A parenthesis on transition systems and their logics

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November 11, 2026

Labelled Transition Systems

A Labelled Transition System (LTS) over a set Act of names is a pair $(W, \longrightarrow)$ where

- $W = \{p_0, p_1, p_2, ...\}$ is a set of states
- $\longrightarrow \subseteq W \times Act \times W$ is the transition relation, often given as an Act-indexed family of binary relations

$$s \xrightarrow{a} s' \equiv (s, a, s') \in \longrightarrow$$

A morphism relating two LTS over Act, $(W, Act, \longrightarrow)$ and $(W', Act, \longrightarrow')$, is a function $h : W \longrightarrow W'$ st

$$p \xrightarrow{a} p' \quad \Rightarrow \quad h(p) \xrightarrow{a}{}' h(p')$$

i.e. morphisms preserve transitions.

Simulation & Bissimulation

Given $(W_1, Act, \longrightarrow_1)$ and $(W_2, Act, \longrightarrow_2)$ over Act, relation $R \subseteq W_1 \times W_2$ is a simulation iff, for all $(p, q) \in R$ and $a \in Act$,

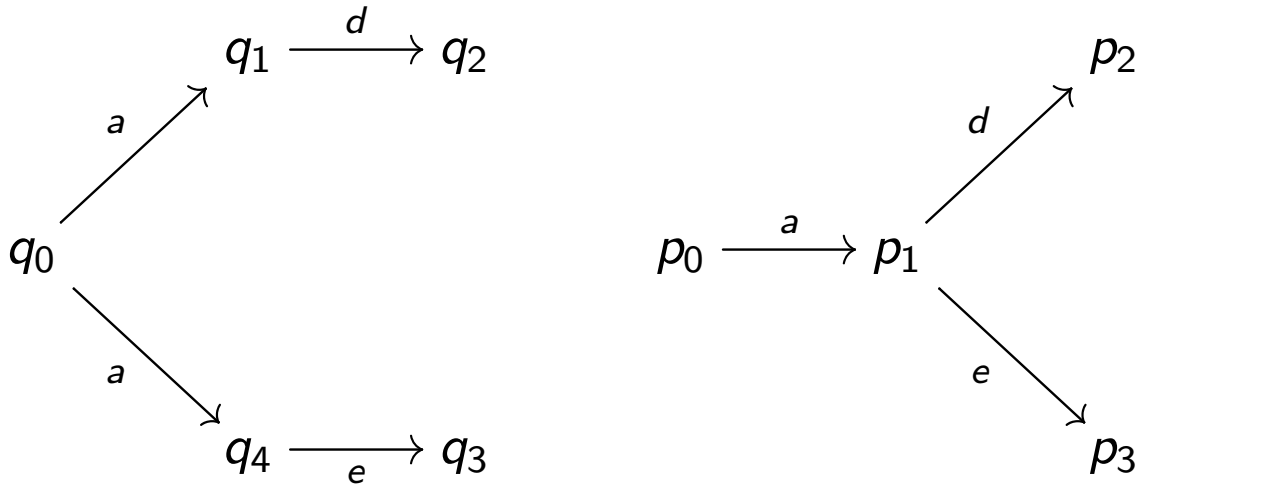
$$(1)\quad p \xrightarrow{a}_1 p' \;\Rightarrow\; (\exists\, q' \;:\; q' \in S_2 :\; q \xrightarrow{a}_2 q' \wedge (p', q') \in R)$$

$$\begin{array}{ccc} p & R & q \\ \downarrow a & & \\ p' & & \end{array} \qquad \Rightarrow \qquad \begin{array}{ccc} & & q \\ & & \downarrow a \\ p' & R & q' \end{array}$$

Is is a bisimulation iff both R and its converse R° are simulations, i.e.

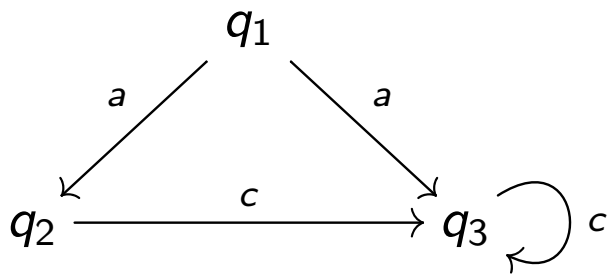
$$(1) \text{ and } (2)\quad q \xrightarrow{a}_2 q' \;\Rightarrow\; (\exists\, p' \;:\; p' \in S_1 :\; p \xrightarrow{a}_1 p' \wedge (p', q') \in R)$$

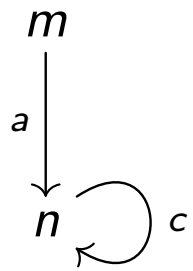
Example (Simulation)

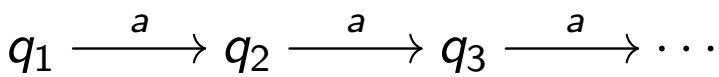


$q_0 \lesssim p_0$ cf. $\{(q_0, p_0), (q_1, p_1), (q_4, p_1), (q_2, p_2), (q_3, p_3)\}$

Examples: Bisimulation









A modal logic

Motivation: System's correctness wrt a specification

- equivalence or refinement checking (through simulation and bisimulation)
- are not enough to check properties such as

  can the system perform action a followed by b?

  which are best answered by exploring the system's state space

Modal logic

- a very old idea (from the Middle Ages):

  truth is relative to where I am ...

- a zoo of logics: e.g. temporal, dynamic, epistemic, deontic, ...

A modal logic

Syntax

For each elementary proposition $p \in \mathit{Prop}$ and $a \in \mathit{Act}$,

$$\phi \ ::= \ p \mid \mathit{true} \mid \mathit{false} \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \rightarrow \phi_2 \mid \langle a \rangle \phi \mid [a]\phi$$

Semantics

A model $\mathfrak{M}$ for the language is a pair consists of a

- Act-labelled transition system typically written as $(W, \{R_a\}_{a \in \mathit{Act}})$ (also called a Kripke frame),
- a valuation of elementary propositions in each state: $V : W \times \mathit{Prop} \longrightarrow \mathbf{2}$.

The language

Satisfaction: for a model $\mathfrak{M}$ and a state w

$\mathfrak{M}, w \models \mathit{true}$		
$\mathfrak{M}, w \not\models \mathit{false}$		
$\mathfrak{M}, w \models p$	iff	$w \in V(p)$
$\mathfrak{M}, w \models \neg\phi$	iff	$\mathfrak{M}, w \not\models \phi$
$\mathfrak{M}, w \models \phi_1 \wedge \phi_2$	iff	$\mathfrak{M}, w \models \phi_1$ and $\mathfrak{M}, w \models \phi_2$
$\mathfrak{M}, w \models \phi_1 \rightarrow \phi_2$	iff	$\mathfrak{M}, w \not\models \phi_1$ or $\mathfrak{M}, w \models \phi_2$
$\mathfrak{M}, w \models \langle a \rangle \phi$	iff	there exists $v \in W$ st $wR_a v$ and $\mathfrak{M}, v \models \phi$
$\mathfrak{M}, w \models [a]\phi$	iff	for all $v \in W$ st $wR_a v$ and $\mathfrak{M}, v \models \phi$

Example: Some typical properties

- After a followed by b, c may occur: $[a][b]\langle c\rangle$
- inevitability of a: $\langle -\rangle true \wedge [-a] false$
- progress: $\langle -\rangle true$
- deadlock or termination: $[-] false$
- what about

$$\langle -\rangle false \quad \text{and} \quad [-] true \quad ?$$

Note that satisfaction is decided by unfolding the definition of $\models$: no need to compute the transition graph

Can this setting be tunned to specific applications?

- Explore the underlying graph structure to express the system's structure (e.g. non deterministic, finitely branching, image finite, ...)
- Develop the graph into higher dimensions (e.g. reactive transition systems)
- Put more structure on the labels (e.g. time instants, atomic actions, programs)
- Generalize the meaning of edges (e.g. to probabilisitc or weighted relations)
- Develop appropriate (modal) logics
- ...