i_2)
$(\bot, b)|(0.4, 0.6)$
$(a, b)|(0.4, 0.6)$
$(\bot, b)|(0.4, 0.6)$
(i_1, v)
(w, v)
$(a, \bot)|(0.7, 0.2)$

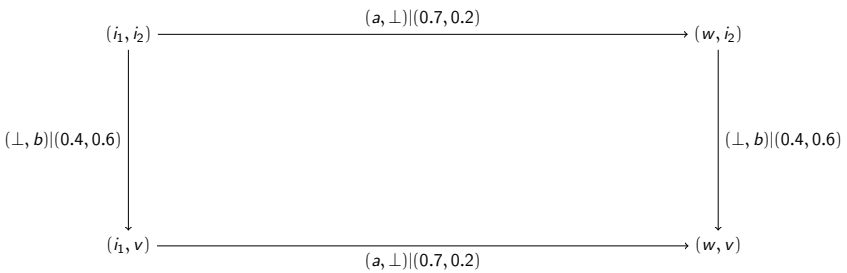
New PLTS from old

Interleaving: $T_1 \,|||\, T_2 = (T_1 \times T_2) \restriction \lambda$

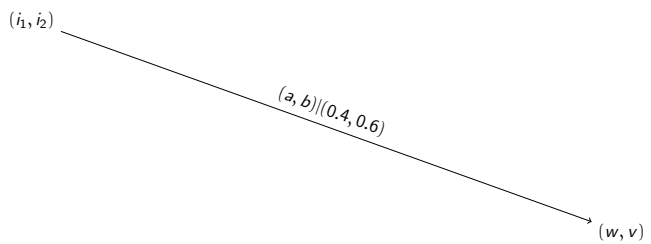
with the inclusion $\lambda : \{(a, \bot) \mid a \in Act_1\} \cup \{(\bot, b) \mid b \in Act_2\} \to Act_1 \times_\bot Act_2$

Synchronous product: $T_1 \otimes T_2 = (T_1 \times T_2) \restriction \lambda$

taking $\{(a, b) \mid a \in Act_1 \text{ and } b \in Act_2\}$ as the domain of λ



$T_1 \,|||\, T_2$



$T_1 \otimes T_2$

New PLTS from old

Choice

$T_1 + T_2$ is the categorical coproduct in $Pt_{\boldsymbol{A}}$:

$$T_1 + T_2 = \langle W, (i_1, i_2), R \rangle$$

over $Act = Act_1 \cup Act_2$, where

- $W = (W_1 \times \{i_2\}) \cup (\{i_1\} \times W_2)$
- $R_a((w_1, w_2), (v_1, v_2)) = (\alpha, \beta)$ iff
  $(R_1)_a(w_1, v_1) = (\alpha, \beta)$ or $(R_2)_a(w_2, v_2) = (\alpha, \beta)$

$$(i_1, v) \xleftarrow{b|(0.4,0.6)} (i_1, i_2) \xrightarrow{a|(0.7,0.2)} (w, i_2)$$

New PLTS from old

Other Operators

- Sequential composition as prefixing
- Functorial extension of operations from the underlying iMTL-algebra (e.g. to operate on weights)

... leading to a sort of (paraconsistent) process algebra

$PML(\boldsymbol{A})$: A multimodal logic for PLTS

Given an iMTL-algebra $\boldsymbol{A}$, a set of proposition symbols *Prop* and a set of action symbols *Act*:

$$\varphi \coloneqq \bot \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid [a]\varphi$$

where $p \in Prop$ and $a \in Act$.

Abbreviations:

- $\top = \neg\bot$
- $\varphi \vee \varphi' = \neg(\neg\varphi \wedge \neg\varphi')$
- $\varphi \rhd \varphi' = \neg\varphi \vee \varphi'$
- $\varphi \lhd\rhd \varphi' = \varphi \rhd \varphi' \wedge \varphi' \rhd \varphi$
- $\langle a\rangle\varphi = \neg[a]\neg\varphi$

The logic $PML(\boldsymbol{A})$

Satisfaction: $\models : M \times Sen(Prop, Act) \rightarrow A \times A$

$$(M \models \varphi) = \prod_{w \in W} (M, w \models \varphi)$$

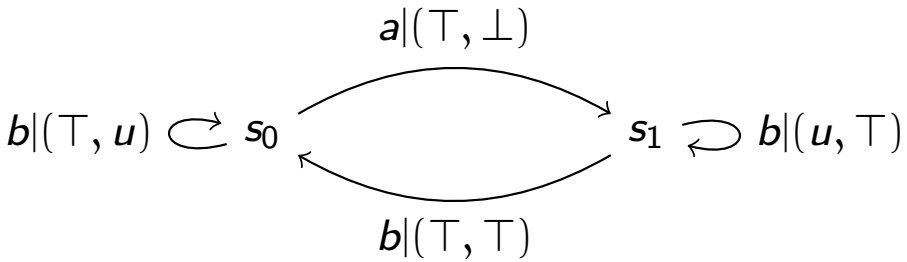
- $(M, w \models \bot) = (0, 1)$
- $(M, w \models p) = V(w, p)$
- $(M, w \models \neg\varphi) = /\!/(M, w \models \varphi)$
- $(M, w \models \varphi \wedge \varphi') = (M, w \models \varphi) \amalg (M, w \models \varphi')$
- $(M, w \models [a]\varphi) = \prod_{v \in W} (R_a(w, v) \Rightarrow M, v \models \varphi)$

The logic $PML(\boldsymbol{A})$

Deriving $\langle a \rangle \varphi$ (the role of the adjunction)

$$
\begin{aligned}
(M, w \models \langle a \rangle \varphi) &= (M, w \models \neg [a] \neg \varphi) \\
&= /\!\!/ \, (M, w \models [a] \neg \varphi) \\
&= /\!\!/ \left(\prod_{v \in W} \left(R_a(w, v) \Rightarrow (M, v \models \neg \varphi) \right) \right) \\
&= /\!\!/ \left(\prod_{v \in W} \left(R_a(w, v) \Rightarrow /\!\!/ (M, v \models \varphi) \right) \right) \\
&= \coprod_{v \in W} /\!\!/ \left(R_a(w, v) \Rightarrow /\!\!/ (M, v \models \varphi) \right) \\
&= \coprod_{v \in W} \left(R_a(w, v) \otimes (M, v \models \varphi) \right)
\end{aligned}
$$

Example over 3



V	p	q	r
s_0	$(\top,\top)$	$(\bot,u)$	(u,u)
s_1	$(\bot,u)$	$(\bot,\bot)$	(u,u)

$$\boxed{M, s_1 \models r \triangleright (p \vee q)}$$

$$\begin{aligned} M, s_0 \models r \triangleright (p \vee q) &= /\!\!/ \, (M, s_0 \models r) \sqcup (M, s_0 \models (p \vee q)) \\ &= /\!\!/ \, V(s_0, r) \sqcup (V(s_0, p) \sqcup V(s_0, q)) \\ &= (u,u) \sqcup (\top,\top) \sqcup (\bot,u) \\ &= (u \vee_3 \top \vee_3 \bot, u \wedge_3 \top \wedge_3 u) \\ &= (\top, u) \end{aligned}$$

Example over 3

$$M, s_1 \models \langle b \rangle p$$

$$\begin{aligned}
M, s_1 \models \langle b \rangle p &= M, s_1 \models /\!/[b] /\!/ p \\
&= /\!/ \left(\prod_{s \in W} \widehat{R}_b(s_1, s) \Rightarrow /\!/(M, s \models p) \right) \\
&= /\!/ \left(\left(R_b(s_1, s_0) \Rightarrow /\!/ V(s_0, p) \right) \sqcap \left(R_b(s_1, s_1) \Rightarrow /\!/ V(s_1, p) \right) \right) \\
&= /\!/ \left(\left((\top, \top) \Rightarrow (\top, \top) \right) \sqcap \left((u, \top) \Rightarrow (u, \bot) \right) \right) \\
&= /\!/ (\top \wedge_3 \top, \top \vee_3 \bot) \\
&= (\top, \top)
\end{aligned}$$

Bisimulation and Modal Invariance

- S crisp bisimulation:

$$\langle w, w' \rangle \in S \text{ entails } (M, w \models \varphi) = (M', w' \models \varphi)$$

- G graded bsimulation:

$$G(w, w') \preccurlyeq \Big((M, w \models \varphi) \Leftrightarrow (M', w' \models \varphi) \Big)$$

Some references

Theory & Applications

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The algebraic counterpart

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LSFA'24 (in print), 2026.

Concluding: Paraconsistency is everywhere ...

When the hills are flat,
The rivers are all dry.

When it thunders in winter,
When it snows in summer.
When heaven and earth mingle,

Not till then will I part.

Yuefu poems, Han dynasty
(206 BC - 220 AD)

