THE SYNTACTIC SIDE OF AUTONOMOUS CATEGORIES ENRICHED OVER GENERALISED METRIC SPACES

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Abstract. Programs with a continuous state space or that interact with physical processes often require notions of equivalence going beyond the standard binary setting in which equivalence either holds or does not hold. In this paper we explore the idea of equivalence taking values in a quantale $\mathcal{V}$, which covers the cases of (in)equations and (ultra)metric equations among others.

Our main result is the introduction of a $\mathcal{V}$-*equational deductive system* for linear λ-calculus together with a proof that it is sound and complete. In fact we go further than this, by showing that linear λ-theories based on this $\mathcal{V}$-equational system form a category that is equivalent to a category of autonomous categories enriched over 'generalised metric spaces'. If we instantiate this result to inequations, we get an equivalence with autonomous categories enriched over partial orders. In the case of (ultra)metric equations, we get an equivalence with autonomous categories enriched over (ultra)metric spaces. We additionally show that this syntax-semantics correspondence extends to the affine setting.

We use our results to develop examples of inequational and metric equational systems for higher-order programming in the setting of real-time, probabilistic, and quantum computing.

1. Introduction

Programs frequently act over a *continuous* state space or interact with physical processes like time progression or the movement of a vehicle. Such features naturally call for notions of approximation and refinement integrated in different aspects of program equivalence. Our paper falls in this line of research. Specifically, our aim is to integrate notions of approximation and refinement into the *equational system* of linear λ-calculus [BBdPH92, MRA93, MMDPR05].

The core idea that we explore in this paper is to have equations $t =_q s$ labelled by elements q of a quantale $\mathcal{V}$. This covers a wide range of situations, among which the cases of (in)equations [KV17, AFMS20] and metric equations [MPP16, MPP17]. The latter case

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* This paper is an extended version of [DN22]. It includes proofs omitted in the *op. cit.* and new examples. It also includes new technical results: among other things an equivalence theorem and an extension (from the linear setting to the affine one) of the results established in the *op. cit.*

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is perhaps less known: it consists of equations $t =_\epsilon s$ labelled by a non-negative rational number ϵ which represents the 'maximum distance' that the two terms t and s can be from each other. In order to illustrate metric equations, consider a programming language with a (ground) type X and a signature of operations $\Sigma = \{\mathtt{wait_n} : X \to X \mid n \in \mathbb{N}\}$ that model time progression over computations of type X. Specifically, $\mathtt{wait_n}(x)$ reads as "add a latency of n seconds to the computation x". In this context, the following axioms involving metric equations arise naturally:

$$\mathtt{wait_0}(x) =_0 x \qquad \mathtt{wait_n}(\mathtt{wait_m}(x)) =_0 \mathtt{wait_{n+m}}(x) \qquad \frac{\epsilon = |m - n|}{\mathtt{wait_n}(x) =_\epsilon \mathtt{wait_m}(x)} \tag{1.1}$$

An equation $t =_0 s$ states that the terms t and s are exactly the same and equations $t =_\epsilon s$ state that t and s differ by *at most* ϵ seconds in their execution time.

Contributions. In this paper we introduce an equational deductive system for linear λ-calculus in which equations are labelled by elements of a quantale $\mathcal{V}$. By using key features of a quantale's structure, we show that this deductive system is *sound and complete* for a class of enriched symmetric monoidal closed categories (*i.e.* enriched *autonomous* categories). In particular, if we fix $\mathcal{V}$ to be the Boolean quantale this class of categories consists of autonomous categories enriched over partial orders. If we fix $\mathcal{V}$ to be the (ultra)metric quantale, then this class of categories consists of autonomous categories enriched over (ultra)metric spaces. The aforementioned example of wait calls fits in the setting in which $\mathcal{V}$ is the metric quantale. Our result provides this example with a sound and complete metric equational system, where the models are all those autonomous categories enriched over metric spaces that can soundly interpret the axioms of wait calls (1.1).

The next contribution of our paper falls in a major topic of categorical logic: to establish *logical descriptions* of certain classes of categories (in a nutshell, this allows to translate categorical assertions or constructions into logical ones and *vice-versa*). A famous result of this kind is the correspondence between λ-calculus and Cartesian closed categories, formalised in terms of an equivalence between categories respective to both structures. Intuitively, the equivalence states that λ-theories are the syntactic counterpart of Cartesian closed categories and that the latter are the semantic counterpart of the former [LS88, Cro93]. An analogous result is known to exist for linear λ-calculus and autonomous categories [MRA93, MMDPR05]. Here we extend the latter result to the setting of $\mathcal{V}$-equations (*i.e.* equations labelled by elements of a quantale $\mathcal{V}$). Specifically, we prove the existence of an equivalence between linear λ-theories based on $\mathcal{V}$-equations and autonomous categories enriched over 'generalised metric spaces'.

Outline. §2 recalls linear λ-calculus and its equational system together with corresponding proofs of soundness, completeness, and the aforementioned equivalence with a category of autonomous categories (in fact a 'quasi-category' [AHS09]). The contents of this section are slight adaptations of results presented in [BBdPH92, MRA93, Cro93, MMDPR05], the main difference being that we forbid the exchange rule to be explicitly part of linear λ-calculus (instead it is only admissible). This choice is important to ensure that judgements in the calculus have *unique* derivations, which allows us to refer to their interpretations unambiguously [Shu19]. §3 presents the main contributions of this paper. It walks a path analogous to § 2, but now in the setting of $\mathcal{V}$-equations. As we will see, the semantic counterpart of moving from equations to $\mathcal{V}$-equations is to move from ordinary categories to categories enriched over $\mathcal{V}$-*categories*. The latter, often regarded as generalised metric spaces, are central entities in a fruitful area of enriched category theory that aims to treat uniformly different kinds of 'structured sets', such as partial orders, fuzzy partial orders, and (ultra)metric spaces [Law73, Stu14, VKB19]. Our results are applicable to all these cases. § 4 presents some examples of $\mathcal{V}$-equational axioms (and corresponding models) for three computational paradigms, namely real-time, probabilistic, and quantum computing (in all these paradigms notions of approximation take a central role). Specifically, in § 4 we will revisit the axioms of wait calls (1.1) and consider an inequational variant. Then we will study a metric axiom for probabilistic programs and show that the category of Banach spaces and short linear maps is a model for the resulting metric theory. Next we turn our attention to quantum computing and introduce a metric axiom that reflects the fact that implementations of quantum operations can only approximate the intended behaviour. We build a corresponding model over a certain category of presheaves based on the concept of a quantum channel. We will use both probabilistic and quantum examples to illustrate how our deductive system allows to compute an approximate distance between two probabilistic/quantum programs easily as opposed to computing an exact distance 'semantically' which tends to involve quite complex operators.

Finally §5 provides two concluding notes: first a proof that our results extend from the linear to the affine case. Second the presentation of a functorial connection (in terms of adjunctions) between our results and previous well-known semantics of linear logic [DP99, MMDPR05]. §5 then ends with a brief exposition of future work. We assume knowledge of λ-calculus and category theory [MRA93, MMDPR05, LS88, ML98].

Related work. Several approaches to incorporating quantitative information to programming languages have been explored in the literature. Closest to this work are various approaches targeted at λ-calculi. In [CDL15, CDL17] a notion of distance called *context distance* is developed, first for an affine, then for a more general λ-calculus, with probabilistic programs as the main motivation. [Gav18] considers a notion of quantale-valued *applicative (bi)similarity*, an operational *coinductive* technique used for showing contextual equivalence between two programs. Recently, [Pis21] presented several Cartesian closed categories of generalised metric spaces that provide a quantitative semantics to simply-typed λ-calculus based on a generalisation of logical relations. None of these examples reason about distances in a quantitative equational system, and in this respect our work is closer to the metric universal algebra developed in [MPP16, MPP17].

A different approach consists in encoding quantitative information via a type system. In particular, graded (modal) types [GSS92, GKO$^+$16, OLEI19] have found applications in *e.g.* differential privacy [RP10] and information flow [ABHR99]. This approach is to some extent orthogonal to ours as it mainly aims to model coeffects, whilst we aim to reason about the intrinsic quantitative nature of λ-terms acting *e.g.* on continuous or ordered spaces.

Quantum programs provide an interesting example of intrinsically quantitative programs, by which we mean that the metric structure on quantum states does not arise from (co)effects. Recently, [HHZ$^+$19] showed how the issue of noise in a quantum while-language can be handled by developing a *deductive system* to determine how similar a quantum program is from its idealised, noise-free version; an approach very much in the spirit of this work.

2. Background: linear λ-theories and autonomous categories

In this section we briefly recall linear λ-calculus, which can be regarded as a term assignment system for the exponential free, multiplicative fragment of *intuitionistic linear logic*. Then we recall that it is sound and complete w.r.t. autonomous categories – we mention only what is needed to present our results, the interested reader will find a more detailed exposition in [MRA93, BBdPH92, MMDPR05]. Subsequently we present a quasi-category [AHS09] of linear λ-theories, a quasi-category of autonomous categories, and show that they are equivalent. The same result is detailed in [MMDPR05], with the main difference that here we have decided to take the more general approach of allowing functors to preserve autonomous structures only *up-to isomorphism* (*i.e.* more in the style of [Cro93] which the studies the Cartesian variant). For that reason we will present this result with some detail.

2.1. **Linear λ-calculus, soundness, and completeness.** Let us start by fixing a *class* G of ground types. The grammar of types for linear λ-calculus is given by:

$$\mathbb{A} ::= X \in G \mid \mathbb{I} \mid \mathbb{A} \otimes \mathbb{A} \mid \mathbb{A} \multimap \mathbb{A}$$

We also fix a *class* Σ of sorted operation symbols $f : \mathbb{A}_1, \ldots, \mathbb{A}_n \to \mathbb{A}$ with $n \geq 1$. As usual, we use Greek letters $\Gamma, \Delta, E, \ldots$ to denote *typing contexts*, *i.e.* lists $x_1 : \mathbb{A}_1, \ldots, x_n : \mathbb{A}_n$ of typed variables such that each variable $x_i, 1 \leq i \leq n$, occurs at most once in $x_1, \ldots, x_n$.

We will use the notion of a *shuffle* to build a linear typing system such that the *exchange rule* is admissible and each judgement $\Gamma \rhd v : \mathbb{A}$ has a *unique derivation* – this will allow us to refer to a judgement's denotation $[\![\Gamma \rhd v : \mathbb{A}]\!]$ unambiguously. By shuffle we mean a permutation of typed variables in a context sequence $\Gamma_1, \ldots, \Gamma_n$ such that for all $i \leq n$ the relative order of the variables in Γ_i is preserved [Shu19]. For example, if $\Gamma_1 = x : \mathbb{A}, y : \mathbb{B}$ and $\Gamma_2 = z : \mathbb{C}$ then $z : \mathbb{C}, x : \mathbb{A}, y : \mathbb{B}$ is a shuffle of Γ_1, Γ_2 but $y : \mathbb{B}, x : \mathbb{A}, z : \mathbb{C}$ is *not*, because we changed the order in which x and y appear in Γ_1. As explained in [Shu19] (and also in Thm. 2.1), such a restriction on relative orders is crucial for judgements having unique derivations. We denote by $\mathrm{Sf}(\Gamma_1; \ldots; \Gamma_n)$ the set of shuffles on $\Gamma_1, \ldots, \Gamma_n$.

The term formation rules of linear λ-calculus are listed in Fig. 1. They correspond to the natural deduction rules of the exponential free, multiplicative fragment of intuitionistic linear logic. Substitution is defined as expected, yielding a particularly well-behaved calculus.

$$\frac{\Gamma_i \rhd v_i : \mathbb{A}_i \quad f : \mathbb{A}_1, \ldots, \mathbb{A}_n \to \mathbb{A} \in \Sigma \quad E \in \mathrm{Sf}(\Gamma_1; \ldots; \Gamma_n)}{E \rhd f(v_1, \ldots, v_n) : \mathbb{A}}\ (\mathbf{ax}) \qquad \frac{}{x : \mathbb{A} \rhd x : \mathbb{A}}\ (\mathbf{hyp})$$

$$\frac{}{- \rhd * : \mathbb{I}}\ (\mathbb{I}_\mathbf{i}) \qquad \frac{\Gamma \rhd v : \mathbb{A} \otimes \mathbb{B} \quad \Delta, x : \mathbb{A}, y : \mathbb{B} \rhd w : \mathbb{C} \quad E \in \mathrm{Sf}(\Gamma; \Delta)}{E \rhd \mathtt{pm}\ v\ \mathtt{to}\ x \otimes y.\ w : \mathbb{C}}\ (\otimes_\mathbf{e})$$

$$\frac{\Gamma \rhd v : \mathbb{A} \quad \Delta \rhd w : \mathbb{B} \quad E \in \mathrm{Sf}(\Gamma; \Delta)}{E \rhd v \otimes w : \mathbb{A} \otimes \mathbb{B}}\ (\otimes_\mathbf{i}) \qquad \frac{\Gamma \rhd v : \mathbb{I} \quad \Delta \rhd w : \mathbb{A} \quad E \in \mathrm{Sf}(\Gamma; \Delta)}{E \rhd v\ \mathtt{to}\ *.\ w : \mathbb{A}}\ (\mathbb{I}_\mathbf{e})$$

$$\frac{\Gamma, x : \mathbb{A} \rhd v : \mathbb{B}}{\Gamma \rhd \lambda x : \mathbb{A}.\, v : \mathbb{A} \multimap \mathbb{B}}\ (\multimap_\mathbf{i}) \qquad \frac{\Gamma \rhd v : \mathbb{A} \multimap \mathbb{B} \quad \Delta \rhd w : \mathbb{A} \quad E \in \mathrm{Sf}(\Gamma; \Delta)}{E \rhd v\, w : \mathbb{B}}\ (\multimap_\mathbf{e})$$

FIGURE 1. Term formation rules of linear λ-calculus.

Theorem 2.1. *The calculus defined by the rules of Fig. 1 enjoys the following properties:*

(1) *(Unique typing) For any two judgements $\Gamma \rhd v : \mathbb{A}$ and $\Gamma \rhd v : \mathbb{A}'$, we have $\mathbb{A} = \mathbb{A}'$;*

(2) *(Unique derivation) Every judgement* $\Gamma \rhd v : \mathbb{A}$ *has a unique derivation;*
(3) *(Exchange) For every judgement* $\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \rhd v : \mathbb{C}$ *we can derive* $\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \rhd v : \mathbb{C}$*;*
(4) *(Substitution) For all judgements* $\Gamma, x : \mathbb{A} \rhd v : \mathbb{B}$ *and* $\Delta \rhd w : \mathbb{A}$ *we can derive* $\Gamma, \Delta \rhd v[w/x] : \mathbb{B}$*.*

Proof. (1) follows straightforwardly by induction over the derivation, thanks to the explicit annotation of types in λ-abstractions.

(2) follows by induction over the derivation: let us only consider the case $f(v_1, \dots, v_n)$, because the other cases follow analogously. Suppose that $E \rhd f(v_1, \dots, v_n) : \mathbb{A}$. Then according to the typing system it is necessarily the case that the previous derivations were $\Gamma_i \rhd v_i : \mathbb{A}_i$ for all $i \leq n$ with $E \in \mathrm{Sf}(\Gamma_1; \dots; \Gamma_n)$ for some family of contexts $(\Gamma_i)_{i \leq n}$. The only room for choice is therefore in choosing the contexts Γ_i. We will show that even this choice is unique. Consider two families $(\Gamma_i)_{i \leq n}$ and $(\Gamma'_i)_{i \leq n}$ such that $\Gamma_i \rhd v_i : \mathbb{A}_i$ and $\Gamma'_i \rhd v_i : \mathbb{A}_i$ for all $i \leq n$, and moreover $E \in \mathrm{Sf}(\Gamma_1; \dots; \Gamma_n)$ and $E \in \mathrm{Sf}(\Gamma'_1; \dots; \Gamma'_n)$. Since $\Gamma_i \rhd v_i : \mathbb{A}_i$ and $\Gamma'_i \rhd v_i : \mathbb{A}_i$ we deduce (by linearity) that Γ_i is a permutation of Γ'_i. Consequently, since $E \in \mathrm{Sf}(\Gamma_1, \dots, \Gamma_n)$, $E \in \mathrm{Sf}(\Gamma'_1, \dots, \Gamma'_n)$ and E (by the definition of a shuffle) cannot change the relative order of the elements in Γ_i and Γ'_i for all $i \leq n$, it must be the case that $\Gamma_i = \Gamma'_i$ for all $i \leq n$. In other words, the choice of $(\Gamma_i)_{i \leq n}$ is fixed *a priori*. The proof now follows by applying the induction hypothesis to each $\Gamma_i \rhd v_i : \mathbb{A}_i$.

(3) follows by induction over the derivation. Again, we only consider the case $\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \rhd f(v_1, \dots, v_n) : \mathbb{C}$, the other cases follow analogously. Suppose that $\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \rhd f(v_1, \dots, v_n) : \mathbb{C}$ with $\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \in \mathrm{Sf}(\Gamma_1; \dots; \Gamma_n)$. We proceed by case distinction: assume first that both $x : \mathbb{A}$ and $y : \mathbb{B}$ are in some Γ_i, with $i \leq n$. We can thus decompose Γ_i into $\Gamma_i^1, x : \mathbb{A}, y : \mathbb{B}, \Gamma_i^2$. Then we apply the induction hypothesis on $\Gamma_i \rhd v_i : \mathbb{A}_i$ and proceed by observing that if $\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \in \mathrm{Sf}(\Gamma_1; \dots; (\Gamma_i^1, x : \mathbb{A}, y : \mathbb{B}, \Gamma_i^2); \dots; \Gamma_n)$ then it is also the case that $\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \in \mathrm{Sf}(\Gamma_1; \dots; (\Gamma_i^1, y : \mathbb{B}, x : \mathbb{A}, \Gamma_i^2); \dots; \Gamma_n)$. Assume now that $x : \mathbb{A}$ is in some Γ_i and $y : \mathbb{B}$ is in some Γ_j with $i \neq j$. Then since $\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \in \mathrm{Sf}(\Gamma_1; \dots; \Gamma_n)$ it must be the case that $\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \in \mathrm{Sf}(\Gamma_1; \dots; \Gamma_n)$ so we only need to apply rule (**ax**).

Finally, (4) follows from the exchange property that was just proved and induction over $\Gamma, x : \mathbb{A} \rhd v : \mathbb{B}$. Again, we exemplify this with rule (**ax**). Suppose that $\Gamma, x : \mathbb{A} \rhd f(v_1, \dots, v_n) : \mathbb{B}$. Then for all $i \leq n$ we have $\Gamma_i \rhd v_i : \mathbb{A}_i$ and $\Gamma, x : \mathbb{A} \in \mathrm{Sf}(\Gamma_1; \dots; \Gamma_n)$. By linearity and by the definition of a shuffle there exists exactly one Γ_i that can be decomposed into $\Gamma_i = \Gamma'_i, x : \mathbb{A}$. We then use the induction hypothesis to obtain $\Gamma'_i, \Delta \rhd v_i[w/x] : \mathbb{A}_i$. Now observe that if $\Gamma, x : \mathbb{A} \in \mathrm{Sf}(\Gamma_1; \dots; (\Gamma'_i, x : \mathbb{A}); \dots; \Gamma_n)$ then $\Gamma, \Delta \in \mathrm{Sf}(\Gamma_1; \dots; (\Gamma'_i, \Delta); \dots; \Gamma_n)$. We use this last observation to build $\Gamma, \Delta \rhd f(v_1, \dots, v_i[w/x], \dots, v_n) = f(v_1, \dots, v_n)[w/x] : \mathbb{B}$. ◻

We now recall the interpretation of judgements $\Gamma \rhd v : \mathbb{A}$ in a symmetric monoidal closed (autonomous) category C. We start by fixing some notation. For all C-objects X, Y, Z, $\mathsf{sw}_{X,Y} : X \otimes Y \to Y \otimes X$ denotes the symmetry morphism, $\lambda_X : \mathbb{I} \otimes X \to X$ the left unitor, $\rho_X : X \otimes \mathbb{I} \to X$ the right unitor, and $\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$ the left associator. Moreover for all C-morphisms $f : X \otimes Y \to Z$ we denote the corresponding curried version (right transpose) by $\overline{f} : X \to (Y \multimap Z)$.

Next, for all ground types $X \in G$ we postulate an interpretation $[\![X]\!]$ as a C-object. Types are then interpreted inductively using the unit I, the tensor $\otimes$ and the internal hom $\multimap$ of autonomous categories. Given a non-empty context $\Gamma = \Gamma', x : \mathbb{A}$, its interpretation is defined by $[\![\Gamma', x : \mathbb{A}]\!] = [\![\Gamma']\!] \otimes [\![\mathbb{A}]\!]$ if Γ' is non-empty and $[\![\Gamma', x : \mathbb{A}]\!] = [\![\mathbb{A}]\!]$ otherwise. The empty context $-$ is interpreted as $[\![-]\!] = I$. Given $X_1, \dots, X_n \in \mathsf{C}$ we write $X_1 \otimes \dots \otimes X_n$ for the n-tensor $(\dots(X_1 \otimes X_2) \otimes \dots) \otimes X_n$, and similarly for C-morphisms. Moreover we will often omit subscripts in components of natural transformations if no ambiguities arise.

We will also need 'housekeeping' morphisms to handle interactions between context interpretation and the autonomous structure of C. Given $\Gamma_1, \dots, \Gamma_n$ we denote by $\mathsf{sp}_{\Gamma_1;\dots;\Gamma_n} : [\![\Gamma_1, \dots, \Gamma_n]\!] \to [\![\Gamma_1]\!] \otimes \dots \otimes [\![\Gamma_n]\!]$ the morphism that splits $[\![\Gamma_1, \dots, \Gamma_n]\!]$ into $[\![\Gamma_1]\!] \otimes \dots \otimes [\![\Gamma_n]\!]$ which is defined as follows. Given Γ_1 and Γ_2, $\mathsf{sp}_{\Gamma_1;\Gamma_2} : [\![\Gamma_1, \Gamma_2]\!] \to [\![\Gamma_1]\!] \otimes [\![\Gamma_2]\!]$ is defined by

$$\mathsf{sp}_{-;\Gamma} = \lambda^{-1} \qquad \mathsf{sp}_{\Gamma;-} = \rho^{-1} \qquad \mathsf{sp}_{\Gamma;x:\mathbb{A}} = \mathsf{id} \qquad \mathsf{sp}_{\Gamma;(\Delta,x:\mathbb{A})} = \alpha \cdot (\mathsf{sp}_{\Gamma;\Delta} \otimes \mathsf{id})$$

For $n > 2$, $\mathsf{sp}_{\Gamma_1;\dots;\Gamma_n} : [\![\Gamma_1, \dots, \Gamma_n]\!] \to [\![\Gamma_1]\!] \otimes \dots \otimes [\![\Gamma_n]\!]$ is defined via the previous definition and induction on the size of n:

$$\mathsf{sp}_{\Gamma_1;\dots;\Gamma_n} = (\mathsf{sp}_{\Gamma_1;\dots;\Gamma_{n-1}} \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma_1,\dots,\Gamma_{n-1};\Gamma_n}$$

We denote by $\mathsf{jn}_{\Gamma_1;\dots;\Gamma_n}$ the inverse of $\mathsf{sp}_{\Gamma_1;\dots;\Gamma_n}$. Next, given $\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta$ we denote by $\mathsf{exch}_{\Gamma,\underline{x:\mathbb{A},y:\mathbb{B}},\Delta} : [\![\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta]\!] \to [\![\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta]\!]$ the morphism permuting x and y:

$$\mathsf{exch}_{\Gamma;\underline{x:\mathbb{A},y:\mathbb{B}},\Delta} = \mathsf{jn}_{\Gamma;y:\mathbb{B},x:\mathbb{A};\Delta} \cdot (\mathsf{id} \otimes \mathsf{sw} \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;x:\mathbb{A},y:\mathbb{B};\Delta}$$

The shuffling morphism $\mathsf{sh}_E : [\![E]\!] \to [\![\Gamma_1, \dots, \Gamma_n]\!]$ corresponds to a sequence of permutations and is thus defined as a suitable composition of exchange morphisms. Whenever convenient we will drop variables or even the whole subscript of the housekeeping morphisms.

$$\frac{[\![\Gamma_i \rhd v_i : \mathbb{A}_i]\!] = m_i \quad f : \mathbb{A}_1, \dots, \mathbb{A}_n \to \mathbb{A} \in \Sigma \quad E \in \mathrm{Sf}(\Gamma_1 \dots \Gamma_n)}{[\![E \rhd f(v_1, \dots, v_n) : \mathbb{A}]\!] = [\![f]\!] \cdot (m_1 \otimes \dots \otimes m_n) \cdot \mathsf{sp}_{\Gamma_1;\dots;\Gamma_n} \cdot \mathsf{sh}_E} \qquad \frac{}{[\![x : \mathbb{A} \rhd x : \mathbb{A}]\!] = \mathsf{id}_{[\![\mathbb{A}]\!]}}$$

$$\frac{}{[\![- \rhd * : \mathbb{I}]\!] = \mathsf{id}_{[\![\mathbb{I}]\!]}} \qquad \frac{[\![\Gamma \rhd v : \mathbb{A} \otimes \mathbb{B}]\!] = m \quad [\![\Delta, x : \mathbb{A}, y : \mathbb{B} \rhd w : \mathbb{C}]\!] = n \quad E \in \mathrm{Sf}(\Gamma; \Delta)}{[\![E \rhd \mathtt{pm}\ v\ \mathtt{to}\ x \otimes y.\ w : \mathbb{C}]\!] = n \cdot \mathsf{jn}_{\Delta;\mathbb{A};\mathbb{B}} \cdot \alpha \cdot \mathsf{sw} \cdot (m \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E}$$

$$\frac{[\![\Gamma \rhd v : \mathbb{A}]\!] = m \quad [\![\Delta \rhd w : \mathbb{B}]\!] = n \quad E \in \mathrm{Sf}(\Gamma; \Delta)}{[\![E \rhd v \otimes w : \mathbb{A} \otimes \mathbb{B}]\!] = (m \otimes n) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E} \qquad \frac{[\![\Gamma \rhd v : \mathbb{I}]\!] = m \quad [\![\Delta \rhd w : \mathbb{A}]\!] = n \quad E \in \mathrm{Sf}(\Gamma; \Delta)}{[\![E \rhd v\ \mathtt{to}\ *.\ w : \mathbb{A}]\!] = n \cdot \lambda \cdot (m \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E}$$

$$\frac{[\![\Gamma, x : \mathbb{A} \rhd v : \mathbb{B}]\!] = m}{[\![\Gamma \rhd \lambda x : \mathbb{A}.\, v : \mathbb{A} \multimap \mathbb{B}]\!] = \overline{(m \cdot \mathsf{jn}_{\Gamma;\mathbb{A}})}} \qquad \frac{[\![\Gamma \rhd v : \mathbb{A} \multimap \mathbb{B}]\!] = m \quad [\![\Delta \rhd w : \mathbb{A}]\!] = n \quad E \in \mathrm{Sf}(\Gamma; \Delta)}{[\![E \rhd v\, w : \mathbb{B}]\!] = \mathrm{app} \cdot (m \otimes n) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E}$$

FIGURE 2. Judgement interpretation on an autonomous category C.

For every operation symbol $f : \mathbb{A}_1, \dots, \mathbb{A}_n \to \mathbb{A}$ in Σ we postulate an interpretation $[\![f]\!] : [\![\mathbb{A}_1]\!] \otimes \dots \otimes [\![\mathbb{A}_n]\!] \to [\![\mathbb{A}]\!]$ as a C-morphism. The interpretation of judgements is defined by induction over derivations according to the rules in Fig. 2.

As detailed in [BBdPH92, MRA93, MMDPR05], linear λ-calculus comes equipped with a class of equations, given in Fig. 3, specifically *equations-in-context* $\Gamma \rhd v = w : \mathbb{A}$, that corresponds to the axiomatics of autonomous categories. As usual, we omit the context and typing information of the equations in Fig. 3, which can be reconstructed in the usual way.

The next step is to prove that the equational schema listed in Fig. 3 is sound w.r.t. autonomous categories. For that effect, we will use the following *exchange and substitution lemma* (in order to keep calculations in proofs legible we will sometimes abbreviate a denotation $[\![\Gamma \rhd v : \mathbb{A}]\!]$ to $[\![\Gamma \rhd v]\!]$ or even just $[\![v]\!]$).

Monoidal structure	Higher-order structure
$\mathtt{pm}\ v \otimes w\ \mathtt{to}\ x \otimes y.\ u \;=\; u[v/x, w/y]$ $\mathtt{pm}\ v\ \mathtt{to}\ x \otimes y.\ u[x \otimes y/z] \;=\; u[v/z]$ $*\ \mathtt{to}\ *.\ v \;=\; v$ $v\ \mathtt{to}\ *.\ w[*/z] \;=\; w[v/z]$	$(\lambda x : \mathbb{A}.\ v)\ w \;=\; v[w/x]$ $\lambda x : \mathbb{A}.(v\ x) \;=\; v$
Commuting conversions	
$u[v\ \mathtt{to} *.\ w/z] \;=\; v\ \mathtt{to}\ *.\ u[w/z]$ $u[\mathtt{pm}\ v\ \mathtt{to}\ x \otimes y.\ w/z] \;=\; \mathtt{pm}\ v\ \mathtt{to}\ x \otimes y.\ u[w/z]$	

FIGURE 3. Equations corresponding to the axiomatics of autonomous categories.

Lemma 2.2 (Exchange and Substitution)**.** *For any judgements $\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \rhd v : \mathbb{C}$, $\Gamma, x : \mathbb{A} \rhd v : \mathbb{B}$, and $\Delta \rhd w : \mathbb{A}$, the following equations hold in every autonomous category C:*

$$[\![\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \rhd v : \mathbb{C}]\!] = [\![\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \rhd v : \mathbb{C}]\!] \cdot \mathsf{exch}_{\Gamma,\underline{\mathbb{A},\mathbb{B}},\Delta}$$

$$[\![\Gamma, \Delta \rhd v[w/x] : \mathbb{B}]\!] = [\![\Gamma, x : \mathbb{A} \rhd v : \mathbb{B}]\!] \cdot \mathsf{jn}_{\Gamma;\mathbb{A}} \cdot (\mathsf{id} \otimes [\![\Delta \rhd w : \mathbb{A}]\!]) \cdot \mathsf{sp}_{\Gamma;\Delta}$$

Proof sketch. By induction over the derivations. We only consider rule (**ax**), the others follow analogously. In the derivations we mark steps using naturality and the coherence theorem of symmetric monoidal categories [ML98] with $\{c\}$. We start with the exchange property. Suppose that $\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \rhd f(v_1, \ldots, v_n) : \mathbb{C}$. We proceed by case distinction. First, consider the case in which $x : \mathbb{A} \in \Gamma_i$ and $y : \mathbb{B} \in \Gamma_j$ with $i \neq j$. The proof then follows by observing that the shuffling morphisms $\mathsf{sh}_{\Gamma,\mathbb{A},\mathbb{B},\Delta} : [\![\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta]\!] \to [\![\Gamma_1, \ldots, \Gamma_n]\!]$ and $\mathsf{sh}_{\Gamma,\mathbb{B},\mathbb{A},\Delta} : [\![\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta]\!] \to [\![\Gamma_1, \ldots, \Gamma_n]\!]$ satisfy $\mathsf{sh}_{\Gamma,\mathbb{B},\mathbb{A},\Delta} \cdot \mathsf{exch}_{\Gamma,\underline{\mathbb{A},\mathbb{B}},\Delta} = \mathsf{sh}_{\Gamma,\mathbb{A},\mathbb{B},\Delta}$. Consider now the case where $x : \mathbb{A} \in \Gamma_i$ and $y : \mathbb{B} \in \Gamma_i$ for some $i \leq n$. We calculate:

$$\begin{aligned}
&[\![\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \rhd f(v_1, \ldots, v_n) : \mathbb{C}]\!] \\
&= [\![f]\!] \cdot ([\![v_1]\!] \otimes \cdots \otimes [\![v_i]\!] \otimes \cdots \otimes [\![v_n]\!]) \cdot \mathsf{sp}_{\Gamma_1;\ldots;\Gamma_n} \cdot \mathsf{sh}_{\Gamma,\mathbb{A},\mathbb{B},\Delta} \\
&= [\![f]\!] \cdot ([\![v_1]\!] \otimes \cdots \otimes ([\![v_i]\!] \cdot \mathsf{exch}_{\Gamma_i^1,\underline{\mathbb{A},\mathbb{B}},\Gamma_i^2}) \otimes \cdots \otimes [\![v_n]\!]) \cdot \mathsf{sp}_{\Gamma_1;\ldots;\Gamma_n} \cdot \mathsf{sh}_{\Gamma,\mathbb{A},\mathbb{B},\Delta} \\
&= [\![f]\!] \cdot ([\![v_1]\!] \otimes \cdots \otimes [\![v_n]\!]) \cdot (\mathsf{id} \otimes \cdots \otimes \mathsf{exch}_{\Gamma_i^1,\underline{\mathbb{A},\mathbb{B}},\Gamma_i^2} \otimes \cdots \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma_1;\ldots;\Gamma_n} \cdot \mathsf{sh}_{\Gamma,\mathbb{A},\mathbb{B},\Delta} \\
&= [\![f]\!] \cdot ([\![v_1]\!] \otimes \cdots \otimes [\![v_n]\!]) \cdot \mathsf{sp}_{\Gamma_1';\ldots;\Gamma_n'} \cdot \mathsf{exch}_{\Gamma_1,\ldots,\Gamma_i^1,\underline{\mathbb{A},\mathbb{B}},\Gamma_i^2,\ldots\Gamma_n} \cdot \mathsf{sh}_{\Gamma,\mathbb{A},\mathbb{B},\Delta} \\
&= [\![f]\!] \cdot ([\![v_1]\!] \otimes \cdots \otimes [\![v_n]\!]) \cdot \mathsf{sp}_{\Gamma_1';\ldots;\Gamma_n'} \cdot \mathsf{sh}_{\Gamma,\mathbb{B},\mathbb{A},\Delta} \cdot \mathsf{exch}_{\Gamma,\underline{\mathbb{A},\mathbb{B}},\Delta} && \{c\} \\
&= [\![\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \rhd f(v_1, \ldots, v_n) : \mathbb{C}]\!] \cdot \mathsf{exch}_{\Gamma,\underline{\mathbb{A},\mathbb{B}},\Delta}
\end{aligned}$$

Let us now focus on proving the substitution lemma for Rule (**ax**):

$$\begin{aligned}
&[\![\Gamma, \Delta \rhd f(v_1, \ldots, v_n)[w/x] : \mathbb{B}]\!] \\
&= [\![\Gamma, \Delta \rhd f(v_1, \ldots, v_i[w/x], \ldots, v_n) : \mathbb{B}]\!] \\
&= [\![f]\!] \cdot ([\![v_1]\!] \otimes \cdots \otimes [\![v_i[w/x]]\!] \otimes \cdots \otimes [\![v_n]\!]) \cdot \mathsf{sp}_{\Gamma_1;\ldots;\Gamma_n} \cdot \mathsf{sh}_{\Gamma,\Delta} \\
&= [\![f]\!] \cdot ([\![v_1]\!] \otimes \cdots \otimes ([\![v_i]\!] \cdot \mathsf{jn}_{\Gamma_i';\mathbb{A}} \cdot (\mathsf{id} \otimes [\![w]\!]) \cdot \mathsf{sp}_{\Gamma_i';\Delta}) \otimes \cdots \otimes [\![v_n]\!]) \cdot \mathsf{sp}_{\Gamma_1;\ldots;\Gamma_n} \cdot \mathsf{sh}_{\Gamma,\Delta} \\
&= [\![f]\!] \cdot ([\![v_1]\!] \otimes \cdots \otimes [\![v_n]\!]) \cdot (\mathsf{id} \otimes \cdots \otimes (\mathsf{jn}_{\Gamma_i';\mathbb{A}} \cdot (\mathsf{id} \otimes [\![w]\!]) \cdot \mathsf{sp}_{\Gamma_i';\Delta}) \otimes \cdots \otimes \mathsf{id}) \cdot \ldots \\
&\ldots\ \mathsf{sp}_{\Gamma_1;\ldots;\Gamma_n} \cdot \mathsf{sh}_{\Gamma,\Delta} \\
&= [\![f]\!] \cdot ([\![v_1]\!] \otimes \cdots \otimes [\![v_n]\!]) \cdot \mathsf{sp}_{\Gamma_1;\ldots;\Gamma_i',\mathbb{A};\ldots\Gamma_n} \cdot \mathsf{sh}_{\Gamma,\mathbb{A}} \cdot \mathsf{jn}_{\Gamma;\mathbb{A}} \cdot (\mathsf{id} \otimes [\![w]\!]) \cdot \mathsf{sp}_{\Gamma;\Delta} && \{c\} \\
&= [\![\Gamma, x : \mathbb{A} \rhd f(v_1, \ldots, v_n)]\!] \cdot \mathsf{jn}_{\Gamma;\mathbb{A}} \cdot (\mathsf{id} \otimes [\![w]\!]) \cdot \mathsf{sp}_{\Gamma;\Delta}
\end{aligned}$$

□

Theorem 2.3. *The equations presented in Fig. 3 are sound w.r.t. judgement interpretation. More specifically if* $\Gamma \rhd v = w : \mathbb{A}$ *is one of the equations in Fig. 3 then* $[\![\Gamma \rhd v : \mathbb{A}]\!] = [\![\Gamma \rhd w : \mathbb{A}]\!]$.

Proof sketch. The proof follows from Lemma 2.2, the coherence theorem for symmetric monoidal categories, and naturality. We exemplify this with one of the commuting conversions.

$$
\begin{aligned}
& [\![\Gamma, \Delta, E \rhd u[v \;\texttt{to}\; * .\; w/x] : \mathbb{B}]\!] \\
&= [\![u]\!] \cdot \mathsf{jn}_{\Gamma;\mathbb{A}} \cdot (\mathsf{id} \otimes [\![v \;\texttt{to}\; * .\; w]\!]) \cdot \mathsf{sp}_{\Gamma;\Delta,E} && \{\text{Lemma 2.2}\} \\
&= [\![u]\!] \cdot \mathsf{jn}_{\Gamma;\mathbb{A}} \cdot (\mathsf{id} \otimes ([\![w]\!] \cdot \lambda \cdot ([\![v]\!] \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Delta;E})) \cdot \mathsf{sp}_{\Gamma;\Delta,E} \\
&= [\![u]\!] \cdot \mathsf{jn}_{\Gamma;\mathbb{A}} \cdot (\mathsf{id} \otimes [\![w]\!]) \cdot (\mathsf{id} \otimes (\lambda \cdot ([\![v]\!] \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Delta;E})) \cdot \mathsf{sp}_{\Gamma;\Delta,E} \\
&= [\![u]\!] \cdot \mathsf{jn}_{\Gamma;\mathbb{A}} \cdot (\mathsf{id} \otimes [\![w]\!]) \cdot \mathsf{sp}_{\Gamma;E} \cdot \mathsf{jn}_{\Gamma;E} \cdot (\mathsf{id} \otimes (\lambda \cdot ([\![v]\!] \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Delta;E})) \cdot \mathsf{sp}_{\Gamma;\Delta,E} \\
&= [\![u[w/x]]\!] \cdot \mathsf{jn}_{\Gamma;E} \cdot (\mathsf{id} \otimes (\lambda \cdot ([\![v]\!] \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Delta;E})) \cdot \mathsf{sp}_{\Gamma;\Delta,E} && \{\text{Lemma 2.2}\} \\
&= [\![u[w/x]]\!] \cdot \lambda \cdot ([\![v]\!] \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Delta;\Gamma,E} \cdot \mathsf{sh}_{\Gamma,\Delta,E} && \{\text{c}\} \\
&= [\![\Gamma, \Delta, E \rhd v \;\texttt{to}\; * .\; u[w/x] : \mathbb{B}]\!]
\end{aligned}
$$

□

Definition 2.4 (Linear λ-theories)**.** Consider a tuple (G, Σ) consisting of a class G of ground types and a class Σ of sorted operation symbols. A *linear λ-theory* $((G, \Sigma), Ax)$ is a triple such that Ax is a class of equations-in-context over linear λ-terms built from (G, Σ).

The elements of Ax are called the *axioms* of the theory. Let $Th(Ax)$ be the smallest congruence that contains Ax, the equations listed in Fig. 3, and that is closed under exchange and substitution (Thm. 2.1). We call the elements of $Th(Ax)$ the *theorems* of the theory.

Definition 2.5 (Models of linear λ-theories)**.** Consider a linear λ-theory $((G, \Sigma), Ax)$ and also an autonomous category C. Suppose that for each $X \in G$ we have an interpretation $[\![X]\!]$ that is a C-object and analogously for the operation symbols. This interpretation structure is a *model* of the theory if all axioms are satisfied by the interpretation.

Theorem 2.6 (Soundness & Completeness)**.** *Consider a linear λ-theory $\mathscr{T}$. Then an equation* $\Gamma \rhd v = w : \mathbb{A}$ *is a theorem of $\mathscr{T}$ iff it is satisfied by all models of the theory.*

Proof sketch. Soundness follows by induction over the rules that define $Th(Ax)$ and by Thm. 2.3. Completeness is based on the idea of a *Lindenbaum-Tarski algebra*: it follows from building the *syntactic category* $\mathrm{Syn}(\mathscr{T})$ of $\mathscr{T}$ (also known as term model), showing that it possesses an autonomous structure and also that equality $[\![\Gamma \rhd v : \mathbb{A}]\!] = [\![\Gamma \rhd w : \mathbb{A}]\!]$ in the syntactic category is equivalent to provability $\Gamma \rhd v = w : \mathbb{A}$ in the theory. The syntactic category of $\mathscr{T}$ has as objects the types of $\mathscr{T}$ and as morphisms $\mathbb{A} \to \mathbb{B}$ the equivalence classes (w.r.t. provability) of terms v for which we can derive $x : \mathbb{A} \rhd v : \mathbb{B}$. □

2.2. **Equivalence theorem between linear λ-theories and autonomous categories.** As mentioned in the beginning of §2, we will now focus on presenting a quasi-category [AHS09] of λ-theories, a quasi-category of autonomous categories, and then showing that they are equivalent to each other. In order to prepare the stage, we will start by establishing a bijective correspondence (up-to isomorphism) between models of a λ-theory $\mathscr{T}$ on a category C and *autonomous* functors $\mathrm{Syn}(\mathscr{T}) \to \mathsf{C}$. Although not strictly necessary for establishing the aforementioned equivalence, this bijection has multiple benefits: first it will allow us to formally see models as functors and thus opens up the possibility of applying functorial constructions to them. For example, the notion of a natural isomorphism (between functors) carries to the notion of a model isomorphism. Second, it will later on help us motivate the notion of a morphism and equivalence between λ-theories. Third, it will help us establish the aforementioned equivalence of quasi-categories. Our proof of the bijective correspondence between models and autonomous functors is inspired on an analogous one [Cro93] for the Cartesian case.

We first recall the definition of an autonomous functor. We will use I_{C} to denote the unit of a monoidal category C, and drop the subscript whenever no ambiguities arise.

Definition 2.7. A functor $F : \mathsf{C} \to \mathsf{D}$ between two monoidal categories C and D is called *monoidal* if it is equipped with a morphism $\mathsf{u} : I_{\mathsf{D}} \to F(I_{\mathsf{C}})$ and a natural transformation $\mu_{X,Y} : FX \otimes_{\mathsf{D}} FY \to F(X \otimes_{\mathsf{C}} Y)$ such that the following diagrams commute:

$$\begin{tikzcd}
FX \otimes I \arrow[r, "\mathsf{id} \otimes \mathsf{u}"] \arrow[d, "\rho"'] & FX \otimes FI \arrow[d, "\mu"] \\
FX & F(X \otimes I) \arrow[l, "F\rho"]
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes FX \arrow[r, "\mathsf{u} \otimes \mathsf{id}"] \arrow[d, "\lambda"'] & FI \otimes FX \arrow[d, "\mu"] \\
FX & F(I \otimes X) \arrow[l, "F\lambda"]
\end{tikzcd}$$

$$\begin{tikzcd}
(FX \otimes FY) \otimes FZ \arrow[r, "\alpha^{-1}"] \arrow[d, "\mu \otimes \mathsf{id}"'] & FX \otimes (FY \otimes FZ) \arrow[d, "\mathsf{id} \otimes \mu"] \\
F(X \otimes Y) \otimes FZ \arrow[d, "\mu"'] & FX \otimes F(Y \otimes Z) \arrow[d, "\mu"] \\
F((X \otimes Y) \otimes Z) \arrow[r, "F\alpha^{-1}"'] & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

The functor $F : \mathsf{C} \to \mathsf{D}$ is *strong monoidal* if $\mathsf{u} : I_{\mathsf{D}} \to F(I_{\mathsf{C}})$ is an isomorphism and if $\mu_{X,Y} : FX \otimes_{\mathsf{D}} FY \to F(X \otimes_{\mathsf{C}} Y)$ is a natural isomorphism. We call F *strict* if u is the identity and $\mu_{X,Y}$ is also the identity for all C-objects X, Y. Assume next that both categories C and D are symmetric monoidal. Then we say that F is *symmetric monoidal* if it is monoidal and moreover the diagram below commutes.

$$\begin{tikzcd}
FX \otimes FY \arrow[r, "\mathsf{sw}"] \arrow[d, "\mu"'] & FY \otimes FX \arrow[d, "\mu"] \\
F(X \otimes Y) \arrow[r, "F\,\mathsf{sw}"'] & F(Y \otimes X)
\end{tikzcd}$$

Finally consider the following D-morphism:

$$F(X \multimap Y) \otimes FX \xrightarrow{\mu} F((X \multimap Y) \otimes X) \xrightarrow{F\mathrm{app}} FY$$

We call F *autonomous* if it is symmetric strong monoidal and if the right transpose of the previous morphism is an isomorphism.

Next, for a linear λ-theory $\mathscr{T}$ we will show that autonomous functors $F : \mathsf{C} \to \mathsf{D}$ send models of $\mathscr{T}$ on C to models of $\mathscr{T}$ on D. In other words, the autonomous functor F permits a '$\mathscr{T}$-respecting' change of interpretation domain. This mapping will be useful for proving the aforementioned bijection between autonomous functors and models. So let us consider an interpretation M of a linear λ-theory $\mathscr{T}$ over an autonomous category C and an autonomous functor $F : \mathsf{C} \to \mathsf{D}$. Then we define the following interpretation F_*M of $\mathscr{T}$ over D:

$$[\![G]\!]_{F_*M} = F[\![G]\!]_M \qquad\qquad [\![f]\!]_{F_*M} = F[\![f]\!]_M$$

We wish to show that this interpretation is a model of $\mathscr{T}$ over D whenever M is a model of $\mathscr{T}$ over C. To achieve that, we will need the following construction: we build an isomorphism $h_{\mathbb{A}} : [\![\mathbb{A}]\!]_{F_*M} \to F[\![\mathbb{A}]\!]_M$ for all types $\mathbb{A}$ by induction over the type structure of linear λ-calculus. If $\mathbb{A}$ is a ground type then $[\![\mathbb{A}]\!]_{F_*M} = F[\![\mathbb{A}]\!]_M$ and thus $h_{\mathbb{A}}$ is the identity. If $\mathbb{A}$ is $\mathbb{I}$ then $h_{\mathbb{I}} = \mathsf{u} : [\![\mathbb{I}]\!]_{F_*M} = I_{\mathsf{D}} \to FI_{\mathsf{C}} = F[\![\mathbb{I}]\!]_M$. If $\mathbb{A}$ has the form $\mathbb{A}_1 \otimes \mathbb{A}_2$ then $h_{\mathbb{A}_1 \otimes \mathbb{A}_2}$ arises from the following composite: $[\![\mathbb{A}_1 \otimes \mathbb{A}_2]\!]_{F_*M} = [\![\mathbb{A}_1]\!]_{F_*M} \otimes [\![\mathbb{A}_2]\!]_{F_*M} \cong F[\![\mathbb{A}_1]\!]_M \otimes F[\![\mathbb{A}_2]\!]_M \cong F([\![\mathbb{A}_1]\!]_M \otimes [\![\mathbb{A}_2]\!]_M) = F[\![\mathbb{A}_1 \otimes \mathbb{A}_2]\!]_M$, where the last step is given by the monoidal structure of F and the penultimate step is given by induction; so formally $h_{\mathbb{A}_1 \otimes \mathbb{A}_2}$ is equal to $\mu \cdot (h_{\mathbb{A}_1} \otimes h_{\mathbb{A}_2})$. An analogous reasoning applies to the case in which $\mathbb{A}$ has the form $\mathbb{A}_1 \multimap \mathbb{A}_2$. Specifically, $h_{\mathbb{A}_1 \multimap \mathbb{A}_2}$ takes the form $m^{-1} \cdot (- \cdot h_{\mathbb{A}_1}^{-1}) \cdot (h_{\mathbb{A}_2} \cdot -)$ where $m : F([\![\mathbb{A}_1]\!]_M \multimap [\![\mathbb{A}_2]\!]_M) \to (F[\![\mathbb{A}_1]\!]_M \multimap F[\![\mathbb{A}_2]\!]_M)$ is the right transpose of $F(\mathrm{app}) \cdot \mu$. Whenever no ambiguities arise we will drop the subscript in the isomorphism $h_{\mathbb{A}}$. For a context Γ let us define $[\![\Gamma]\!]^F_M$ via the equations $[\![-]\!]^F_M = I_{\mathsf{D}}$, $[\![\Gamma, x : \mathbb{A}]\!]^F_M = [\![\Gamma]\!]^F_M \otimes F[\![\mathbb{A}]\!]_M$ if Γ is non-empty and $[\![\Gamma, x : \mathbb{A}]\!]^F_M = F[\![\mathbb{A}]\!]_M$ otherwise. Intuitively, $[\![\Gamma]\!]^F_M$ corresponds to a pointwise application of F to the tuple of objects in the tensor $[\![\Gamma]\!]_M$. We can then extend the isomorphism $h : [\![\mathbb{A}]\!]_{F_*M} \to F[\![\mathbb{A}]\!]_M$ to contexts $h[\Gamma] : [\![\Gamma]\!]_{F_*M} \to [\![\Gamma]\!]^F_M$ via induction.

In order to show that F_*M is indeed a model of $\mathscr{T}$ whenever M is, we need to fix extra notation. Recall the natural transformation $\mu_{X,Y} : FX \otimes FY \to F(X \otimes Y)$. Then for $n \geq 0$, let us define the morphism $\mu[X_1, \ldots, X_n] : FX_1 \otimes \cdots \otimes FX_n \to F(X_1 \otimes \cdots \otimes X_n)$ by setting $\mu[-] = \mathsf{u} : I_{\mathsf{D}} \to F(I_{\mathsf{C}})$, $\mu[X] = \mathsf{id}$ and $\mu[X_1, \ldots, X_n] = \mu \cdot (\mu[X_1, \ldots, X_{n-1}] \otimes \mathsf{id})$ for $n \geq 2$. Given a context $\Gamma = x_1 : \mathbb{A}_1, \ldots, x_n : \mathbb{A}_n$ we use $\mu[\Gamma]$ to denote the morphism $\mu[[\![\mathbb{A}_1]\!]_M, \ldots, [\![\mathbb{A}_n]\!]_M] : [\![\Gamma]\!]^F_M \to F[\![\Gamma]\!]_M$. Finally, note that for n contexts $\Gamma_1, \ldots, \Gamma_n$ we can build a 'split' morphism $\mathsf{sp}^F_{\Gamma_1;\ldots;\Gamma_n} : [\![\Gamma_1, \ldots, \Gamma_n]\!]^F_M \to [\![\Gamma_1]\!]^F_M \otimes \cdots \otimes [\![\Gamma_n]\!]^F_M$ analogous to the morphism $\mathsf{sp}_{\Gamma_1;\ldots;\Gamma_n} : [\![\Gamma_1, \ldots, \Gamma_n]\!]_M \to [\![\Gamma_1]\!]_M \otimes \cdots \otimes [\![\Gamma_n]\!]_M$. We will also need the following lemma which states that the morphisms $h_{\mathbb{A}}$ and the monoidal structure of $F : \mathsf{C} \to \mathsf{D}$ commute with the housekeeping morphisms used for judgement interpretation.

Lemma 2.8. *The following diagrams commute:*

$$\begin{tikzcd}
{[\![\Gamma_1, \ldots, \Gamma_n]\!]_{F_*M}} \arrow[r, "\mathsf{sp}_{\Gamma_1;\ldots;\Gamma_n}"] \arrow[d, "{h[\Gamma_1,\ldots,\Gamma_n]}"'] & {[\![\Gamma_1]\!]_{F_*M} \otimes \ldots \otimes [\![\Gamma_n]\!]_{F_*M}} \arrow[d, "{h[\Gamma_1] \otimes \ldots \otimes h[\Gamma_n]}"] \\
{[\![\Gamma_1, \ldots, \Gamma_n]\!]^F_M} \arrow[r, "\mathsf{sp}^F_{\Gamma_1;\ldots;\Gamma_n}"'] & {[\![\Gamma_1]\!]^F_M \otimes \cdots \otimes [\![\Gamma_n]\!]^F_M}
\end{tikzcd} \tag{2.1}$$

$$\begin{tikzcd}
\llbracket \Gamma_1, \ldots, \Gamma_n \rrbracket^F_M \arrow[r, "\mathsf{sp}^F_{\Gamma_1;\ldots;\Gamma_n}"] \arrow[dd, "{\mu[\Gamma_1,\ldots,\Gamma_n]}"'] & \llbracket \Gamma_1 \rrbracket^F_M \otimes \cdots \otimes \llbracket \Gamma_n \rrbracket^F_M \arrow[d, "{\mu[\Gamma_1] \otimes \ldots \otimes \mu[\Gamma_n]}"] \\
& F\llbracket \Gamma_1 \rrbracket_M \otimes \cdots \otimes F\llbracket \Gamma_n \rrbracket_M \arrow[d, "{\mu[\llbracket \Gamma_1 \rrbracket_M, \ldots, \llbracket \Gamma_n \rrbracket_M]}"] \\
F\llbracket \Gamma_1, \ldots, \Gamma_n \rrbracket_M \arrow[r, "F\,\mathsf{sp}_{\Gamma_1;\ldots;\Gamma_n}"'] & F(\llbracket \Gamma_1 \rrbracket_M \otimes \cdots \otimes \llbracket \Gamma_n \rrbracket_M)
\end{tikzcd} \tag{2.2}$$

$$\begin{tikzcd}
\llbracket E \rrbracket_{F_*M} \arrow[r, "\mathsf{sh}_E"] \arrow[d, "{h[E]}"'] & \llbracket \Gamma_1, \ldots, \Gamma_n \rrbracket_{F_*M} \arrow[d, "{h[\Gamma_1,\ldots,\Gamma_n]}"] \\
\llbracket E \rrbracket^F_M \arrow[r, "\mathsf{sh}^F_E"'] & \llbracket \Gamma_1, \ldots, \Gamma_n \rrbracket^F_M
\end{tikzcd} \tag{2.3}$$

$$\begin{tikzcd}
\llbracket E \rrbracket^F_M \arrow[r, "\mathsf{sh}^F_E"] \arrow[d, "{\mu[E]}"'] & \llbracket \Gamma_1, \ldots, \Gamma_n \rrbracket^F_M \arrow[d, "{\mu[\Gamma_1,\ldots,\Gamma_n]}"] \\
F\llbracket E \rrbracket_M \arrow[r, "F\,\mathsf{sh}_E"'] & F\llbracket \Gamma_1, \ldots, \Gamma_n \rrbracket_M
\end{tikzcd} \tag{2.4}$$

Proof. For (2.1), we start by fixing $n = 2$ and proving that the diagram below commutes.

$$\begin{tikzcd}
\llbracket \Gamma_1, \Gamma_2 \rrbracket_{F_*M} \arrow[r, "\mathsf{sp}_{\Gamma_1;\Gamma_2}"] \arrow[d, "{h[\Gamma_1,\Gamma_2]}"'] & \llbracket \Gamma_1 \rrbracket_{F_*M} \otimes \llbracket \Gamma_2 \rrbracket_{F_*M} \arrow[d, "{h[\Gamma_1] \otimes h[\Gamma_2]}"] \\
\llbracket \Gamma_1, \Gamma_2 \rrbracket^F_M \arrow[r, "\mathsf{sp}^F_{\Gamma_1;\Gamma_2}"'] & \llbracket \Gamma_1 \rrbracket^F_M \otimes \llbracket \Gamma_2 \rrbracket^F_M
\end{tikzcd}$$

The proof is obtained by inspecting all possible cases used in the definition of the morphism $\mathsf{sp}_{\Gamma_1;\Gamma_2}$ (resp. $\mathsf{sp}^F_{\Gamma_1;\Gamma_2}$). Specifically the cases $-;\Gamma_2$, $\Gamma_1;-$, and $\Gamma_1;\Gamma_2$ with both Γ_1 and Γ_2 non-empty. The first two cases follow from the naturality of λ^{-1} and ρ^{-1}. The last case follows straightforwardly from induction on the size of Γ_2 (with size one as the base case) and the naturality of α. We can then use this last result to prove that Diagram (2.1) commutes using induction on the size of n with size two as the base case.

Let us now focus on Diagram (2.2). Analogously to before we start by fixing $n = 2$ and then prove that the diagram below commutes.

$$\begin{tikzcd}
\llbracket \Gamma_1, \Gamma_2 \rrbracket^F_M \arrow[r, "\mathsf{sp}^F_{\Gamma_1;\Gamma_2}"] \arrow[dd, "{\mu[\Gamma_1,\Gamma_2]}"'] & \llbracket \Gamma_1 \rrbracket^F_M \otimes \llbracket \Gamma_2 \rrbracket^F_M \arrow[d, "{\mu[\Gamma_1] \otimes \mu[\Gamma_2]}"] \\
& F\llbracket \Gamma_1 \rrbracket_M \otimes F\llbracket \Gamma_2 \rrbracket_M \arrow[d, "\mu"] \\
F\llbracket \Gamma_1, \Gamma_2 \rrbracket_M \arrow[r, "F\,\mathsf{sp}_{\Gamma_1;\Gamma_2}"'] & F(\llbracket \Gamma_1 \rrbracket_M \otimes \llbracket \Gamma_2 \rrbracket_M)
\end{tikzcd}$$

Again the proof is obtained by inspecting all possible cases in the definition of the morphism $\mathsf{sp}_{\Gamma_1;\Gamma_2}$ (resp. $\mathsf{sp}^F_{\Gamma_1;\Gamma_2}$). For the case $-;\Gamma_2$ the previous diagram is instantiated into the outer square below.

$$\begin{tikzcd}
\llbracket -, \Gamma_2 \rrbracket^F_M \arrow[rr, "\lambda^{-1}"] \arrow[dd, "\mu[\Gamma_2]"'] & & I_{\mathsf{D}} \otimes \llbracket \Gamma_2 \rrbracket^F_M \arrow[dl, "\mathsf{id} \otimes \mu[\Gamma_2]"'] \arrow[d, "\mathsf{u} \otimes \mu[\Gamma_2]"] \\
& I_{\mathsf{D}} \otimes F\llbracket \Gamma_2 \rrbracket_M \arrow[r, "\mathsf{u} \otimes \mathsf{id}"] \arrow[dl, "\lambda"'] & F(I_{\mathsf{C}}) \otimes F\llbracket \Gamma_2 \rrbracket_M \arrow[d, "\mu"] \\
F\llbracket -, \Gamma_2 \rrbracket_M \arrow[rr, "F(\lambda^{-1})"'] & & F(I_{\mathsf{C}} \otimes \llbracket \Gamma_2 \rrbracket_M)
\end{tikzcd}$$

The proof that the top left triangle commutes follows from λ^{-1} being a natural transformation; and the one for the top right triangle arises from the axiomatics of monoidal categories. The proof that the bottom right quadrilateral commutes follows from the fact that F is a monoidal functor. The proof for the case $\Gamma_1; -$ is analogous. Next we focus on the case $\Gamma_1; \Gamma_2$ with both Γ_1 and Γ_2 non-empty. This follows from induction on the size of Γ_2. More specifically, the case in which Γ_2 has size one is direct. Then the inductive step uses the naturality of α, the associativity diagram corresponding to F being monoidal, and the naturality of μ. We can then prove straightforwardly that Diagram (2.2) commutes by induction on the size of n with $n = 2$ as the base case.

Let us now focus on Diagram (2.3). Since sh_E amounts to a sequential composition of exchange morphisms we only need to prove that the following diagram commutes.

$$\begin{tikzcd}
\llbracket \Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \rrbracket_{F_*M} \arrow[r, "\mathsf{exch}_{\Gamma; x:\mathbb{A}, y:\mathbb{B}; \Delta}"] \arrow[d, "{h[\Gamma, x:\mathbb{A}, y:\mathbb{B}, \Delta]}"'] & \llbracket \Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \rrbracket_{F_*M} \arrow[d, "{h[\Gamma, y:\mathbb{B}, x:\mathbb{A}, \Delta]}"] \\
\llbracket \Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \rrbracket^F_M \arrow[r, "\mathsf{exch}^F_{\Gamma; x:\mathbb{A}, y:\mathbb{B}; \Delta}"'] & \llbracket \Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \rrbracket^F_M
\end{tikzcd}$$

Its commutativity follows from the fact that the diagram belows commutes (due to the naturality of sw), and also from the commutativity of Diagram (2.1).

$$\begin{tikzcd}
\llbracket x : \mathbb{A}, y : \mathbb{B} \rrbracket_{F_*M} \arrow[r, "\mathsf{sw}"] \arrow[d, "{h[x:\mathbb{A}, y:\mathbb{B}]}"'] & \llbracket y : \mathbb{B}, x : \mathbb{A} \rrbracket_{F_*M} \arrow[d, "{h[y:\mathbb{B}, x:\mathbb{A}]}"] \\
\llbracket x : \mathbb{A}, y : \mathbb{B} \rrbracket^F_M \arrow[r, "\mathsf{sw}^F"'] & \llbracket y : \mathbb{B}, x : \mathbb{A} \rrbracket^F_M
\end{tikzcd}$$

Finally we focus on Diagram (2.4). Its commutativity follows from the fact that the diagram below commutes (due to F being symmetric monoidal) and the commutativity of Diagram (2.2).

$$\begin{tikzcd}
\llbracket x : \mathbb{A}, y : \mathbb{B} \rrbracket^F_M \arrow[r, "\mathsf{sw}^F"] \arrow[d, "{\mu[x:\mathbb{A}, y:\mathbb{B}]}"'] & \llbracket y : \mathbb{B}, x : \mathbb{A} \rrbracket^F_M \arrow[d, "{\mu[y:\mathbb{B}, x:\mathbb{A}]}"] \\
F\llbracket x : \mathbb{A}, y : \mathbb{B} \rrbracket_M \arrow[r, "F\,\mathsf{sw}"'] & F\llbracket y : \mathbb{B}, x : \mathbb{A} \rrbracket_M
\end{tikzcd}$$

$\square$

Corollary 1. *The following equations hold.*

$$\mu[[\![\Gamma_1]\!]_M, \dots, [\![\Gamma_n]\!]_M] \cdot (\mu[\Gamma_1] \otimes \cdots \otimes \mu[\Gamma_n]) \cdot (h[\Gamma_1] \otimes \cdots \otimes h[\Gamma_n]) \cdot \mathsf{sp}_{\Gamma_1;\dots;\Gamma_n} \cdot \mathsf{sh}_E$$
$$= F\,\mathsf{sp}_{\Gamma_1;\dots;\Gamma_n} \cdot F\,\mathsf{sh}_E \cdot \mu[E] \cdot h[E]$$

$$F\,\mathsf{jn}_{\Gamma_1;\dots;\Gamma_n} \cdot \mu[[\![\Gamma_1]\!]_M, \dots, [\![\Gamma_n]\!]_M] \cdot (\mu[\Gamma_1] \otimes \cdots \otimes \mu[\Gamma_n]) \cdot (h[\Gamma_1] \otimes \cdots \otimes h[\Gamma_n])$$
$$= F\,\mathsf{sh}_E \cdot \mu[E] \cdot h[E] \cdot \mathsf{sh}_E^{-1} \cdot \mathsf{jn}_{\Gamma_1;\dots;\Gamma_n}$$

Proof. The first equation amounts to putting together the commutative diagrams obtained in the previous lemma. The second equation follows from the first equation and from $\mathsf{jn}_{\Gamma_1;\dots;\Gamma_n}$ being the inverse of $\mathsf{sp}_{\Gamma_1;\dots;\Gamma_n}$. □

Proposition 1. *If M is a model of the λ-theory $\mathscr{T}$, then F_*M is also model of $\mathscr{T}$.*

Proof. Let us assume for a moment that for all judgements $\Gamma \rhd v : \mathbb{A}$ the following equation holds.

$$[\![\Gamma \rhd v : \mathbb{A}]\!]_{F_*M} = h_{\mathbb{A}}^{-1} \cdot F[\![\Gamma \rhd v : \mathbb{A}]\!]_M \cdot \mu[\Gamma] \cdot h[\Gamma]$$

Diagrammatically this corresponds to the commutativity of the diagram,

$$\begin{tikzcd}
[\![\Gamma]\!]_{F_*M} \arrow[r, "[\![v]\!]_{F_*M}"] \arrow[d, "\mu[\Gamma] \,\cdot\, h[\Gamma]"'] & [\![\mathbb{A}]\!]_{F_*M} \arrow[d, bend left, "h"] \\
F[\![\Gamma]\!]_M \arrow[r, "F[\![v]\!]_M"'] & F[\![\mathbb{A}]\!]_M \arrow[u, bend left, "h^{-1}"]
\end{tikzcd}$$

From this assumption we reason as follows:

$v = w$
$\Rightarrow [\![v]\!]_M = [\![w]\!]_M$
$\Rightarrow F[\![v]\!]_M = F[\![w]\!]_M$
$\Rightarrow h^{-1} \cdot F[\![v]\!]_M = h^{-1} \cdot F[\![w]\!]_M$
$\Rightarrow h^{-1} \cdot F[\![v]\!]_M \cdot \mu[\Gamma] \cdot h[\Gamma] = h^{-1} \cdot F[\![w]\!]_M \cdot \mu[\Gamma] \cdot h[\Gamma]$
$\Rightarrow [\![v]\!]_{F_*M} = [\![w]\!]_{F_*M}$ {Assumption}

which indeed entails our claim. The rest of the proof amounts to showing that our assumption holds. This follows from induction over the semantic rules of linear λ-calculus. We present a selection of some cases, the other ones are obtained in an analogous manner.

$[\![v \otimes w]\!]_{F_*M}$
$= ([\![v]\!]_{F_*M} \otimes [\![w]\!]_{F_*M}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E$
$= ((h^{-1} \cdot F[\![v]\!] \cdot \mu[\Gamma] \cdot h[\Gamma]) \otimes (h^{-1} \cdot F[\![w]\!] \cdot \mu[\Delta] \cdot h[\Delta])) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E$
$= h^{-1} \cdot \mu \cdot ((F[\![v]\!] \cdot \mu[\Gamma] \cdot h[\Gamma]) \otimes (F[\![w]\!] \cdot \mu[\Delta] \cdot h[\Delta])) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E$ {Def. h on $\mathbb{A} \otimes \mathbb{B}$}
$= h^{-1} \cdot \mu \cdot (F[\![v]\!] \otimes F[\![w]\!]) \cdot ((\mu[\Gamma] \cdot h[\Gamma]) \otimes (\mu[\Delta] \cdot h[\Delta])) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E$
$= h^{-1} \cdot F([\![v]\!] \otimes [\![w]\!]) \cdot \mu \cdot ((\mu[\Gamma] \cdot h[\Gamma]) \otimes (\mu[\Delta] \cdot h[\Delta])) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E$ {Naturality of μ}
$= h^{-1} \cdot F([\![v]\!] \otimes [\![w]\!]) \cdot \mu \cdot (\mu[\Gamma] \otimes \mu[\Delta]) \cdot (h[\Gamma] \otimes h[\Delta]) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E$
$= h^{-1} \cdot F([\![v]\!] \otimes [\![w]\!]) \cdot F\,\mathsf{sp}_{\Gamma;\Delta} \cdot F\,\mathsf{sh}_E \cdot \mu[E] \cdot h[E]$ {Corollary 1}
$= h^{-1} \cdot F[\![v \otimes w]\!] \cdot \mu[E] \cdot h[E]$

$$\begin{aligned}
&[\![\lambda x : \mathbb{A}.\ v]\!]_{F_*M} \\
&= \overline{[\![v]\!]_{F_*M} \cdot \mathsf{jn}_{\Gamma;x:\mathbb{A}}} \\
&= \overline{h^{-1} \cdot F[\![v]\!] \cdot \mu[\Gamma, x : \mathbb{A}] \cdot h[\Gamma, x : \mathbb{A}] \cdot \mathsf{jn}_{\Gamma;x:\mathbb{A}}} \\
&= \overline{h^{-1} \cdot F[\![v]\!] \cdot F\,\mathsf{jn}_{\Gamma;x:\mathbb{A}} \cdot \mu \cdot ((\mu[\Gamma] \cdot h[\Gamma]) \otimes h))} && \{\text{Corollary 1}\} \\
&= \overline{h^{-1} \cdot F[\![v]\!] \cdot F\,\mathsf{jn}_{\Gamma;x:\mathbb{A}} \cdot \mu \cdot (\mathsf{id} \otimes h)} \cdot \mu[\Gamma] \cdot h[\Gamma] && \left\{\overline{(f \cdot (g \otimes \mathsf{id})} = \overline{f} \cdot g\right\} \\
&= (h^{-1} \cdot -) \cdot \overline{F[\![v]\!] \cdot F\,\mathsf{jn}_{\Gamma;x:\mathbb{A}} \cdot \mu \cdot (\mathsf{id} \otimes h)} \cdot \mu[\Gamma] \cdot h[\Gamma] && \{\overline{g \cdot f} = (g \cdot -) \cdot \overline{f}\} \\
&= (h^{-1} \cdot -) \cdot (- \cdot h) \cdot \overline{F[\![v]\!] \cdot F\,\mathsf{jn}_{\Gamma;x:\mathbb{A}} \cdot \mu} \cdot \mu[\Gamma] \cdot h[\Gamma] && \left\{\overline{g \cdot (\mathsf{id} \otimes f)} = (- \cdot f) \cdot \overline{g}\right\} \\
&= (h^{-1} \cdot -) \cdot (- \cdot h) \cdot \overline{F([\![v]\!] \cdot \mathsf{jn}_{\Gamma;x:\mathbb{A}}) \cdot \mu} \cdot \mu[\Gamma] \cdot h[\Gamma] \\
&= (h^{-1} \cdot -) \cdot (- \cdot h) \cdot m \cdot F(\overline{[\![v]\!] \cdot \mathsf{jn}_{\Gamma;x:\mathbb{A}}}) \cdot \mu[\Gamma] \cdot h[\Gamma] && \{\overline{Ff \cdot \mu} = m \cdot F\overline{f}\} \\
&= h^{-1} \cdot F(\overline{[\![v]\!] \cdot \mathsf{jn}_{\Gamma;x:\mathbb{A}}}) \cdot \mu[\Gamma] \cdot h[\Gamma] && \{\text{Def. } h \text{ on } \mathbb{A} \multimap \mathbb{B}\} \\
&= h^{-1} \cdot F[\![\lambda x : \mathbb{A}.\ v]\!] \cdot \mu[\Gamma] \cdot h[\Gamma]
\end{aligned}$$

□

Finally, we use the previous result to formally establish the bijective correspondence (up-to isomorphism) that was discussed in the beginning of the current subsection.

Theorem 2.9. *Let $\mathscr{T}$ be a λ-theory. Every autonomous functor $F : \mathrm{Syn}(\mathscr{T}) \to \mathsf{C}$ induces a model $F_*\mathrm{Syn}(\mathscr{T})$ of $\mathscr{T}$ and every model M of $\mathscr{T}$ induces a strict autonomous functor $([v] \mapsto [\![v]\!]_M) : \mathrm{Syn}(\mathscr{T}) \to \mathsf{C}$. Furthermore these constructions are inverse to each other up-to isomorphism, in the sense that,*

$$([v] \mapsto [\![v]\!]_{F_*\mathrm{Syn}(\mathscr{T})}) \cong F \qquad \textit{and} \qquad ([v] \mapsto [\![v]\!]_M)_*\mathrm{Syn}(\mathscr{T}) = M$$

Proof. Let us first focus on the mapping that sends functors to models. Consider an autonomous functor $F : \mathrm{Syn}(\mathscr{T}) \to \mathsf{C}$. Then observe that $\mathrm{Syn}(\mathscr{T})$ corresponds to a model of $\mathscr{T}$ and thus by Proposition 1 we conclude that $F_*\mathrm{Syn}(\mathscr{T})$ must be a model of $\mathscr{T}$. For the inverse direction, we start with a model M and build the functor $\mathrm{Syn}(\mathscr{T}) \to \mathsf{C}$ that sends the equivalence class $[v]$ into $[\![x : \mathbb{A} \rhd v : \mathbb{B}]\!]_M$. It is straightforward to prove that this last functor is strict autonomous.

Our next step is to prove the existence of a natural isomorphism $([v] \mapsto [\![v]\!]_{F_*\mathrm{Syn}(\mathscr{T})}) \cong F$. For that effect observe that for all types $\mathbb{A}$ we already have,

$$[\![\mathbb{A}]\!]_{F_*\mathrm{Syn}(\mathscr{T})} \overset{h}{\cong} F[\![\mathbb{A}]\!]_{\mathrm{Syn}(\mathscr{T})} = F\mathbb{A}$$

Observe as well that the corresponding naturality square,

$$\begin{tikzcd}
{[\![\mathbb{A}]\!]_{F_*\mathrm{Syn}(\mathscr{T})}} \arrow[r, shift left, "h"] \arrow[d, "{[\![v]\!]_{F_*\mathrm{Syn}(\mathscr{T})}}"'] & {F[\![\mathbb{A}]\!]_{\mathrm{Syn}(\mathscr{T})}} \arrow[l, shift left, "h^{-1}"] \arrow[d, "{F[\![v]\!]_{\mathrm{Syn}(\mathscr{T})} = F[v]}"] \\
{[\![\mathbb{B}]\!]_{F_*\mathrm{Syn}(\mathscr{T})}} \arrow[r, shift left, "h"] & {F[\![\mathbb{B}]\!]_{\mathrm{Syn}(\mathscr{T})}} \arrow[l, shift left, "h^{-1}"]
\end{tikzcd}$$

is an instance of equation $[\![v]\!]_{F_* M} = h^{-1} \cdot F[\![v]\!]_M \cdot \mu[\Gamma] \cdot h[\Gamma]$ (specifically, with $|\Gamma| = 1$ and $M = \mathrm{Syn}(\mathscr{T})$) which was proved in Proposition 1. Finally, we show that the equation $([v] \mapsto [\![v]\!]_M)_* \mathrm{Syn}(\mathscr{T}) = M$ holds. First,

$$[\![G]\!]_{([v]\mapsto[\![v]\!]_M)_*\mathrm{Syn}(\mathscr{T})} = ([v] \mapsto [\![v]\!]_M)([\![G]\!]_{\mathrm{Syn}(\mathscr{T})}) = ([v] \mapsto [\![v]\!]_M)(G) = [\![G]\!]_M$$

and second,

$$\begin{aligned}
&[\![f]\!]_{([v]\mapsto[\![v]\!]_M)_*\mathrm{Syn}(\mathscr{T})} \\
&= ([v] \mapsto [\![v]\!]_M)([\![f]\!]_{\mathrm{Syn}(\mathscr{T})}) \\
&= ([v] \mapsto [\![v]\!]_M)([\mathtt{pm}\ x\ \mathtt{to}\ x_1 \otimes \cdots \otimes x_n.\ f(x_1, \dots, x_n)]) \\
&= [\![\mathtt{pm}\ x\ \mathtt{to}\ x_1 \otimes \cdots \otimes x_n.\ f(x_1, \dots, x_n)]\!]_M \\
&= [\![f]\!]_M
\end{aligned}$$

□

We are now in the right setting to present the equivalence between linear λ-theories and autonomous categories that was discussed in the paper's introduction and beginning of § 2. So let Aut be the *quasi-category* of autonomous categories and autonomous functors (as usual the term 'quasi-category' can be replaced by 'category' if one restricts to small autonomous categories). Consider as well the quasi-category $\mathsf{Aut}_{/\cong}$ whose objects are autonomous categories and morphisms are *isomorphism classes* of autonomous functors (such a quasi-category is well-defined because isomorphisms form an equivalence relation that is closed w.r.t. pre- and post-composition [ML98, II.8]). We will show that the latter quasi-category is equivalent to a certain quasi-category $\lambda\text{-}\mathsf{Th}$ whose objects are linear λ-theories:

$$\lambda\text{-}\mathsf{Th} \underset{\longleftarrow}{\xrightarrow{\ \simeq\ }} \mathsf{Aut}_{/\cong} \tag{2.5}$$

We need to define the notion of a morphism in the quasi-category $\lambda\text{-}\mathsf{Th}$. Following traditions in type theory [Cro93] we set $\lambda\text{-}\mathsf{Th}(\mathscr{T}_1, \mathscr{T}_2) := \mathsf{Aut}_{/\cong}(\mathrm{Syn}(\mathscr{T}_1), \mathrm{Syn}(\mathscr{T}_2))$. In words, a morphism $\mathscr{T}_1 \to \mathscr{T}_2$ between λ-theories $\mathscr{T}_1$ and $\mathscr{T}_2$ is exactly an isomorphism class of autonomous functors $\mathrm{Syn}(\mathscr{T}_1) \to \mathrm{Syn}(\mathscr{T}_2)$ – which by Thm. 2.9 are in bijective correspondence (up-to isomorphism) to models of $\mathscr{T}_1$ on the category $\mathrm{Syn}(\mathscr{T}_2)$.

Note that an isomorphism $\mathscr{T}_1 \cong \mathscr{T}_2$ in $\lambda\text{-}\mathsf{Th}$ is equivalent to the corresponding syntactic categories being equivalent, which provides a bijection,

$$\mathsf{Aut}_{/\cong}(\mathrm{Syn}(\mathscr{T}_1), \mathsf{C}) \cong \mathsf{Aut}_{/\cong}(\mathrm{Syn}(\mathscr{T}_2), \mathsf{C})$$

natural on C. This echoes the notion of *Morita equivalence* in universal algebra [ASS06, Joh02], which states that two theories are equivalent provided that the corresponding categories of varieties are equivalent (in our case, the variety of a λ-theory $\mathscr{T}_1$ on C is simply regarded as the set $\mathsf{Aut}_{/\cong}(\mathrm{Syn}(\mathscr{T}_1), \mathsf{C})$).

Next we set the stage for describing the functor $\mathsf{Aut}_{/\cong} \to \lambda\text{-}\mathsf{Th}$ that is part of the equivalence (2.5). As we will see, it corresponds to an *internal language construct i.e.* it maps an autonomous category C to a linear λ-theory $\mathrm{Lng}(\mathsf{C})$ that encodes the former completely. Among other things, this is useful for seeing morphisms of a category C equivalently as λ-terms, or more generally for seeing C from a set-theoretic lens without loss of information (a deeper discussion about the notion of an internal language can be consulted for example in [Pit01]).

Definition 2.10 (Internal language)**.** An autonomous category C induces a linear λ-theory $\mathrm{Lng}(\mathsf{C})$ whose ground types $X \in G$ are the objects of C and whose signature Σ of operation symbols consists of all the morphisms in C plus certain isomorphisms that we describe in (2.6). The axioms of $\mathrm{Lng}(\mathsf{C})$ are all the equations satisfied by the obvious interpretation on C. In order to explicitly distinguish the autonomous structure of C from the type structure of $\mathrm{Lng}(\mathsf{C})$ let us denote the tensor of C by $\otimes_{\mathsf{C}}$, the unit by I_{C}, and the exponential by $\multimap_{\mathsf{C}}$. Consider then the following map on types:

$$i(\mathbb{I}) = I_{\mathsf{C}} \qquad i(X) = X \qquad i(\mathbb{A} \otimes \mathbb{B}) = i(\mathbb{A}) \ \otimes_{\mathsf{C}} i(\mathbb{B}) \qquad i(\mathbb{A} \multimap \mathbb{B}) = i(\mathbb{A}) \ \multimap_{\mathsf{C}} i(\mathbb{B}) \tag{2.6}$$

For each type $\mathbb{A}$ we add an isomorphism $i_{\mathbb{A}} : \mathbb{A} \cong i(\mathbb{A})$ to the theory $\mathrm{Lng}(\mathsf{C})$.

The following theorem is the core of the proof that establishes the equivalence (2.5) and formalises the fact that the λ-theory $\mathrm{Lng}(\mathsf{C})$ associated to a category C describes the latter completely *i.e.* it is the internal language of C.

Theorem 2.11. *For every autonomous category* C *there exists an equivalence* $\mathrm{Syn}(\mathrm{Lng}(\mathsf{C})) \simeq \mathsf{C}$ *and both functors witnessing this equivalence are autonomous. The functor going from the left to right direction is additionally strict.*

Proof. By construction, we have an interpretation of $\mathrm{Lng}(\mathsf{C})$ in C which behaves as the identity for operation symbols and ground types. This interpretation is by definition a model of $\mathrm{Lng}(\mathsf{C})$ on C and by Thm. 2.9 we obtain a strict autonomous functor $\mathrm{Syn}(\mathrm{Lng}(\mathsf{C})) \to \mathsf{C}$. The functor in the opposite direction behaves as the identity on objects and sends a C-morphism f into $[f(x)]$. The equivalence of categories is then shown by using the aforementioned isomorphisms which connect the type constructors of $\mathrm{Lng}(\mathsf{C})$ with the autonomous structure of C.

To finish the proof we need to show that the functor $\mathsf{C} \to \mathrm{Syn}(\mathrm{Lng}(\mathsf{C}))$ is autonomous (Def. 2.7). Our first step is to establish an isomorphism $\mathbb{I} \to I_{\mathsf{C}}$ living in the category $\mathrm{Syn}(\mathrm{Lng}(\mathsf{C}))$ which we define as $a : \mathbb{I} \rhd [i_{\mathbb{I}}(a)] : I_{\mathsf{C}}$. Analogously, for all C-objects X and Y the required isomorphism $X \otimes Y \to X \otimes_{\mathsf{C}} Y$ living in the same category is provided by $a : X \otimes Y \rhd [i_{X \otimes Y}(a)] : X \otimes_{\mathsf{C}} Y$. Next we prove that the thus obtained transformation is natural on X and Y. By analysing the corresponding diagram we see that it corresponds to proving the equality $[i_{X' \otimes Y'}(b)] \cdot [\mathtt{pm}\ a\ \mathtt{to}\ x \otimes y.\ f(x) \otimes g(y)] = [f \otimes_{\mathsf{C}} g\,(x)] \cdot [i_{X \otimes Y}(a)]$. Then note that, by the definition of syntactic category and by substitution, the previous equality is entailed by $i_{X' \otimes Y'}(\mathtt{pm}\ a\ \mathtt{to}\ x \otimes y.\ f(x) \otimes g(y)) = f \otimes_{\mathsf{C}} g\ (i_{X \otimes Y}(a))$. According to the definition of $\mathrm{Lng}(\mathsf{C})$ (Definition 2.10), this last equality is entailed by,

$$[\![i_{X' \otimes Y'}(\mathtt{pm}\ a\ \mathtt{to}\ x \otimes y.\ f(x) \otimes g(y))]\!] = [\![f \otimes_{\mathsf{C}} g\,(i_{X \otimes Y}(a))]\!]$$

on C and the proof that the latter holds is straightforward.

Next let us show that the left unitality diagram commutes. An inspection of this diagram reveals that it corresponds to the equality $[\mathtt{pm}\ a\ \mathtt{to}\ i \otimes x.\ i\ \mathtt{to}\ \ast\,.\ x] = [\lambda_{\mathsf{C}}(c)] \cdot [i_{I_{\mathsf{C}} \otimes X}(b)] \cdot [\mathtt{pm}\ a\ \mathtt{to}\ i \otimes x.\ i_{\mathbb{I}}(i) \otimes x]$. Analogously to the previous reasoning we will prove that,

$$[\![\mathtt{pm}\ a\ \mathtt{to}\ i \otimes x.\ i\ \mathtt{to}\ \ast\,.\ x]\!] = [\![\lambda_{\mathsf{C}}(i_{I_{\mathsf{C}} \otimes X}(\mathtt{pm}\ a\ \mathtt{to}\ i \otimes x.\ i_{\mathbb{I}}(i) \otimes x))]\!]$$

We start by simplifying the right-hand side of equation,

$$\begin{aligned}\llbracket \lambda_{\mathsf{C}}(i_{I_{\mathsf{C}}\otimes X}(\mathtt{pm}\ a\ \mathtt{to}\ i\otimes x.\ i_{\mathbb{I}}(i)\otimes x))\rrbracket &= \llbracket \lambda_{\mathsf{C}}\rrbracket \cdot \llbracket i_{I_{\mathsf{C}}\otimes X}\rrbracket \cdot \llbracket \mathtt{pm}\ a\ \mathtt{to}\ i\otimes x.\ i_{\mathbb{I}}(i)\otimes x\rrbracket \\ &= \llbracket \lambda_{\mathsf{C}}\rrbracket \cdot \mathsf{id}\cdot \mathsf{id} \\ &= \lambda_{\mathsf{C}}\end{aligned}$$

The simplified equation amounts to stating that the left unitor of Syn(Lng(C)) (the term on the left-hand side of the equation) is *strictly* preserved by the interpretation of Lng(C) on C. This is clearly the case because the functor Syn(Lng(C)) $\to \mathsf{C}$ corresponding to this interpretation is *strict* autonomous (Thm. 2.9). This reasoning is also applicable to the diagrams concerning right unitality, symmetry, and associativity.

The final step is to prove that the right transpose of the composite,

$$(X \multimap_{\mathsf{C}} Y)\otimes X \xrightarrow{i_{(X\multimap_{\mathsf{C}}Y)\otimes X}} (X\multimap_{\mathsf{C}} Y)\otimes_{\mathsf{C}} X \xrightarrow{\mathrm{app}_{\mathsf{C}}} Y$$

is an isomorphism in the category Syn(Lng(C)). To that effect note that the aforementioned right transpose is given by $f : X \multimap_{\mathsf{C}} Y \rhd [\lambda x : X.\ \mathrm{app}_{\mathsf{C}}\ (i_{X\multimap_{\mathsf{C}} Y\otimes_{\mathsf{C}} X}\ (f\otimes x))] : X\multimap Y$. It is straightforward to prove that this term is interpreted as the identity on C. Next note the existence of the term $x : X\multimap Y \rhd [i_{X\multimap Y}(x)] : X\multimap_{\mathsf{C}} Y$ which is given in Definition 2.10 and is also interpreted as the identity on the category C. Since both terms are interpreted as the identity, by Definition 2.10 they are indeed inverses of each other. □

Theorem 2.12. *There exists an equivalence,*

$$\lambda\text{-}\mathsf{Th} \underset{\mathrm{Lng}}{\overset{\mathrm{Syn}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \mathsf{Aut}_{/\cong}$$

Proof. The functor Syn : λ-$\mathsf{Th} \to \mathsf{Aut}_{/\cong}$ sends a λ-theory to its syntactic category and acts as the identity on morphisms. For the inverse direction, recall that for every autonomous category C Thm. 2.11 provides autonomous functors $e_{\mathsf{C}} : \mathsf{C} \to$ Syn(Lng(C)) and e'_{C} : Syn(Lng(C)) $\to \mathsf{C}$. Then we define Lng : $\mathsf{Aut}_{/\cong} \to \lambda$-$\mathsf{Th}$ as the functor that sends an autonomous category C to its theory Lng(C) and that sends an isomorphism class $[F]$ of autonomous functors of type $\mathsf{C}\to\mathsf{D}$ into $[e_{\mathsf{D}}\cdot F\cdot e'_{\mathsf{C}}]$. This last mapping is well-defined because if $F\cong F'$ then $e_{\mathsf{D}}\cdot F\cdot e'_{\mathsf{C}} \cong e_{\mathsf{D}}\cdot F'\cdot e'_{\mathsf{C}}$. Furthermore it respects the functorial laws because $e_{\mathsf{C}}\cdot \mathrm{Id}\cdot e'_{\mathsf{C}}\cong \mathrm{Id}$ and thus $[e_{\mathsf{C}}\cdot\mathrm{Id}\cdot e'_{\mathsf{C}}] = [\mathrm{Id}]$, also for all autonomous functors $F : \mathsf{C}\to\mathsf{A}$ and $G : \mathsf{A}\to\mathsf{D}$ we have the isomorphism $e_{\mathsf{D}}\cdot G\cdot F\cdot e'_{\mathsf{C}} \cong e_{\mathsf{D}}\cdot G\cdot e'_{\mathsf{A}}\cdot e_{\mathsf{A}}\cdot F\cdot e'_{\mathsf{C}}$ and therefore $[e_{\mathsf{D}}\cdot G\cdot F\cdot e'_{\mathsf{C}}] = [e_{\mathsf{D}}\cdot G\cdot e'_{\mathsf{A}}\cdot e_{\mathsf{A}}\cdot F\cdot e'_{\mathsf{C}}]$.

The next step is to prove the existence of isomorphisms Syn(Lng(C)) $\cong \mathsf{C}$ in $\mathsf{Aut}_{/\cong}$ and Lng(Syn($\mathscr{T}$)) $\cong \mathscr{T}$ in λ-Th. The former arises directly from Thm. 2.11 and the fact that we are now working with isomorphism classes of autonomous functors. For the latter we appeal to Thm. 2.11 to establish an equivalence Syn(Lng(Syn($\mathscr{T}$))) $\simeq$ Syn($\mathscr{T}$); then we resort to the previous remark on isomorphisms of λ-theories and Morita equivalence to obtain Lng(Syn($\mathscr{T}$)) $\cong \mathscr{T}$. The fact that the thus established isomorphisms are natural on C and $\mathscr{T}$ follows from routine calculations. □

3. From equations to $\mathcal{V}$-equations

In this section we extend the results of §2 to the setting of $\mathcal{V}$-equations.

3.1. **Preliminaries.** Let $\mathcal{V}$ denote a commutative and unital quantale, $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ the corresponding binary operation, and k the corresponding unit [PR00]. We start by recalling two definitions concerning ordered structures [GHK+03, Gou13] and then explain their relevance to our work.

Definition 3.1. Consider a complete lattice L. For every $x, y \in L$ we say that y is *way-below* x (in symbols, $y \ll x$) if for every subset $X \subseteq L$ whenever $x \leq \bigvee X$ there exists a *finite* subset $A \subseteq X$ such that $y \leq \bigvee A$. The lattice L is called *continuous* iff for every $x \in L$,

$$x = \bigvee \{y \mid y \in L \text{ and } y \ll x\}$$

Definition 3.2. Let L be a complete lattice. A *basis* B of L is a subset $B \subseteq L$ such that for every $x \in L$ the set $B \cap \{y \mid y \in L \text{ and } y \ll x\}$ is directed and has x as the least upper bound.

From now on we assume that the underlying lattice of $\mathcal{V}$ is continuous and has a basis B which is closed under finite joins, the multiplication of the quantale $\otimes$ and contains the unit k. These assumptions will allow us to work *only* with a specified subset of $\mathcal{V}$-equations chosen *e.g.* for computational reasons, such as the *finite* representation of values $q \in \mathcal{V}$.

Example 3.3. The Boolean quantale $((\{0 \leq 1\}, \vee), \otimes := \wedge)$ is *finite* and thus continuous [GHK+03]. Since it is continuous, $\{0, 1\}$ itself is a basis for the quantale that satisfies the conditions above. For the Gödel t-norm [DEW13] $(([0,1], \vee), \otimes := \wedge)$, the way-below relation is the strictly-less relation $<$ with the exception that $0 < 0$. A basis for the underlying lattice that satisfies the conditions above is the set $\mathbb{Q} \cap [0,1]$. Note that, unlike real numbers, rationals numbers always have a finite representation. For the metric quantale (also known as Lawvere quantale) $(([0,\infty], \wedge), \otimes := +)$, the way-below relation corresponds to the *strictly greater* relation with $\infty > \infty$, and a basis for the underlying lattice that satisfies the conditions above is the set of extended non-negative rational numbers. The latter also serves as basis for the ultrametric quantale $(([0,\infty], \wedge), \otimes := \max)$.

From now on we assume that $\mathcal{V}$ is *integral*, *i.e.* that the unit k is the top element of $\mathcal{V}$. This will allow us to establish a smoother theory of $\mathcal{V}$-equations, whilst still covering *e.g.* all the examples above. This assumption is common in quantale theory [VKB19].

3.2. **A $\mathcal{V}$-equational deductive system.** As mentioned in the introduction, $\mathcal{V}$ induces the notion of a $\mathcal{V}$-equation, *i.e.* an equation $t =_q s$ labelled by an element q of $\mathcal{V}$. This subsection explores this concept by introducing a $\mathcal{V}$-equational deductive system for linear λ-calculus and a notion of a linear $\mathcal{V}\lambda$-theory. Recall the term formation rules of linear λ-calculus from Fig. 1. A $\mathcal{V}$-equation-in-context is an expression $\Gamma \rhd v =_q w : \mathbb{A}$ with $q \in B$ (the basis of $\mathcal{V}$), $\Gamma \rhd v : \mathbb{A}$ and $\Gamma \rhd w : \mathbb{A}$. Let $\top$ be the top element in $\mathcal{V}$. An equation-in-context $\Gamma \rhd v = w : \mathbb{A}$ now denotes the particular case in which both $\Gamma \rhd v =_\top w : \mathbb{A}$ and $\Gamma \rhd w =_\top v : \mathbb{A}$. For the case of the Boolean quantale, $\mathcal{V}$-equations are labelled by $\{0,1\}$. We will see that $\Gamma \rhd v =_1 w : \mathbb{A}$ can be effectively treated as an inequation $\Gamma \rhd v \leq w : \mathbb{A}$, whilst $\Gamma \rhd v =_0 w : \mathbb{A}$ corresponds to a trivial $\mathcal{V}$-equation, *i.e.* a $\mathcal{V}$-equation that always holds. For the Gödel t-norm, we can choose $\mathbb{Q} \cap [0,1]$ as basis and then obtain what we call *fuzzy inequations*. For the metric quantale, we can choose the set of extended non-negative rational numbers as basis and then obtain *metric equations* in the spirit of [MPP16, MPP17]. Similarly, by choosing the ultrametric quantale

$(([0,\infty],\wedge),\otimes := \max)$ with the set of extended non-negative rational numbers as basis we obtain what we call *ultrametric equations*.

Definition 3.4 ($\mathcal{V}\lambda$-theories)**.** Consider a tuple (G,Σ) consisting of a class G of ground types and a class of sorted operation symbols $f : \mathbb{A}_1,\ldots,\mathbb{A}_n \to \mathbb{A}$ with $n \geq 1$. A linear $\mathcal{V}\lambda$-theory $((G,\Sigma), Ax)$ is a tuple such that Ax is a class of $\mathcal{V}$-equations-in-context over linear λ-terms built from (G,Σ).

$$\frac{}{v =_\top v}\ \textbf{(refl)} \qquad \frac{v =_q w \qquad w =_r u}{v =_{q\otimes r} u}\ \textbf{(trans)} \qquad \frac{v =_q w \qquad r \leq q}{v =_r w}\ \textbf{(weak)}$$

$$\frac{\forall r \ll q.\ v =_r w}{v =_q w}\ \textbf{(arch)} \qquad \frac{\forall i \leq n.\ v =_{q_i} w}{v =_{\vee q_i} w}\ \textbf{(join)} \qquad \frac{v =_q w \quad v' =_r w'}{v \otimes v' =_{q\otimes r} w \otimes w'}$$

$$\frac{\forall i \leq n.\ v_i =_{q_i} w_i}{f(v_1,\ldots,v_n) =_{\otimes q_i} f(w_1,\ldots,w_n)} \qquad \frac{v =_q w \qquad v' =_r w'}{v\ \mathtt{to}\ *.\ v' =_{q\otimes r} w\ \mathtt{to}\ *.\ w'} \qquad \frac{v =_q w}{\lambda x : \mathbb{A}.\ v =_q \lambda x : \mathbb{A}.\ w}$$

$$\frac{v =_q w \qquad v' =_r w'}{\mathtt{pm}\ v\ \mathtt{to}\ x \otimes y.\ v' =_{q\otimes r} \mathtt{pm}\ w\ \mathtt{to}\ x \otimes y.\ w'} \qquad \frac{v =_q w \quad v' =_r w'}{v\, v' =_{q\otimes r} w\, w'}$$

$$\frac{\Gamma \rhd v =_q w : \mathbb{A} \qquad \Delta \in \mathrm{perm}(\Gamma)}{\Delta \rhd v =_q w : \mathbb{A}} \qquad \frac{v =_q w \qquad v' =_r w'}{v[v'/x] =_{q\otimes r} w[w'/x]}$$

FIGURE 4. $\mathcal{V}$-congruence rules.

The elements of Ax are called axioms of the theory. Let $Th(Ax)$ be the smallest class that contains Ax and that is closed under the rules of Fig. 3 and of Fig. 4 (as usual we omit the context and typing information). The elements of $Th(Ax)$ are called theorems of the theory.

Let us examine the rules in Fig. 4 in more detail. They can be seen as a generalisation of the notion of a congruence. The rules **(refl)** and **(trans)** are a generalisation of equality's reflexivity and transitivity. Rule **(weak)** encodes the principle that the higher the label in the $\mathcal{V}$-equation, the 'tighter' is the relation between the two terms in the $\mathcal{V}$-equation. In other words, $v =_r w$ is subsumed by $v =_q w$, for $r \leq q$. This can be seen clearly *e.g.* with the metric quantale by reading $v =_q w$ as "the terms v and w are *at most* at distance q from each other" (recall that in the metric quantale the usual order is reversed, *i.e.* $\leq := \geq_{[0,\infty]}$). Rule **(arch)** is essentially a generalisation of the Archimedean rule in [MPP16, MPP17]. It says that if $v =_r w$ for all *approximations* r of q then it is also the case that $v =_q w$. Rule **(join)** says that deductions are closed under finite joins, and in particular it is always the case that $v =_\perp w$. All other rules correspond to a generalisation of *compatibility* to a $\mathcal{V}$-equational setting.

The reader may have noticed that the rules in Fig. 4 do not contain a $\mathcal{V}$-generalisation of symmetry w.r.t. standard equality. Such a generalisation would be:

$$\frac{v =_q w}{w =_q v}$$

This rule is not present in Fig. 4 because in some quantales $\mathcal{V}$ it forces too many $\mathcal{V}$-equations. For example, in the Boolean quantale the condition $v \leq w$ would automatically entail $w \leq v$ (due to symmetry); in fact, for this particular case symmetry forces the notion of inequation to collapse into the classical notion of equation. On the other hand, symmetry is desirable in the (ultra)metric case because (ultra)metrics need to respect the symmetry equation [Gou13].

Definition 3.5 (Symmetric linear $\mathcal{V}\lambda$-theories)**.** A symmetric linear $\mathcal{V}\lambda$-theory is a linear $\mathcal{V}\lambda$-theory whose set of theorems is closed under symmetry.

We end this subsection by briefly commenting on linear $\mathcal{V}\lambda$-theories where $\mathcal{V}$ is a quantale with a *linear order*. This is relevant for comparing our general $\mathcal{V}$-equational system to that of metric equations [MPP16, MPP17] which tacitly uses a quantale with a linear order, namely the metric quantale.

Theorem 3.6. *Assume that the underlying order of $\mathcal{V}$ is linear and consider a (symmetric) linear $\mathcal{V}\lambda$-theory. Substituting the rule below on the left by the one below on the right does not change the theory.*

$$\frac{\forall i \leq n.\ v =_{q_i} w}{v =_{\vee q_i} w} \qquad\qquad \frac{}{v =_{\perp} w}$$

Proof. Clearly, the rule on the left subsumes the one on the right by choosing $n = 0$. So we only need to show the inverse direction under the assumption that $\mathcal{V}$ is linear. Thus, assume that $\forall i \leq n.\ v =_{q_i} w$. We proceed by case distinction. If $n = 0$ then we need to show that $v =_{\perp} w$ which is given already by the rule on the right. Suppose now that $n > 0$. Then since the order of $\mathcal{V}$ is linear the value $\vee q_i$ must already be one of the values q_i and $v =_{q_i} w$ is already part of the theory. In other words, in case of $n > 0$ the rule on the left is redundant. □

The above result is in accordance with metric universal algebra [MPP16, MPP17] which also does not include rule **(join)**. Interestingly, however, we still have $v =_{\perp} w$ for all λ-terms v and w and the counterpart of such a rule is not present in [MPP16, MPP17]. This is explained by the fact that metric equations in [MPP16, MPP17] are labelled *only* by non-negative rational numbers whilst we also permit infinity to be a label (in our case, labels are given by a basis B which for the metric case corresponds to the *extended* non-negative rational numbers). All remaining rules of our $\mathcal{V}$-equational system instantiated to the metric case find a counterpart in the metric equational system presented in [MPP16, MPP17].

3.3. **Semantics of $\mathcal{V}$-equations.** In this subsection we set the necessary background for presenting a sound and complete class of models for (symmetric) linear $\mathcal{V}\lambda$-theories. We start by recalling basics concepts of $\mathcal{V}$-categories, which are central in a field initiated by Lawvere in [Law73] and can be intuitively seen as generalised metric spaces [Stu14, HN20, VKB19]. As we will show, $\mathcal{V}$-categories provide structure to suitably interpret $\mathcal{V}$-equations.

Definition 3.7. A (small) $\mathcal{V}$-category is a pair (X, a) where X is a class (set) and $a : X \times X \to \mathcal{V}$ is a function that satisfies:

$$k \leq a(x, x) \qquad \text{and} \qquad a(x, y) \otimes a(y, z) \leq a(x, z) \qquad\qquad (x, y, z \in X)$$

For two $\mathcal{V}$-categories (X, a) and (Y, b), a $\mathcal{V}$-functor $f : (X, a) \to (Y, b)$ is a function $f : X \to Y$ that satisfies the inequality $a(x, y) \leq b(f(x), f(y))$ for all $x, y \in X$.

Small $\mathcal{V}$-categories and $\mathcal{V}$-functors form a category which we denote by $\mathcal{V}$-Cat. A $\mathcal{V}$-category (X, a) is called *symmetric* if $a(x, y) = a(y, x)$ for all $x, y \in X$. We denote by $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym}}$ the full subcategory of $\mathcal{V}$-Cat whose objects are symmetric. Every $\mathcal{V}$-category carries a natural order defined by $x \leq y$ whenever $k \leq a(x, y)$. A $\mathcal{V}$-category is called *separated* if its natural order is anti-symmetric. We denote by $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$ the full subcategory of $\mathcal{V}$-Cat whose objects are separated.

Example 3.8. For $\mathcal{V}$ the Boolean quantale, $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$ is the category Pos of partially ordered sets and monotone maps; $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym,sep}}$ is simply the category Set of sets and functions. For $\mathcal{V}$ the metric quantale, $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym,sep}}$ is the category Met of extended metric spaces and non-expansive maps. In what follows we omit the qualifier 'extended' in 'extended (ultra)metric spaces'. For $\mathcal{V}$ the ultrametric quantale, $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym,sep}}$ is the category of ultrametric spaces and non-expansive maps.

The inclusion functor $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}} \hookrightarrow \mathcal{V}$-$\mathsf{Cat}$ has a left adjoint [HN20]. It is constructed first by defining the equivalence relation $x \sim y$ whenever $x \leq y$ and $y \leq x$ (for $\leq$ the natural order introduced earlier). Then this relation induces the separated $\mathcal{V}$-category $(X/_\sim, \tilde{a})$ where $\tilde{a}$ is defined as $\tilde{a}([x], [y]) = a(x, y)$ for every $[x], [y] \in X/_\sim$. The left adjoint of the inclusion functor $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}} \hookrightarrow \mathcal{V}$-$\mathsf{Cat}$ sends every $\mathcal{V}$-category (X, a) to $(X/_\sim, \tilde{a})$. This quotienting construct preserves symmetry, and therefore we automatically obtain the following result.

Theorem 3.9. *The inclusion functor* $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym,sep}} \hookrightarrow \mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym}}$ *has a left adjoint.*

Next, we recall notions of enriched category theory [Kel82] instantiated into the setting of *autonomous categories enriched over $\mathcal{V}$-categories*. We will use the enriched structure to give semantics to $\mathcal{V}$-equations between linear λ-terms (the latter as usual being morphisms in the category itself). First, note that every category $\mathcal{V}$-Cat is autonomous with the tensor $(X, a) \otimes (Y, b) := (X \times Y, a \otimes b)$ where $a \otimes b$ is defined as,

$$(a \otimes b)((x, y), (x', y')) = a(x, x') \otimes b(y, y')$$

and the set of $\mathcal{V}$-functors $\mathcal{V}$-$\mathsf{Cat}((X, a), (Y, b))$ is equipped with the map,

$$(f, g) \mapsto \bigwedge_{x \in X} b(f(x), g(x))$$

Theorem 3.10. *The categories* $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym}}$*,* $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$*, and* $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym,sep}}$ *inherit the autonomous structure of* $\mathcal{V}$-Cat *whenever* $\mathcal{V}$ *is integral.*

Proof. The proof follows by showing that the closed monoidal structure of $\mathcal{V}$-Cat preserves symmetry and separation. It is immediate for symmetry. For separation, note that since $\mathcal{V}$ is integral the inequation $x \otimes y \leq x$ holds for all $x, y \in \mathcal{V}$. It follows that the monoidal structure preserves separation. The fact that the closed structure also preserves separation uses the implication $x \leq \bigwedge A \Rightarrow \forall a \in A.\ x \leq a$ for all $x \in X, A \subseteq X$. $\square$

Since we assume that $\mathcal{V}$ is integral, this last theorem states that all the categories mentioned therein are suitable bases of enrichment [Kel82].

Definition 3.11. A category C is $\mathcal{V}$-Cat-enriched (or simply, a $\mathcal{V}$-Cat-category) if for all C-objects X and Y the hom-set $\mathsf{C}(X, Y)$ is a *$\mathcal{V}$-category* and if the composition of C-morphisms,

$$(\,\cdot\,) : \mathsf{C}(X, Y) \otimes \mathsf{C}(Y, Z) \longrightarrow \mathsf{C}(X, Z)$$

is a $\mathcal{V}$-functor. Given two $\mathcal{V}$-Cat-categories C and D and a functor $F : \mathsf{C} \to \mathsf{D}$, we call F a $\mathcal{V}$-Cat-enriched functor (or simply, $\mathcal{V}$-Cat-functor) if for all C-objects X and Y the map $F_{X,Y} : \mathsf{C}(X,Y) \to \mathsf{D}(FX, F,Y)$ is a $\mathcal{V}$-functor. An adjunction $\mathsf{C} : F \dashv G : \mathsf{D}$ is called $\mathcal{V}$-Cat-enriched if for all objects $X \in |\mathsf{C}|$ and $Y \in |\mathsf{D}|$ there exists a $\mathcal{V}$-isomorphism $\mathsf{D}(FX, Y) \cong \mathsf{C}(X, GY)$ natural in X and Y. We obtain analogous notions of enrichment by substituting $\mathcal{V}$-Cat with $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$, $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym}}$, or $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym,sep}}$.

If C is a $\mathcal{V}$-Cat-category then $\mathsf{C} \times \mathsf{C}$ is also a $\mathcal{V}$-Cat-category via the tensor operation $\otimes$ in $\mathcal{V}$-Cat. We take advantage of this fact in the following definition.

Definition 3.12. A $\mathcal{V}$-Cat-enriched autonomous category C is an autonomous and $\mathcal{V}$-Cat-enriched category C such that the bifunctor $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ is a $\mathcal{V}$-Cat-functor and the adjunction $(- \otimes X) \dashv (X \multimap -)$ is a $\mathcal{V}$-Cat-adjunction. We obtain analogous notions of enriched autonomous category by replacing $\mathcal{V}$-Cat (as basis of enrichment) with $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$, $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym}}$, or $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym,sep}}$.

Example 3.13. Recall that $\mathsf{Pos} \cong \mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$ when $\mathcal{V}$ is the Boolean quantale. According to Thm. 3.10 the category Pos is autonomous. It follows by general results that the category is Pos-enriched [Bor94]. It is also easy to see that its tensor is Pos-enriched and that the adjunction $(- \otimes X) \dashv (X \multimap -)$ is Pos-enriched. Therefore, Pos is an instance of Definition 3.12. Note also that $\mathsf{Set} \cong \mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym,sep}}$ for $\mathcal{V}$ the Boolean quantale and that Set is trivially an instance of Definition 3.12.

Recall that $\mathsf{Met} \cong \mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym,sep}}$ when $\mathcal{V}$ is the metric quantale. Thus, the category Met is autonomous (Thm. 3.10) and Met-enriched [Bor94]. It follows as well from routine calculations that its tensor is Met-enriched and that the adjunction $(- \otimes X) \dashv (X \multimap -)$ is Met-enriched. Therefore Met is an instance of Definition 3.12. An analogous reasoning tells that the category of ultrametric spaces (enriched over itself) is also an instance of Definition 3.12. We present further examples of $\mathcal{V}$-Cat-enriched autonomous categories in § 4.

Finally, let us recall the interpretation of linear λ-terms on an autonomous category C (Section 2) and assume that C is $\mathcal{V}$-Cat-enriched. Then we say that a $\mathcal{V}$-equation $\Gamma \triangleright v =_q w : \mathbb{A}$ is *satisfied* by this interpretation if $a(\llbracket \Gamma \triangleright v : \mathbb{A} \rrbracket, \llbracket \Gamma \triangleright w : \mathbb{A} \rrbracket) \geq q$ where $a : \mathsf{C}(\llbracket \Gamma \rrbracket, \llbracket \mathbb{A} \rrbracket) \times \mathsf{C}(\llbracket \Gamma \rrbracket, \llbracket \mathbb{A} \rrbracket) \to \mathcal{V}$ is the underlying function of the $\mathcal{V}$-category $\mathsf{C}(\llbracket \Gamma \rrbracket, \llbracket \mathbb{A} \rrbracket)$.

Theorem 3.14. *The rules listed in Fig. 3 and in Fig. 4 are sound for $\mathcal{V}$-Cat-enriched autonomous categories C. Specifically if $\Gamma \triangleright v =_q w : \mathbb{A}$ results from the rules in Fig. 3 and Fig. 4 then $a(\llbracket \Gamma \triangleright v : \mathbb{A} \rrbracket, \llbracket \Gamma \triangleright w : \mathbb{A} \rrbracket) \geq q$.*

Proof. Let us focus first on the equations listed in Fig. 3. Recall that an equation $\Gamma \triangleright v = w : \mathbb{A}$ abbreviates the $\mathcal{V}$-equations $\Gamma \triangleright v =_\top w : \mathbb{A}$ and $\Gamma \triangleright w =_\top v : \mathbb{A}$. Moreover, we already know that the equations listed in Fig. 3 are sound for autonomous categories, specifically if $v = w$ is an equation of Fig. 3 then $\llbracket v \rrbracket = \llbracket w \rrbracket$ in C (Thm. 2.6). Thus, by the definition of a $\mathcal{V}$-category (Definition 3.7) and by the assumption of $\mathcal{V}$ being integral ($k = \top$) we obtain $a(\llbracket v \rrbracket, \llbracket w \rrbracket) \geq k = \top$ and $a(\llbracket w \rrbracket, \llbracket v \rrbracket) \geq k = \top$.

Let us now focus on the rules listed in Fig. 4. The first three rules follow from the definition of a $\mathcal{V}$-category and the transitivity property of $\leq$. Rule **(arch)** follows from the continuity of $\mathcal{V}$, specifically from the fact that q is the *least* upper bound of all elements r that are way-below q. Rule **(join)** follows from the definition of least upper bound. The remaining rules follow from the definition of the tensor functor $\otimes$ in $\mathcal{V}$-Cat, the fact that C

is $\mathcal{V}$-Cat-enriched, $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ is a $\mathcal{V}$-Cat-functor, and the fact that $(- \otimes X) \dashv (X \multimap -)$ is a $\mathcal{V}$-Cat-adjunction. For example, for the compatibility rule concerning (**ax**) we reason as follows:

$$\begin{aligned}
&a([\![f(v_1,\dots,v_n)]\!],[\![f(w_1,\dots,w_n)]\!])\\
&= a([\![f]\!]\cdot([\![v_1]\!]\otimes\cdots\otimes[\![v_n]\!])\cdot \mathsf{sp}_{\Gamma_1;\dots;\Gamma_n}\cdot \mathsf{sh}_E, [\![f]\!]\cdot([\![w_1]\!]\otimes\cdots\otimes[\![w_n]\!])\cdot \mathsf{sp}_{\Gamma_1;\dots;\Gamma_n}\cdot \mathsf{sh}_E)\\
&\geq a([\![f]\!]\cdot([\![v_1]\!]\otimes\cdots\otimes[\![v_n]\!]), [\![f]\!]\cdot([\![w_1]\!]\otimes\cdots\otimes[\![w_n]\!]))\\
&\geq a([\![v_1]\!]\otimes\cdots\otimes[\![v_n]\!]),([\![w_1]\!]\otimes\cdots\otimes[\![w_n]\!])\\
&\geq a([\![v_1]\!],[\![w_1]\!])\otimes\cdots\otimes a([\![v_n]\!],[\![w_n]\!])\\
&\geq q_1\otimes\cdots\otimes q_n
\end{aligned}$$

where the second step follows from the fact that $\mathsf{sp}_{\Gamma_1;\dots;\Gamma_n}\cdot \mathsf{sh}_E$ is a morphism in C and that C is $\mathcal{V}$-Cat-enriched; the third step follows from an analogous reasoning; the fourth step follows from the fact that $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ is a $\mathcal{V}$-Cat-functor; the last step follows from the premise of the rule in question. As another example, the proof for the substitution rule proceeds similarly:

$$\begin{aligned}
&a([\![v[v'/x]]\!],[\![w[w'/x]]\!])\\
&= a([\![v]\!]\cdot \mathsf{jn}_{\Gamma,\mathbb{A}}\cdot(\mathsf{id}\otimes[\![v']\!])\cdot \mathsf{sp}_{\Gamma;\Delta}, [\![w]\!]\cdot \mathsf{jn}_{\Gamma,\mathbb{A}}\cdot(\mathsf{id}\otimes[\![w']\!])\cdot \mathsf{sp}_{\Gamma;\Delta})\\
&\geq a([\![v]\!]\cdot \mathsf{jn}_{\Gamma,\mathbb{A}}\cdot(\mathsf{id}\otimes[\![v']\!]), [\![w]\!]\cdot \mathsf{jn}_{\Gamma,\mathbb{A}}\cdot(\mathsf{id}\otimes[\![w']\!]))\\
&\geq a(\mathsf{id}\otimes[\![v']\!],\mathsf{id}\otimes[\![w']\!])\otimes a([\![v]\!]\cdot \mathsf{jn}_{\Gamma,\mathbb{A}}, [\![w]\!]\cdot \mathsf{jn}_{\Gamma,\mathbb{A}})\\
&\geq a(\mathsf{id}\otimes[\![v']\!],\mathsf{id}\otimes[\![w']\!])\otimes a([\![v]\!],[\![w]\!])\\
&\geq a(\mathsf{id},\mathsf{id})\otimes a([\![v']\!],[\![w']\!])\otimes a([\![v]\!],[\![w]\!])\\
&= a([\![v']\!],[\![w']\!])\otimes a([\![v]\!],[\![w]\!])\\
&\geq q\otimes r
\end{aligned}$$

The proof for the rule concerning $(\multimap_i)$ additionally requires the following two facts: if a $\mathcal{V}$-functor $f : (X, a) \to (Y, b)$ is an isomorphism then $a(x, x') = b(f(x), f(x'))$ for all $x, x' \in X$. For a context Γ, the morphism $\mathsf{jn}_{\Gamma;x:\mathbb{A}} : [\![\Gamma]\!] \otimes [\![\mathbb{A}]\!] \to [\![\Gamma, x : \mathbb{A}]\!]$ is an isomorphism in C. The proof for the rule concerning the permutation of variables (exchange) also makes use of the fact that $[\![\Delta]\!] \to [\![\Gamma]\!]$ is an isomorphism. □

3.4. **$\mathcal{V}\lambda$-calculus, soundness, and completeness.** We now establish a soundness and completeness result for $\mathcal{V}\lambda$-calculus introduced in §3.2. A key construct in this result is the quotienting of a $\mathcal{V}$-category into a *separated* $\mathcal{V}$-category: we will use it to identify linear λ-terms when generating a syntactic category (from a linear $\mathcal{V}\lambda$-theory) that satisfies the axioms of autonomous categories. This naturally leads to the following notion of a model for linear $\mathcal{V}\lambda$-theories.

Definition 3.15 (Models of linear $\mathcal{V}\lambda$-theories)**.** Consider a linear $\mathcal{V}\lambda$-theory $((G, \Sigma), Ax)$ and a $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-enriched autonomous category C. Suppose that for each $X \in G$ we have an interpretation $[\![X]\!]$ as a C-object and analogously for the operation symbols. This interpretation structure is a model of the theory if all axioms in Ax are satisfied by the interpretation.

Let us then focus on establishing a completeness result for $\mathcal{V}\lambda$-calculus. The first important observation is that in §2 we did not assume that autonomous categories are locally small. In particular, linear λ-theories are able to generate non-(locally small) categories. However, now we need to be stricter because $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sep}}$-enriched autonomous categories are always locally small (recall the definition of $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sep}}$). Thus for two types $\mathbb{A}$ and $\mathbb{B}$ of a $\mathcal{V}\lambda$-theory $\mathscr{T}$, consider the class $\mathsf{Values}(\mathbb{A},\mathbb{B})$ of values v such that $x : \mathbb{A} \rhd v : \mathbb{B}$. We equip $\mathsf{Values}(\mathbb{A},\mathbb{B})$ with the function $a : \mathsf{Values}(\mathbb{A},\mathbb{B}) \times \mathsf{Values}(\mathbb{A},\mathbb{B}) \to \mathcal{V}$ defined by,

$$a(v,w) = \bigvee \{q \mid v =_q w \text{ is a theorem of } \mathscr{T}\}$$

It is easy to see that $(\mathsf{Values}(\mathbb{A},\mathbb{B}), a)$ is a (possibly large) $\mathcal{V}$-category. We then quotient this $\mathcal{V}$-category into a *separated* $\mathcal{V}$-category which we suggestively denote by $\mathsf{C}(\mathbb{A},\mathbb{B})$ (as detailed in the proof of the next theorem, $\mathsf{C}(\mathbb{A},\mathbb{B})$ will serve as a hom-object of a syntactic category C generated from a linear $\mathcal{V}\lambda$-theory). Adapting the nomenclature of [Lin66] to our setting, we call $\mathscr{T}$ *varietal* if $\mathsf{C}(\mathbb{A},\mathbb{B})$ is a *small* $\mathcal{V}$-category. In the rest of the paper we will only work with varietal theories and locally small categories.

Theorem 3.16 (Soundness & Completeness)**.** *Consider a varietal $\mathcal{V}\lambda$-theory. A $\mathcal{V}$-equation-in-context $\Gamma \rhd v =_q w : \mathbb{A}$ is a theorem iff it holds in all models of the theory.*

Proof. Soundness follows by induction over the rules that define the class $Th(Ax)$ and by an appeal to Thm. 3.14. For completeness, we use a strategy similar to the proof of Thm. 2.6, and take advantage of the quotienting of a $\mathcal{V}$-category into a separated $\mathcal{V}$-category. Recall that we assume that the theory is *varietal* and therefore can safely take $\mathsf{C}(\mathbb{A},\mathbb{B})$ (defined above) to be a small $\mathcal{V}$-category. Note that the quotienting process identifies all terms $x : \mathbb{A} \rhd v : \mathbb{B}$ and $x : \mathbb{A} \rhd w : \mathbb{B}$ such that $v =_\top w$ and $w =_\top v$. Such a relation contains the equations-in-context from Fig. 3 and moreover it is straighforward to show that it is compatible with the term formation rules of linear λ-calculus (Fig. 1). So, analogously to Thm. 2.6 we obtain an autonomous category C whose objects are the types of the language and whose hom-sets are the underlying sets of the $\mathcal{V}$-categories $\mathsf{C}(\mathbb{A},\mathbb{B})$.

Our next step is to show that the category C has a $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sep}}$-enriched autonomous structure. We start by showing that the composition map $\mathsf{C}(\mathbb{A},\mathbb{B}) \otimes \mathsf{C}(\mathbb{B},\mathbb{C}) \to \mathsf{C}(\mathbb{A},\mathbb{C})$ is a $\mathcal{V}$-functor:

$$\begin{aligned}
& a(([v'],[v]),([w'],[w])) \\
& = a([v],[w]) \otimes a([v'],[w']) \\
& = a(v,w) \otimes a(v',w') \\
& = \bigvee \{q \mid v =_q w\} \otimes \bigvee \{r \mid v' =_r w'\} \\
& = \bigvee \{q \otimes r \mid v =_q w, v' =_r w'\} && \{\text{Defn. of quantale}\} \\
& \leq \bigvee \{q \mid v[v'/x] =_q w[w'/x]\} && \left\{A \subseteq B \Rightarrow \bigvee A \leq \bigvee B\right\} \\
& = a(v[v'/x], w[w'/x]) \\
& = a([v[v'/x]], [w[w'/x]]) \\
& = a([v] \cdot [v'], [w] \cdot [w'])
\end{aligned}$$

The fact that $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ is a $\mathcal{V}\text{-}\mathsf{Cat}$-functor follows by an analogous reasoning. Next, we need to show that $(- \otimes X) \dashv (X \multimap -)$ is a $\mathcal{V}\text{-}\mathsf{Cat}$-adjunction. It is straightforward to show that both functors are ($\mathcal{V}$-Cat)-functors, and from a similar reasoning it follows that the isomorphism $\mathsf{C}(\mathbb{B}, \mathbb{A} \multimap \mathbb{C}) \cong \mathsf{C}(\mathbb{B} \otimes \mathbb{A}, \mathbb{C})$ is a $\mathcal{V}$-isomorphism.

The final step is to show that if an equation $\Gamma \rhd v =_q w : \mathbb{A}$ with $q \in B$ is satisfied by C then it is a theorem of the linear $\mathcal{V}\lambda$-theory. By assumption $a([v],[w]) = a(v,w) = \bigvee\{r \mid v =_r w\} \geq q$. It follows from the definition of the way-below relation that for all $x \in B$ with $x \ll q$ there exists a *finite* set $A \subseteq \{r \mid v =_r w\}$ such that $x \leq \bigvee A$. Then by an application of rule **(join)** in Fig. 4 we obtain $v =_{\bigvee A} w$, and consequently rule **(weak)** in Fig. 4 provides $v =_x w$ for all $x \ll q$. Finally, by an application of rule **(arch)** in Fig. 4 we deduce that $v =_q w$ is part of the theory. □

Thm. 3.16 extends straightforwardly to symmetric linear $\mathcal{V}\lambda$-theories and $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym,sep}}$-autonomous categories.

3.5. **Equivalence theorem between linear $\mathcal{V}\lambda$-theories and $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-autonomous categories.** Analogously to § 2.2 (where we address λ-calculus with just classical equations), in this subsection we will present a quasi-category of $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-autonomous categories, a quasi-category of linear $\mathcal{V}\lambda$-theories, and then prove the existence of an equivalence between both quasi-categories. For reasons analogous to those presented in §2.2, we start by providing a bijective correspondence up-to isomorphism between models of $\mathcal{V}\lambda$-theories $\mathscr{T}$ and $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-autonomous functors $\mathrm{Syn}(\mathscr{T}) \to \mathsf{C}$. The following proposition will help us establish that.

Proposition 2. *Consider a $\mathcal{V}$-Cat-autonomous functor $F : \mathsf{C} \to \mathsf{D}$ and a model M on C of a $\mathcal{V}\lambda$-theory $\mathscr{T}$. Then F_*M is a model of $\mathscr{T}$ on D.*

Proof. Recall from the proof of Proposition 1 the equation,

$$[\![\Gamma \rhd v : \mathbb{A}]\!]_{F_*M} = h_{\mathbb{A}}^{-1} \cdot F[\![\Gamma \rhd v : \mathbb{A}]\!]_M \cdot \mu[\Gamma] \cdot h[\Gamma]$$

which holds for all judgements $\Gamma \rhd v : \mathbb{A}$. We use it to reason in the following manner:

$$\begin{aligned}
& v =_q w \\
\Rightarrow\; & a([\![v]\!]_M, [\![w]\!]_M) \geq q && \{M \text{ is a model of } \mathscr{T}\} \\
\Rightarrow\; & a(F[\![v]\!]_M, F[\![w]\!]_M) \geq q && \{F \text{ is } \mathcal{V}\text{-}\mathsf{Cat}\text{-enriched}\} \\
\Rightarrow\; & a(h^{-1} \cdot F[\![v]\!]_M, h^{-1} \cdot F[\![w]\!]_M) \geq q && \{\mathsf{C} \text{ is } \mathcal{V}\text{-}\mathsf{Cat}\text{-enriched}\} \\
\Rightarrow\; & a(h^{-1} \cdot F[\![v]\!]_M \cdot \mu[\Gamma] \cdot h[\Gamma], h^{-1} \cdot F[\![w]\!]_M \cdot \mu[\Gamma] \cdot h[\Gamma]) \geq q && \{\mathsf{C} \text{ is } \mathcal{V}\text{-}\mathsf{Cat}\text{-enriched}\} \\
\Rightarrow\; & a([\![v]\!]_{F_*M}, [\![w]\!]_{F_*M}) \geq q
\end{aligned}$$

□

Theorem 3.17. *Let $\mathscr{T}$ be a $\mathcal{V}\lambda$-theory. Every $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-autonomous functor $F : \mathrm{Syn}(\mathscr{T}) \to \mathsf{C}$ induces a model $F_*\mathrm{Syn}(\mathscr{T})$ of $\mathscr{T}$ and every model M of $\mathscr{T}$ induces a $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-strict autonomous functor $([v] \mapsto [\![v]\!]_M) : \mathrm{Syn}(\mathscr{T}) \to \mathsf{C}$. Furthermore these constructions are inverse to each other up-to isomorphism, in the sense that,*

$$([v] \mapsto [\![v]\!]_{F_*\mathrm{Syn}(\mathscr{T})}) \cong F \qquad \textit{and} \qquad ([v] \mapsto [\![v]\!]_M)_*\mathrm{Syn}(\mathscr{T}) = M$$

Proof. Let us first focus on the mapping that sends functors to models. Consider a $\mathcal{V}$-Cat-autonomous functor $F : \mathrm{Syn}(\mathscr{T}) \to \mathsf{C}$. Then observe that $\mathrm{Syn}(\mathscr{T})$ corresponds to a model of $\mathscr{T}$ and thus by Proposition 2 we conclude that $F_*\mathrm{Syn}(\mathscr{T})$ must be a model of $\mathscr{T}$. For the inverse direction, consider a model of $\mathscr{T}$ over C. Let a denote the underlying function of the hom-($\mathcal{V}$-categories) in $\mathrm{Syn}(\mathscr{T})$ and b the underlying function of the hom-($\mathcal{V}$-categories) in C. Then note that if $[v] = [w]$ in $\mathrm{Syn}(\mathscr{T})$ then, by completeness, the equations $v =_\top w$ and $w =_\top v$ are theorems of $\mathscr{T}$, which means that $[\![v]\!]_M = [\![w]\!]_M$ by the definition of a model and *separability*. This allows us to define a mapping $F : \mathrm{Syn}(T) \to \mathsf{C}$ that sends each type $\mathbb{A}$ to $[\![\mathbb{A}]\!]_M$ and each morphism $[v]$ to $[\![v]\!]_M$. The fact that this mapping is an autonomous functor follows from an analogous reasoning to the one used in the proof of Thm. 2.9. We now need to show that this functor is $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-enriched. Recall that $a([v],[w]) = \bigvee\{q \mid v =_q w\}$ and observe that for every $v =_q w$ in the previous quantification we have $b([\![v]\!]_M, [\![w]\!]_M) \geq q$ (by the definition of a model), which establishes, by the definition of a least upper bound, $a([v],[w]) = \bigvee\{q \mid v =_q w\} \leq b([\![v]\!]_M, [\![w]\!]_M)$. Finally the proof for showing that,

$$([v] \mapsto [\![v]\!]_{F_*\mathrm{Syn}(\mathscr{T})}) \cong F \qquad \text{and} \qquad ([v] \mapsto [\![v]\!]_M)_*\mathrm{Syn}(\mathscr{T}) = M$$

is completely analogous to that of Thm. 2.9. □

Next, let $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-$\mathsf{Aut}$ be the quasi-category of $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-autonomous categories and $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-autonomous functors. Consider then $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-$\mathsf{Aut}_{/\cong}$ whose objects are $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-autonomous categories and morphisms are *isomorphism classes* of $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-autonomous functors. We will now focus on showing that the latter quasi-category is equivalent to certain quasi-category $\mathcal{V}\lambda$-Th whose objects are linear $\mathcal{V}\lambda$-theories,

$$\mathcal{V}\lambda\text{-}\mathsf{Th} \underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}} \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sep}}\text{-}\mathsf{Aut}_{/\cong} \tag{3.1}$$

Analogously to §2.2, we set $\mathcal{V}\lambda\text{-}\mathsf{Th}(\mathscr{T}_1,\mathscr{T}_2) := \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sep}}\text{-}\mathsf{Aut}_{/\cong}(\mathrm{Syn}(\mathscr{T}_1), \mathrm{Syn}(\mathscr{T}_2))$. In words, this means that a morphism $\mathscr{T}_1 \to \mathscr{T}_2$ between $\mathcal{V}\lambda$-theories $\mathscr{T}_1$ and $\mathscr{T}_2$ is exactly an isomorphism class of $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-autonomous functors $\mathrm{Syn}(\mathscr{T}_1) \to \mathrm{Syn}(\mathscr{T}_2)$, which by Thm. 3.17 are in bijective correspondence (up-to isomorphism) to models of $\mathscr{T}_1$ on the category $\mathrm{Syn}(\mathscr{T}_2)$. Note that our remarks in §2.2 about Morita equivalence hold here as well, with the exception that now an equivalence of theories $\mathscr{T}_1$ and $\mathscr{T}_2$ translates into a $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-equivalence of categories $\mathrm{Syn}(\mathscr{T}_1)$ and $\mathrm{Syn}(\mathscr{T}_2)$ instead of just an ordinary equivalence.

The next step is to set down the necessary constructions for describing the functor $(\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sep}})\text{-}\mathsf{Aut}_{/\cong} \to \mathcal{V}\lambda\text{-}\mathsf{Th}$ that is part of the equivalence (3.1).

Definition 3.18 (Internal language)**.** Consider a $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-enriched autonomous category C. It induces a linear $\mathcal{V}\lambda$-theory $\mathrm{Lng}(\mathsf{C})$ whose ground types and operations symbols are defined as in the case of linear λ-theories (recall Definition 2.10). The axioms of $\mathrm{Lng}(\mathsf{C})$ are all the $\mathcal{V}$-equations-in-context that are satisfied by the obvious interpretation on C plus the $\mathcal{V}$-equations that mark $i_{\mathbb{A}} : \mathbb{A} \to i(\mathbb{A})$ as an isomorphism in the theory $\mathrm{Lng}(\mathsf{C})$.

In conjunction with the proof of Thm. 3.16, a consequence of the following theorem is that $\mathrm{Syn}(\mathrm{Lng}(\mathsf{C}))$ is a $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-enriched category. We will need this to properly formulate the claim that $\mathrm{Lng}(\mathsf{C})$ completely describes C from a syntactical perspective, like we did in §2.2 for classical equations.

Theorem 3.19. *The linear $\mathcal{V}\lambda$-theory* $\mathrm{Lng}(\mathsf{C})$ *is varietal.*

Proof. Let us denote by $\mathrm{Lng}^\lambda(\mathsf{C})$ the *linear λ-theory* generated from C. According to Thm. 2.11, the category $\mathrm{Syn}(\mathrm{Lng}^\lambda(\mathsf{C}))$ (*i.e.* the syntactic category generated from $\mathrm{Lng}^\lambda(\mathsf{C})$) is locally small whenever C is locally small. Then consider two types $\mathbb{A}$ and $\mathbb{B}$. We will prove our claim by taking advantage of the axiom of replacement in ZF set-theory, specifically by presenting a *surjective* map,

$$\mathrm{Syn}(\mathrm{Lng}^\lambda(\mathsf{C}))(\mathbb{A},\mathbb{B}) \longrightarrow \mathrm{Syn}(\mathrm{Lng}(\mathsf{C}))(\mathbb{A},\mathbb{B})$$

The crucial observation is that if $v = w$ in $\mathrm{Lng}^{\lambda}(\mathsf{C})$ then $v =_{\top} w$ and $w =_{\top} v$ in $\mathrm{Lng}(\mathsf{C})$. This is obtained by the definition of a model, the definition of a $\mathcal{V}$-category, and the definition of $\mathrm{Lng}(\mathsf{C})$. This observation allows to establish the surjective map that sends $[v]$ to $[v]$, *i.e.* it sends the equivalence class of v as a λ-term in $\mathrm{Lng}^{\lambda}(\mathsf{C})$ into the equivalence class of v as a λ-term in $\mathrm{Lng}(\mathsf{C})$. □

Finally we state,

Theorem 3.20. *Consider a $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-autonomous category C. There exists a equivalence $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-$\mathrm{Syn}(\mathrm{Lng}(\mathsf{C})) \simeq \mathsf{C}$ and both functors witnessing this equivalence are autonomous. The functor going from the left to right direction is additionally strict.*

Proof. Let a denote the underlying function of the hom-($\mathcal{V}$-categories) in $\mathrm{Syn}(\mathrm{Lng}(\mathsf{C}))$ and b the underlying function of the hom-($\mathcal{V}$-categories) in C. We have, by construction, a model of $\mathrm{Lng}(\mathsf{C})$ on C which acts as the identity in the interpretation of ground types and operation symbols. We can then appeal to Thm. 3.17 to establish a $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-functor $\mathrm{Syn}(\mathrm{Lng}(\mathsf{C})) \to \mathsf{C}$. Next, the functor working on the inverse direction behaves as the identity on objects and sends a morphism f into $[f(x)]$. Let us show that it is $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-enriched. First, observe that if $q \ll b(f,g)$ in C and $q \in B$ then $f(x) =_q g(x)$ is a theorem of $\mathrm{Lng}(\mathsf{C})$, due to the fact that $\ll$ entails $\leq$ and by the definition of $\mathrm{Lng}(\mathsf{C})$. Using the definition of a basis, we thus obtain $b(f,g) = \bigvee\{q \in B \mid q \ll b(f,g)\} \leq \bigvee\{q \in B \mid f(x) =_q g(x)\} = a([f(x)],[g(x)])$. The fact that both functors are autonomous and that the one from left to right direction is additionally strict is proved as in Thm. 2.11. The same applies to showing that the functors define an equivalence of categories. □

Theorem 3.21. *There exists an equivalence,*

$$\mathcal{V}\lambda\text{-}\mathsf{Th} \underset{\mathrm{Lng}}{\overset{\mathrm{Syn}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sep}}\text{-}\mathsf{Aut}_{/\cong} \tag{3.2}$$

Proof. The functor $\mathrm{Syn} : \mathcal{V}\lambda\text{-}\mathsf{Th} \to \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sep}}\text{-}\mathsf{Aut}_{/\cong}$ sends a $\mathcal{V}\lambda$-theory to its syntactic category and acts as the identity on morphisms. For the inverse direction, recall that for every $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-autonomous category C, Thm. 3.20 provides ($\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$)-autonomous functors $e_{\mathsf{C}} : \mathsf{C} \to \mathrm{Syn}(\mathrm{Lng}(\mathsf{C}))$ and $e'_{\mathsf{C}} : \mathrm{Syn}(\mathrm{Lng}(\mathsf{C})) \to \mathsf{C}$. So then we define $\mathrm{Lng} : \mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sep}}\text{-}\mathsf{Aut}_{/\cong} \to \mathcal{V}\lambda\text{-}\mathsf{Th}$ as the functor that sends a $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-autonomous category C to its theory $\mathrm{Lng}(\mathsf{C})$ and that sends an isomorphism class $[F]$ of ($\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$)-autonomous functors of type $\mathsf{C} \to \mathsf{D}$ into $[e_{\mathsf{D}} \cdot F \cdot e'_{\mathsf{C}}]$. This last functor is well-defined for the reasons already detailed in Thm. 2.11. Similarly, the fact that both functors Syn and Lng form an equivalence follows a reasoning analogous to the one detailed in the proof of Thm. 2.12. □

Thm. 3.21 and the preceding results extend straightforwardly to symmetric linear $\mathcal{V}\lambda$-theories and $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym,sep}}$-autonomous categories.

4. Examples of linear $\mathcal{V}\lambda$-theories and their models

4.1. **Real-time computation.** We now return to the example of wait calls and its metric axioms (1.1) sketched in § 1. Let us build a model on Met for this theory. Denote by $\mathbb{N}$ the metric space of natural numbers. Then fix a metric space A, interpret the ground type X as $\mathbb{N} \otimes A$ and the operation symbol $\mathtt{wait_n} : X \to X$ as the non-expansive map,

$[\![\mathtt{wait_n}]\!] : \mathbb{N} \otimes A \to \mathbb{N} \otimes A$, $(i, a) \mapsto (i + n, a)$. Since we already know that Met is a Met-autonomous category (recall Definition 3.12 and Example 3.13) we only need to show that the axioms in (1.1) are satisfied by the proposed interpretation. This can be shown via a few routine calculations.

As an illustration of this theory at work, let us use it to briefly explore what would happen if one freely allowed multiple uses of the same variable in building a judgement. Consider the *non-linear* term $\lambda f, x.\ f(f(\ldots(f\,x)))$ which we abbreviate to $\mathtt{seq^n}$. Intuitively, it applies the operation given as argument n-times. For example in $\mathtt{seq^n}\ (\lambda x.\ \mathtt{wait_1}(x))\ z$ the operation $\lambda x.\ \mathtt{wait_1}(x)$ is applied n-times. If $\mathtt{seq^n}$ was allowed in our calculus then we would obtain problematic metric theorems, such as,

$$\mathtt{seq^n}\ (\lambda x.\ \mathtt{wait_1}(x))\ z =_1 \mathtt{seq^n}\ (\lambda x.\ \mathtt{wait_2}(x))\ z$$

The theorem makes no sense because the left-hand side unfolds to an execution time of n seconds and the right-hand side unfolds to an execution time of $2n$ seconds. In the linear setting, we can rewrite $\mathtt{seq^n}$ into the more general term $\lambda f_1, \ldots, f_n, x.\ f_1(f_2(\ldots(f_n\,x)))$ and then via our $\mathcal{V}$-deductive system obtain,

$$\mathtt{seq^n}\ (\lambda x.\ \mathtt{wait_1}(x))\ \ldots\ (\lambda x.\ \mathtt{wait_1}(x))\ z =_{n\cdot 1} \mathtt{seq^n}\ (\lambda x.\ \mathtt{wait_2}(x))\ \ldots\ (\lambda x.\ \mathtt{wait_2}(x))\ z$$

which is in line with the total execution times of n and $2n$ seconds mentioned above. This brief exploration tells that it is important to track the use of resources in metric λ-theories.

Now, it may be the case that is unnecessary to know the *distance* between the execution time of two programs – instead it suffices to know whether a program finishes its execution *before* another one. This leads us to linear $\mathcal{V}\lambda$-theories where $\mathcal{V}$ is the Boolean quantale. We call such theories *linear ordered λ-theories*. Recall again the language from the introduction with a single ground type X and the signature of wait calls $\Sigma = \{\mathtt{wait_n} : X \to X \mid n \in \mathbb{N}\}$. Then we adapt the metric axioms (1.1) to the case of the Boolean quantale by considering instead:

$$\mathtt{wait_0}(x) = x \qquad \mathtt{wait_n}(\mathtt{wait_m}(x)) = \mathtt{wait_{n+m}}(x) \qquad \frac{n \leq m}{\mathtt{wait_n}(x) \leq \mathtt{wait_m}(x)}$$

where a classical equation $v = w$ is shorthand for $v \leq w$ (*i.e.* $v =_1 w$) and $w \leq v$ (*i.e.* $w =_1 v$). In the resulting theory we can consider for instance (and omitting types for simplicity) the λ-term that defines the composition of two functions $\lambda f.\ \lambda g.\ g\ (f\ z)$, which we denote by v, and show that,

$$v\ (\lambda x.\ \mathtt{wait_1}(x)) \leq v\ (\lambda x.\ \mathtt{wait_1}(\mathtt{wait_1}(x)))$$

This inequation between higher-order programs arises from the argument $\lambda x.\ \mathtt{wait_1}(\mathtt{wait_1}(x))$ being costlier than the argument $\lambda x.\ \mathtt{wait_1}(x)$ – specifically, the former will invoke one more wait call ($\mathtt{wait_1}$) than the latter. Moreover, the inequation entails that for every argument g the execution time of computation $v\ (\lambda x.\ \mathtt{wait_1}(x))\ g$ will always be smaller than that of computation $v\ (\lambda x.\ \mathtt{wait_1}(\mathtt{wait_1}(x)))\ g$ since it invokes one more wait call. Thus in general the inequation tells that costlier programs fed as input to v will result in longer execution times when performing the corresponding computation.

In order to build a model for the ordered theory of wait calls, let now $\mathbb{N}$ be the poset of natural numbers. Then fix a poset A and define a model over Pos by sending X into $\mathbb{N} \otimes A$ and $\mathtt{wait_n} : X \to X$ to the monotone map $[\![\mathtt{wait_n}]\!] : \mathbb{N} \otimes A \to \mathbb{N} \otimes A$, $(i, a) \mapsto (i + n, a)$. Since we already know that Pos is a Pos-autonomous category (recall Definition 3.12 and

Example 3.13) we only need to show that the ordered axioms are satisfied by the proposed interpretation. But again, this can be shown via a few routine calculations.

4.2. **Probabilistic computation.** Let us now analyse an example of a metric λ-theory concerning probabilistic computation. We start by considering ground types $\texttt{real}, \texttt{real}^+$ and $\texttt{unit}$, and a signature of operations consisting of $\{r : \mathbb{I} \to \texttt{real} \mid r \in \mathbb{Q}\} \cup \{r^+ : \mathbb{I} \to \texttt{real}^+ \mid r \in \mathbb{Q}_{\geq 0}\} \cup \{r^u : \mathbb{I} \to \texttt{unit} \mid r \in [0,1] \cap \mathbb{Q}\}$, an operation $+$ of type $\texttt{real}, \texttt{real} \to \texttt{real}$, and *sampling* functions $\texttt{bernoulli} : \texttt{real}, \texttt{real}, \texttt{unit} \to \texttt{real}$ and $\texttt{normal} : \texttt{real}, \texttt{real}^+ \to \texttt{real}$. Whenever no ambiguities arise, we drop the superscripts in r^u and r^+. Operationally, $\texttt{bernoulli}(x, y, p)$ generates a sample from the Bernoulli distribution with parameter p on the set $\{x, y\}$, whilst $\texttt{normal}(x, y)$ generates a normal deviate with mean x and standard deviation y. We then postulate the metric axiom,

$$\frac{p, q \in [0,1] \cap \mathbb{Q}}{\texttt{bernoulli}(x_1, x_2, p(*)) =_{|p-q|} \texttt{bernoulli}(x_1, x_2, q(*))} \tag{4.1}$$

Consider now the following λ-terms (where we abbreviate the constants $0(*), 1(*), p(*), q(*)$ to $0, 1, p, q$, respectively),

$$\texttt{walk1} \triangleq \lambda x : \texttt{real}.\texttt{bernoulli}(0, x + \texttt{normal}(0,1), p)$$
$$\texttt{walk2} \triangleq \lambda x : \texttt{real}.\texttt{bernoulli}(0, x + \texttt{normal}(0,1), q), \qquad p, q \in [0,1] \cap \mathbb{Q}.$$

As the names suggest, these two terms of type $\texttt{real} \multimap \texttt{real}$ correspond to random walks on $\mathbb{R}$. At each call, $\texttt{walk1}$ (resp. $\texttt{walk2}$) performs a jump drawn randomly from a standard normal distribution, or is forced to return to the origin with probability p (resp. q). These are non-standard random walks whose semantics tend to be given by complicated operators (details below), but the simple $\mathcal{V}$-equational system of Fig. 4 and the axiom (4.1) allow us to easily derive $\texttt{walk1} =_{|p-q|} \texttt{walk2}$ without having to compute the semantics of these terms. In other words, the axiom (4.1) is enough to tightly bound the distance between two non-trivial random walks represented as higher-order terms in a probabilistic programming language. Furthermore the tensor in λ-calculus allows to easily scale up this reasoning to random walks on higher dimensions such as $\lambda x : \texttt{real}.\ \texttt{pm}\ x\ \texttt{to}\ x_1 \otimes x_2.\ \texttt{walk1}(x_1) \otimes \texttt{walk2}(x_2) : (\texttt{real} \otimes \texttt{real}) \multimap (\texttt{real} \otimes \texttt{real})$ on $\mathbb{R}^2$.

We now interpret the resulting metric λ-theory in the category Ban of Banach spaces and short operators, *i.e.* the semantics of [DK20, Koz81] without the order structure needed to interpret $\texttt{while}$ loops. This is the usual representation of Markov chains/kernels as matrices/operators. Recall that every operator $T : V \to U$ between Banach spaces V and U has a norm $\|T\|$,

$$\|T\| = \bigvee \{ \|Tv\| \mid \|v\| = 1\}$$

called *operator norm*. Recall also that short operators satisfy the inequation $\|T\| \leq 1$. It is well known that Ban has an autonomous structure [DK20, Koz81] where the tensor product is the projective tensor $\hat{\otimes}_\pi$. So we will just focus on showing that Ban is a Met-enriched autonomous category (*i.e.* an instance of Definition 3.12). First, recall that a norm on a vector space induces a metric [Rya13], in particular we obtain a metric d on the hom-set $\mathsf{Ban}(V, U)$ which is defined by,

$$d(S, T) = \|S - T\|$$

for $S, T \in \mathsf{Ban}(V, U)$. For all operators T, S between Banach spaces, it is also well-known that if S is a short operator then $\|ST\| \leq \|T\|$, and if T is a short operator then $\|ST\| \leq \|S\|$.

Moreover it is the case that $\left\|T \hat{\otimes}_\pi \mathsf{id}\right\| \leq \|T\|$ and $\left\|\mathsf{id} \,\hat{\otimes}_\pi T\right\| \leq \|T\|$ (see [Rya13, §2.1]). From these results it follows that,

Proposition 3. *The category* Ban *is* Met*-enriched and the bifunctor* $\hat{\otimes}_\pi : \mathsf{Ban} \otimes \mathsf{Ban} \to \mathsf{Ban}$ *is* Met*-enriched as well.*

Proof. Let us first show that Ban is Met-enriched. We deduce by unfolding the respective definitions that we need to show the following: for all short operators $T, T' : V \to U$ and $S, S' : U \to W$ the inequation $\|S - S'\| + \|T - T'\| \geq \|ST - S'T'\|$ holds. So we reason in the following way:

$$
\begin{aligned}
& \left\|S - S'\right\| + \left\|T - T'\right\| \\
\geq\ & \left\|(S - S')T\right\| + \left\|S'(T - T')\right\| && \left\{\left\|(S - S')T\right\| \leq \left\|S - S'\right\| \text{ and } \left\|S'(T - T')\right\| \leq \left\|T - T'\right\|\right\} \\
=\ & \left\|ST - S'T\right\| + \left\|S'T - S'T'\right\| \\
\geq\ & \left\|ST - S'T + S'T - S'T'\right\| && \{\text{Triangle inequality}\} \\
=\ & \left\|ST - S'T'\right\|
\end{aligned}
$$

Next, concerning $\hat{\otimes}_\pi$ we can also deduce by unfolding the respective definitions that we need to prove $\|S - S'\| + \|T - T'\| \geq \left\|T \hat{\otimes}_\pi S - T' \hat{\otimes}_\pi S'\right\|$. For this case we calculate,

$$
\begin{aligned}
& \left\|S - S'\right\| + \left\|T - T'\right\| \\
\geq\ & \left\|\mathsf{id} \,\hat{\otimes}_\pi (S - S')\right\| + \left\|(T - T') \hat{\otimes}_\pi \mathsf{id}\right\| && \Big\{\left\|\mathsf{id} \,\hat{\otimes}_\pi (S - S')\right\| \leq \left\|S - S'\right\| \text{ and } \ldots \\
& && \left\|(T - T') \hat{\otimes}_\pi \mathsf{id}\right\| \leq \left\|T - T'\right\|\Big\} \\
=\ & \left\|\mathsf{id} \,\hat{\otimes}_\pi S - \mathsf{id} \,\hat{\otimes}_\pi S'\right\| + \left\|T \hat{\otimes}_\pi \mathsf{id} - T' \hat{\otimes}_\pi \mathsf{id}\right\| \\
\geq\ & \left\|(\mathsf{id} \,\hat{\otimes}_\pi S) \cdot (T \hat{\otimes}_\pi \mathsf{id}) - (\mathsf{id} \,\hat{\otimes}_\pi S') \cdot (T' \hat{\otimes}_\pi \mathsf{id})\right\| && \{\mathsf{Ban} \text{ is } \mathsf{Met}\text{-enriched}\} \\
=\ & \left\|T \hat{\otimes}_\pi S - T' \hat{\otimes}_\pi S'\right\|
\end{aligned}
$$

□

We will next recur to the following general theorem for showing that Ban is a Met-enriched autonomous category.

Theorem 4.1. *Consider an autonomous category* C *such that,*

(1) C *is* $\mathcal{V}$-Cat*-enriched;*
(2) *the functor* $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ *is* $\mathcal{V}$-Cat*-enriched;*
(3) *and for all* C*-objects* X *the functor* $(X \multimap -) : \mathsf{C} \to \mathsf{C}$ *is* $\mathcal{V}$-Cat*-enriched as well.*

Then C *is a* $\mathcal{V}$-Cat*-enriched autonomous category.*

Proof. We first show that for all C-objects X the functor $(- \otimes X) : \mathsf{C}(A, B) \to \mathsf{C}(A \otimes X, B \otimes X)$ defined by $h \mapsto h \otimes \mathsf{id}$ is $\mathcal{V}$-Cat-enriched. This follows directly from (2) and from the fact that the corresponding mapping is given by the composite,

$$
\mathsf{C}(A, B) \cong \mathsf{C}(A, B) \otimes 1 \xrightarrow{\mathsf{id} \otimes (* \mapsto \mathsf{id}_X)} \mathsf{C}(A, B) \otimes \mathsf{C}(X, X) \xrightarrow{\ \otimes\ } \mathsf{C}(A \otimes X, B \otimes X)
$$

We thus obtain that for all C-objects X both functors $(- \otimes X)$ and $(X \multimap -)$ are $(\mathcal{V}$-$\mathsf{Cat})$-enriched. The next step is to prove that the maps witnessing the isomorphism $\mathsf{C}(Y \otimes X, Z) \cong$

$\mathsf{C}(Y, X \multimap Z)$ are $\mathcal{V}$-functors. The left-to-right direction is the composite,

$$\mathsf{C}(Y \otimes X, Z) \xrightarrow{(X \multimap -)} \mathsf{C}(X \multimap (Y \otimes X), X \multimap Z) \cong 1 \otimes \mathsf{C}(X \multimap (Y \otimes X), X \multimap Z)$$
$$\xrightarrow{(* \mapsto \overline{\mathsf{id}_{Y \otimes X}}) \otimes \mathsf{id}} \mathsf{C}(Y, X \multimap (Y \otimes X)) \otimes \mathsf{C}(X \multimap (Y \otimes X), X \multimap Z) \xrightarrow{(\cdot)} \mathsf{C}(Y, X \multimap Z)$$

and so by (3) and (1) it must be a $\mathcal{V}$-functor. The right-to-left direction is the composite,

$$\mathsf{C}(Y, X \multimap Z) \xrightarrow{(- \otimes X)} \mathsf{C}(Y \otimes X, (X \multimap Z) \otimes X) \cong \mathsf{C}(Y \otimes X, (X \multimap Z) \otimes X) \otimes 1 \xrightarrow{\mathsf{id} \otimes (* \mapsto \mathrm{app})}$$
$$\mathsf{C}(Y \otimes X, (X \multimap Z) \otimes X) \otimes \mathsf{C}((X \multimap Z) \otimes X, Z) \xrightarrow{(\cdot)} \mathsf{C}(Y \otimes X, Z)$$

Since $(- \otimes X)$ is $\mathcal{V}$-Cat-enriched, and C is autonomous and $\mathcal{V}$-Cat-enriched, this composite must be a $\mathcal{V}$-functor as well. □

The previous theorem can easily be adjusted to work with the bases of enrichment $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$ and $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym,sep}}$. We then obtain,

Theorem 4.2. *The category* Ban *is a* Met*-enriched autonomous category.*

Proof. Proposition 3 and Thm. 4.1 entail our claim if we show that the functor $(V \multimap -) : \mathsf{Ban} \to \mathsf{Ban}$ is Met-enriched for all Banach spaces V. Thus recall that for all Banach spaces W and U the mapping $(V \multimap -) : \mathsf{Ban}(W, U) \to \mathsf{Ban}(V \multimap W, V \multimap U)$ is defined as $S \mapsto (S \cdot -)$. Then to prove that $(V \multimap -)$ is Met-enriched recall as well that the space $V \multimap W$ is equipped with the operator norm, and that all elements T of this space with $\|T\| = 1$ are necessarily short. Finally observe that for all $S \in \mathsf{Ban}(W, U)$ we have $\|S\| \geq \|(S \cdot -)\|$, because for all operators $T \in V \multimap W$ with $\|T\| = 1$ the inequation $\|S\| \geq \|S \cdot T\|$ holds. It follows that the mapping $(V \multimap -) : \mathsf{Ban}(W, U) \to \mathsf{Ban}(V \multimap W, V \multimap U)$, $S \mapsto (S \cdot -)$ is non-expansive. □

Finally, Ban forms a model of our metric λ-theory via the following interpretation: we define $[\![\mathtt{real}]\!] = \mathcal{M}\mathbb{R}$, the Banach space of finite Borel measures on $\mathbb{R}$ equipped with the total variation norm, and similarly $[\![\mathtt{real}^+]\!] = \mathcal{M}\mathbb{R}^+$ and $[\![\mathtt{unit}]\!] = \mathcal{M}[0,1]$. We have $[\![\mathbb{I}]\!] = \mathbb{R} \ni 1$, and for every $r \in \mathbb{Q}$ we put $[\![r]\!] : \mathbb{R} \to \mathcal{M}\mathbb{R}, x \mapsto x\delta_r$, where δ_r is the Dirac delta over r; thus $[\![r]\!](1) = \delta_r$. We define an analogous interpretation for the operation symbols r^+ and r^u. For $\mu, \upsilon \in \mathcal{M}\mathbb{R}$ we define $[\![+]\!](\mu \otimes \upsilon) = +_*(\mu \otimes \upsilon)$, *i.e.* the pushforward under $+$ of the product measure $\mu \otimes \upsilon$ (seen as an element of $\mathcal{M}\mathbb{R} \otimes \mathcal{M}\mathbb{R}$, see [DK20]). For $\mu, \upsilon, \xi \in \mathcal{M}\mathbb{R}$ we define $[\![\mathtt{bernoulli}]\!](\mu \otimes \upsilon \otimes \xi) = \mathrm{bern}_*(\mu \otimes \upsilon \otimes \xi)$, the pushforward of the product measure $\mu \otimes \upsilon \otimes \xi$ under the Markov kernel $\mathrm{bern} : \mathbb{R}^3 \to \mathbb{R}, (u, v, p) \mapsto p\delta_u + (1-p)\delta_v$, and similarly for $[\![\mathtt{normal}]\!]$ (see [DK20] for the definition of pushforward and pushforward by a Markov kernel). This interpretation is sound because the norm on $\mathcal{M}\mathbb{R}$ is the total variation norm, and the metric axiom (4.1) describes the total variation distance between the corresponding Bernoulli distributions. More formally, for $[\![\mathtt{bernoulli}(x, y, p)]\!]$ and $[\![\mathtt{bernoulli}(x, y, q)]\!]$

with $p, q \in [0,1]$ we calculate,

$$\begin{aligned}
&\bigvee_A \left| \iint p\delta_u(A) + (1-p)\delta_v(A)\ d[\![x]\!](du)d[\![y]\!](dv) - \iint q\delta_u(A) + (1-q)\delta_v(A)\ d[\![x]\!](du)d[\![y]\!](dv) \right| \\
&= \bigvee_A \left| \iint (p-q)\delta_u(A) + ((1-p)-(1-q))\delta_v(A)\ d[\![x]\!](du)d[\![y]\!](dv) \right| \\
&= \bigvee_A |(p-q)[\![x]\!](A) + (q-p)[\![y]\!](A)| \\
&= \bigvee_A |(p-q)([\![x]\!](A) - [\![y]\!](A))| \\
&\leq |p-q|
\end{aligned}$$

4.3. **Quantum computation.** We now turn our attention to quantum computing, a paradigm in which notions of approximation and noise take a central role [NC02, Chapter 9]. Since quantum computing is perhaps less known than the probabilistic case, we start by recalling some basic concepts on the topic. We assume however some familiarity with linear algebra.

Recall that for every natural number $n \geq 1$ the set $\mathbb{C}^n$ is a complex vector space, in fact an inner product space with the inner product $\langle v, u\rangle$ given by $v^\dagger u$ where $(-)^\dagger$ is the adjoint operation. Recall as well that inner products induce a norm $\|v\| = \sqrt{\langle v, v\rangle}$. A quantum state is an element $v \in \mathbb{C}^n$ such that $\|v\| = 1$. We denote the elements of the canonical basis of $\mathbb{C}^n$ by $|0\rangle, \dots, |n-1\rangle$. Consider now an operator $T : \mathbb{C}^n \to \mathbb{C}^m$. If $\|Tv\| = \|v\|$ for all elements $v \in \mathbb{C}^n$ we call T an isometry. Note that isometries are precisely those operators that send quantum states to quantum states (and not to something else), so it makes sense that we focus on isometries at least for now. An important isometry is the so-called *phase* operation $P(\phi) : \mathbb{C}^2 \to \mathbb{C}^2$ (ϕ is an angle in radians) which is defined by $|0\rangle \mapsto |0\rangle$, $|1\rangle \mapsto e^{i\phi}|1\rangle$. This operation can be elegantly explained geometrically in the following way: first, it is well-known that when global phases are ignored we can represent a quantum state $v \in \mathbb{C}^2$ in the form,

$$\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

which corresponds to a point in the unit sphere where θ marks the latitude (*i.e.* the polar angle) and φ marks the longitude (*i.e.* the azimuthal angle). This representation is traditionally called the *Bloch sphere representation*. A point in the latter representation corresponds to the vector in $\mathbb{R}^3$ defined by $(\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$ and often called Bloch vector [NC02, page 174]. Then via routine linear algebra calculations we see that $P(\phi)$ corresponds to a rotation of ϕ radians along the z-axis, *i.e.* the polar angle is preserved and the azimuthal angle changes to $\varphi + \phi$. Another imporant isometry is $R_y(\phi) : \mathbb{C}^2 \to \mathbb{C}^2$ which corresponds to a rotation of ϕ radians along the y-axis and is given by the matrix,

$$\begin{pmatrix} \cos\frac{\phi}{2} & -\sin\frac{\phi}{2} \\ \sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix}$$

Finally an isometry vastly used in quantum computing is the Hadamard operation $H : \mathbb{C}^2 \to \mathbb{C}^2$ which is defined by $H = R_y(\frac{\pi}{2})P(\pi)$.

Quantum random walk on a (discrete) circle. Quantum random walks have been extensively studied in the quantum literature [VA12]. They are part of several quantum algorithms, and although conceptually similar to the probabilistic counterpart they tend to produce vastly different results from the latter. Here we focus on a simple case of a quantum random walk and use it to provide another example of a metric λ-theory.

Let us start by defining a metric λ-theory whose ground types are $\mathtt{qbit}$ and $\mathtt{pos}$. In this example values of type $\mathtt{qbit}$ will be used to decide whether to move left or right, and $\mathtt{pos}$ will be used to mark the current position on a discrete circle. As operations we have $\mathtt{H} : \mathtt{qbit} \to \mathtt{qbit}$ (which will later on be interpreted as the Hadamard operation) and $\mathtt{S} : \mathtt{qbit} \otimes \mathtt{pos} \to \mathtt{qbit} \otimes \mathtt{pos}$, which as alluded above will use a value of type $\mathtt{qbit}$ to move either to the left or right. Here one step of the quantum walk is given by,

$$\lambda x : \mathtt{qbit} \otimes \mathtt{pos}.\ \mathtt{pm}\ x\ \mathtt{to}\ x_1 \otimes x_2.\ \mathtt{S}(\mathtt{H}(x_1) \otimes x_2) : \mathtt{qbit} \otimes \mathtt{pos} \multimap \mathtt{qbit} \otimes \mathtt{pos}$$

which we abbreviate to $\mathtt{step} : \mathtt{qbit} \otimes \mathtt{pos} \multimap \mathtt{qbit} \otimes \mathtt{pos}$. If we wish to perform n-steps, then we can proceed modularly in the following way. First we define the closed judgement $\lambda f_1, \dots, f_n, x.\ f_1(f_2(\dots(f_n x)))$ which we abbreviate to $\mathtt{seq^n}$ (recall the example of wait calls above). Then we can straightforwardly build the closed judgement $\mathtt{seq^n}\ \mathtt{step}\ \dots\ \mathtt{step} : \mathtt{qbit} \otimes \mathtt{pos} \multimap \mathtt{qbit} \otimes \mathtt{pos}$ to represent n-steps in the walk.

Let us now consider a simple metric axiom: we add a new operation $\mathtt{H}^\epsilon : \mathtt{qbit} \to \mathtt{qbit}$ and postulate as axiom that $x : \mathtt{qbit} \rhd \mathtt{H}(x) =_\epsilon \mathtt{H}^\epsilon(x) : \mathtt{qbit}$. The operation $\mathtt{H}^\epsilon$ represents an *imperfect* implementation of $\mathtt{H}$. For example by knowing that the Hadamard operation is the composition $R_y(\frac{\pi}{2}) \cdot P(\pi)$ we may regard $\mathtt{H}^\epsilon$ as the composition $R_y(\frac{\pi}{2}) \cdot P(\pi + \delta)$ which does not rotate along the z-axis precisely π radians but $\pi + \delta$ radians instead. This kind of imperfection is unavoidable in the implementation of quantum operations. Next we denote by $\mathtt{step}^\epsilon$ the judgement that results from replacing $\mathtt{H}$ in $\mathtt{step}$ by $\mathtt{H}^\epsilon$, and can easily obtain $\mathtt{step} =_\epsilon \mathtt{step}^\epsilon$ via our metric deductive system. Finally we can deduce,

$$\mathtt{seq^n}\ \mathtt{step}\ \dots\ \mathtt{step} =_{n \cdot \epsilon} \mathtt{seq^n}\ \mathtt{step}^\epsilon\ \dots\ \mathtt{step}^\epsilon$$

This last metric equation gives the important message that by closing the distance between $\mathtt{H}$ and $\mathtt{H}^\epsilon$ (with $\mathtt{H}^\epsilon$ corresponding to an imperfect implementation of $\mathtt{H}$) we can also close the distance between an idealised quantum walk and its implementation when the walk is bounded by a specific number of steps. This is of course necessary to be able to *actually execute* a quantum walk that is close to our idealisation of it.

Let us now build an actual model for the theory just introduced. An obvious candidate for the interpretation domain is the category Iso whose objects are natural numbers $n \geq 1$ and morphisms $n \to m$ are isometries $\mathbb{C}^n \to \mathbb{C}^m$. This category is strict symmetric monoidal with the tensor given by the usual tensor product of vector spaces and symmetry given by the isometry $v \otimes w \mapsto w \otimes v$ [HS18]. Moreover the set of isometries $\mathsf{Iso}(n, m)$ can be equipped with the metric induced by the *operator norm* which is defined as,

$$\|T\| = \bigvee \{ \|Tv\| \mid \|v\| = 1 \}$$

for $T : n \to m$ an isometry. Unfortunately, the category Iso has two important issues:

(1) it is not monoidal closed – and although higher-order structure is not frequently used in quantum algorithmics we know that it renders program constructions and deductions more modular (as illustrated with the quantum walk above and $\mathtt{seq^n}$);
(2) the metric induced by the operator norm is too fine-grained for quantum computing. Indeed quantum states should be *indistinguishable up to global phase*, but even so we

can define two isometries $T, S : 1 \to 2$ such that $T1 = |0\rangle$, $S1 = -|0\rangle$ and the distance between $\|T - S\|$ will be at least 2.

So we now recall a formalism for quantum computing that brings forth a certain category that fixes item (2) (item (1) will be handled later on). We will need some preliminaries: a matrix $A \in \mathbb{C}^{n\times n}$ is said to be *positive semi-definite* (or just positive), notation $A \geq 0$, if $\langle v, Av\rangle \geq 0$ for all vectors $v \in \mathbb{C}^n$. A matrix $A \in \mathbb{C}^{n\times n}$ is *Hermitian* if $A = A^\dagger$. Positive matrices are Hermitian [NC02, page 71]. A positive matrix $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Tr}\ A = 1$ is called a *density matrix* or a *density operator*. A mixed quantum state is a convex combination $p_1 \cdot v_1 v_1^\dagger + \cdots + p_n \cdot v_n v_n^\dagger$ of quantum states $v_1, \ldots, v_n \in \mathbb{C}^n$ and such a combination corresponds precisely to a density matrix [NC02]. One usually denotes density matrices by the greek letters ρ, σ and so forth. A density matrix encodes uncertainty about the current state of the quantum system at hand. For example, $\frac{1}{2} \cdot |0\rangle|0\rangle^\dagger + \frac{1}{2} \cdot |1\rangle|1\rangle^\dagger$ tells that the current state is either $|0\rangle$ or $|1\rangle$ with probability $\frac{1}{2}$. Note that global phases disappear in this formalism because given a quantum state $e^{i\theta}v$ we have $(e^{i\theta}v)(e^{i\theta}v)^\dagger = e^{i\theta}\overline{e^{i\theta}}vv^\dagger = vv^\dagger$.

An operator between spaces of matrices $T : \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$ is often called a super-operator. A super-operator $T : \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$ is called *positive* if it sends positive matrices to positive matrices, *i.e.* $A \geq 0 \Rightarrow TA \geq 0$. Since density matrices are positive, it is clearly necessary that a physically allowed transformation be represented by a positive operator. In fact more is needed: since one can always extend the space $\mathbb{C}^{n\times n}$ to a space $\mathbb{C}^{n\times n} \otimes \mathbb{C}^{k\times k}$ by adjoining a new quantum system, any physically allowed transformation $T : \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$ must have the property that $T \otimes \mathsf{id}_{\mathbb{C}^{k\times k}} : \mathbb{C}^{n\times n} \otimes \mathbb{C}^{k\times k} \to \mathbb{C}^{m\times m} \otimes \mathbb{C}^{k\times k}$ is positive. A super-operator T satisfying this condition is called *completely positive*. Finally, a super-operator T is called *trace-preserving* if $\mathrm{Tr}\ TA = \mathrm{Tr}\ A$. Completely positive, trace-preserving super-operators are traditionally called *quantum channels*.

It is straightforward to prove that quantum channels are closed under operator composition and tensoring [Sel04a]. In fact, we have the strict symmetric monoidal category CPTP whose objects are natural numbers $n \geq 1$ and morphisms $n \to m$ are quantum channels $\mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$. Note also the existence of a strict symmetric monoidal functor $r : \mathsf{Iso} \to \mathsf{CPTP}$ (formally a reflection) that behaves as the identity on objects and that sends an isometry T to the mapping $A \mapsto TAT^\dagger$ [HS18].

Our next step is to prove that CPTP is a Met-enriched symmetric monoidal category. For that effect we will require more preliminaries: first a matrix $A \in \mathbb{C}^{n\times n}$ is said to be *normal* if $AA^\dagger = A^\dagger A$. Clearly every Hermitian matrix is normal. Note also that for every matrix $A \in \mathbb{C}^{n\times n}$ the matrix $A^\dagger A$ is Hermitian. Next, it is well-known that by appealing to the spectral theorem [NC02], every normal matrix $A \in \mathbb{C}^{n\times n}$ can be expressed as a linear combination $\sum_i \lambda_i b_i b_i^\dagger$ where the set $\{b_1, \ldots, b_n\}$ is an orthonormal basis of $\mathbb{C}^n$. Using this last result we can extend any function $f : \mathbb{C} \to \mathbb{C}$ to normal matrices via,

$$f(A) = \sum_i f(\lambda_i) b_i b_i^\dagger$$

We then obtain the norm $\|A\|_1 = \mathrm{Tr}\sqrt{A^\dagger A}$ for matrices $A \in \mathbb{C}^{n\times n}$. This norm is called the *trace norm* and is also known as the Schatten 1-norm [Wat18]. The trace norm induces a metric on the set of density matrices which is defined by $d(\rho, \sigma) = \|\rho - \sigma\|_1$. In the sequel we will often treat $vv^\dagger$ as if it were simply v. Next, it is well known that the distance $d(vv^\dagger, uu^\dagger)$ between two quantum states v and u is their *Euclidean distance* in the Bloch sphere [Wat18, NC02]. It is thus easy to see for example that the distance between two quantum states,

$$\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle \text{ and } \cos\frac{\theta}{2}|0\rangle + e^{i(\varphi+\epsilon)}\sin\frac{\theta}{2}|1\rangle$$

tends to 0 when ϵ approaches 0, more formally if $\epsilon_n \to 0$ then $f(\epsilon_n) \to 0$ where,

$$\begin{aligned} f(\epsilon) &= \|(\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta) - (\cos(\varphi+\epsilon)\sin\theta, \sin(\varphi+\epsilon)\sin\theta, \cos\theta)\|_2 \\ &= \|((\cos\varphi - \cos(\varphi+\epsilon))\sin\theta, (\sin\varphi - \sin(\varphi+\epsilon))\sin\theta, 0)\|_2 \\ &= |\sin\theta|\|(\cos\varphi - \cos(\varphi+\epsilon), (\sin\varphi - \sin(\varphi+\epsilon), 0)\|_2 \end{aligned}$$

Using results from topology, we know that this holds because $f(0) = 0$ and moreover f is continuous. The definition of f also tells us that for a fixed ϵ the distance between the two states is maximised when $|\sin\ \theta| = 1$ which corresponds to $\theta = \pm\frac{\pi}{2}$ (both states are located at the equator) and minimised when $|\sin\ \theta| = 0$ which corresponds to $\theta = 0$ or $\theta = \pi$ (both states are located at one of the poles which renders the azimuthal angle irrelevant). We now consider the following norm on super-operators $T : \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$ [Wat18]:

$$\|T\|_1 = \max\ \{\|TA\|_1 \mid \|A\|_1 = 1\}$$

Unfortunately, the norm $\|-\|_1$ on super-operators is not stable under tensoring [Wat18], specifically the inequation $\|T \otimes \mathsf{id}\|_1 \leq \|T\|_1$ does not hold which makes impossible to enrich the monoidal structure of CPTP via this norm. So instead we will use the so-called *diamond norm*, which given a super-operator $\mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$ is defined by,

$$\|T\|_\diamond = \|T \otimes \mathsf{id}_n\|_1$$

Then it follows from [Wat18, Proposition 3.44 and Proposition 3.48] that for all super-operators $T : \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$, $S : \mathbb{C}^{m\times m} \to \mathbb{C}^{o\times o}$ if T is a quantum channel the inequation $\|ST\|_\diamond \leq \|S\|_\diamond$ holds and if S is a quantum channel the inequation $\|ST\|_\diamond \leq \|T\|_\diamond$ also holds. Furthermore, via [Wat18, Corollary 3.47] we have both inequations $\|T\|_\diamond \geq \|T \otimes \mathsf{id}\|_\diamond$ and $\|T\|_\diamond \geq \|\mathsf{id} \otimes T\|_\diamond$ for T a super-operator. As we saw previously with the category Ban these are sufficient conditions to prove that CPTP is a Met-enriched symmetric monoidal category (recall Proposition 3).

The exact calculation of distances induced by $\|-\|_\diamond$ tends to be quite complicated, but a useful property for calculating the distance between quantum channels in the image of $r : \mathsf{Iso} \to \mathsf{CPTP}$ is provided in [Wat18, Theorem 3.55]:

Theorem 4.3. *Consider two isometries $T, S : n \to m$. There exists a unit vector $v \in \mathbb{C}^n$ such that,*

$$\left\| r(T)(vv^\dagger) - r(S)(vv^\dagger) \right\|_1 = \|r(T) - r(S)\|_\diamond$$

As an illustration of the theorem at work, note that every isometry $1 \to n$ corresponds to the initialisation of a quantum state. So for two such isometries T and S we can immediately see that the distance $\|r(T) - r(S)\|_\diamond$ between quantum channels $r(T)$ and $r(S)$ of type $1 \to n$ is precisely $\left\|r(T)(11^\dagger) - r(S)(11^\dagger)\right\|_1$. For example the distance between the r-image of the mapping $1 \mapsto \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$ and the mapping $1 \mapsto \cos\frac{\theta}{2}|0\rangle + e^{i(\varphi+\epsilon)}\sin\frac{\theta}{2}|1\rangle$ must be $f(\epsilon) = |\sin\theta|\|(\cos\varphi - \cos(\varphi+\epsilon), (\sin\varphi - \sin(\varphi+\epsilon), 0)\|_2$ which we already know tends to 0 when ϵ approaches 0. Let us now consider a more complex example, which is closely related to the example above of a quantum random walk.

We already know that the phase operation $P(\phi) : \mathbb{C}^2 \to \mathbb{C}^2$ is an isometry defined by $|0\rangle \mapsto |0\rangle$ and $|1\rangle \mapsto e^{i\phi}|1\rangle$. We also know that the implementation of a phase operation can only approximate its idealisation. For example, the implementation of $P(\phi)$ might be $P(\phi + \epsilon)$ for some error ϵ, meaning that we will not rotate along the z-axis precisely ϕ radians but $\phi + \epsilon$ instead. We will now compute an upper bound for the distance $\|r(P(\phi+\epsilon)) - r(P(\phi))\|_\diamond$ and additionally show that it tends to 0 when ϵ approaches 0. The crucial observation here is that for all quantum states $\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle \in \mathbb{C}^2$ we have,

$$\begin{aligned}
&\left\| r(P(\phi))\left(\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right) - r(P(\phi+\epsilon))\left(\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right)\right\|_1 \\
&= \left\| \left(\cos\frac{\theta}{2}|0\rangle + e^{i(\varphi+\phi)}\sin\frac{\theta}{2}|1\rangle\right) - \left(\cos\frac{\theta}{2}|0\rangle + e^{i(\varphi+\phi+\epsilon)}\sin\frac{\theta}{2}|1\rangle\right)\right\|_1 \\
&= |\sin\theta| \|(\cos(\varphi+\phi) - \cos(\varphi+\phi+\epsilon), \sin(\varphi+\phi) - \sin(\varphi+\phi+\epsilon), 0)\|_2 \\
&\leq \|(\cos(\varphi+\phi) - \cos(\varphi+\phi+\epsilon), \sin(\varphi+\phi) - \sin(\varphi+\phi+\epsilon), 0)\|_2 \\
&\leq \|(K_1\epsilon, K_2\epsilon, 0)\|_2 \\
&\leq \max(K_1, K_2)\|(\epsilon, \epsilon, 0)\|_2
\end{aligned}$$

The penultimate step arises from both functions cos and sin being Lipschitz continuous with K_1 and K_2 the corresponding Lipschitz factors. Then by an application of Thm. 4.3 we obtain,

$$\|r(P(\phi+\epsilon)) - r(P(\phi))\|_\diamond \leq \max(K_1, K_2)\|(\epsilon, \epsilon, 0)\|_2 \tag{4.2}$$

Finally the function defined by $f(\epsilon) = \max(K_1, K_2)\|(\epsilon, \epsilon, 0)\|_2$ clearly respects $f(0) = 0$ and is continuous which entails that the distance $\|r(P(\phi+\epsilon)) - r(P(\phi))\|_\diamond$ tends to 0 as ϵ approaches 0.

Let us now address item (1) which concerns the fact that the category Iso is not monoidal closed. Unfortunately, the category CPTP is also not monoidal closed [Sel04b]. We can however use general results of category theory to overcome this issue: specifically, we can *embed* CPTP into a certain Met-autonomous category $[\mathsf{CPTP}^{\mathsf{op}}, \mathsf{Met}]$ of Met-valued presheaves. Before proceeding we need to recall further aspects of (enriched) category theory.

First it is well-known that CPTP is small and Met is (co)complete. Thus by general results [Kel82, Bor94], we obtain a Met-enriched category $[\mathsf{CPTP}^{\mathsf{op}}, \mathsf{Met}]$ whose objects are Met-*enriched* functors $\mathsf{CPTP}^{\mathsf{op}} \to \mathsf{Met}$ and for two such functors F and G the corresponding hom-object is given by the enriched end formula,

$$[\mathsf{CPTP}^{\mathsf{op}}, \mathsf{Met}](F, G) \cong \int_n \mathsf{Met}(Fn, Gn)$$

We also obtain a Met-enriched Yoneda *embedding* functor $\mathrm{Y} : \mathsf{CPTP} \to [\mathsf{CPTP}^{\mathsf{op}}, \mathsf{Met}]$ that sends an object $n \in \mathsf{CPTP}$ to $\mathsf{CPTP}(-, n) : \mathsf{CPTP}^{\mathsf{op}} \to \mathsf{Met}$ and that sends a quantum channel $T : n \to m$ to the morphism $(T \cdot -) : \mathsf{CPTP}(-, n) \to \mathsf{CPTP}(-, m)$. Let us analyse the distance between two quantum channels T and S when in the form $\mathrm{Y}(T)$, $\mathrm{Y}(S)$. To that effect we will recur to the enriched Yoneda lemma [Kel82] which in our setting establishes an isometry,

$$[\mathsf{CPTP}^{\mathsf{op}}, \mathsf{Met}](\mathsf{CPTP}(-, n), F) \cong Fn$$

for every object $n \in \mathsf{CPTP}$ and Met-enriched functor $F : \mathsf{CPTP}^{\mathsf{op}} \to \mathsf{Met}$. By instantiating F with $\mathsf{CPTP}(-, m)$ we obtain an isometry,

$$[\mathsf{CPTP}^{\mathsf{op}}, \mathsf{Met}](\mathsf{CPTP}(-, n), \mathsf{CPTP}(-, m)) \cong \mathsf{CPTP}(n, m)$$

and therefore the distance between $\mathrm{Y}(T)$ and $\mathrm{Y}(S)$ must be equal to that of T and S in $\mathsf{CPTP}(n, m)$. So the Yoneda embedding $\mathrm{Y} : \mathsf{CPTP} \to [\mathsf{CPTP}^{\mathsf{op}}, \mathsf{Met}]$ indeed faithfully embeds the Met-enriched categorical structure of CPTP into $[\mathsf{CPTP}^{\mathsf{op}}, \mathsf{Met}]$. We can actually do better than this by recurring to Day's work on equipping functor categories with enriched biclosed structures [Day70b, Day70a]. Specifically, by virtue of CPTP being small and Met being (co)complete we can equip $[\mathsf{CPTP}^{\mathsf{op}}, \mathsf{Met}]$ with a Met-enriched symmetric monoidal structure given by Day convolution,

$$F \otimes_D G \cong \int^{n,m} Fn \otimes Gm \otimes \mathsf{CPTP}(-, n \otimes m)$$

Theorem 4.4 ([Day70b, Day70a, IK86]). *The Yoneda embedding functor* $\mathrm{Y} : \mathsf{CPTP} \to [\mathsf{CPTP}^{\mathsf{op}}, \mathsf{Met}]$ *is symmetric strong monoidal.*

In particular, we obtain $\mathrm{Y}(n) \otimes_D \mathrm{Y}(m) \cong \mathrm{Y}(n \otimes m)$, and when $\mathrm{Y}(n) \otimes_D \mathrm{Y}(m)$ is seen in the form $\mathrm{Y}(n \otimes m)$ the swap operation corresponds to $\mathrm{Y}(\mathsf{sw}) : \mathrm{Y}(n \otimes m) \to \mathrm{Y}(m \otimes n)$. Finally given two Met-enriched functors $F, G : \mathsf{CPTP}^{\mathsf{op}} \to \mathsf{Met}$ their exponential is defined as,

$$F \multimap_D G \cong \int_n \mathsf{Met}(Fn, G(- \otimes n))$$

This category of enriched presheaves thus overcomes issues (1) and (2) discussed above. We also obtain the sequence of symmetric strong monoidal functors,

$$\mathsf{Iso} \xrightarrow{r} \mathsf{CPTP} \xhookrightarrow{\mathrm{Y}} [\mathsf{CPTP}^{\mathsf{op}}, \mathsf{Met}]$$

We are finally ready to build a model for the metric theory of a quantum random walk presented above. Recall that the class of ground types is $\{\mathtt{qbit}, \mathtt{pos}\}$. We interpret $[\![\mathtt{qbit}]\!] = \mathrm{Y}r(2)$ and $[\![\mathtt{pos}]\!] = \mathrm{Y}r(n)$ where we allow n to be any natural number $n \geq 1$. Then recall that the class of operation symbols is given by $\{\mathtt{H} : \mathtt{qbit} \to \mathtt{qbit},\ \mathtt{H}^{\epsilon} : \mathtt{qbit} \to \mathtt{qbit},\ \mathtt{S} : \mathtt{qbit} \otimes \mathtt{pos} \to \mathtt{qbit} \otimes \mathtt{pos}\}$. We interpret $[\![\mathtt{H}]\!] = \mathrm{Y}r(R_y(\frac{\pi}{2}) \cdot P(\phi))$ and $[\![\mathtt{H}^{\epsilon}]\!] = \mathrm{Y}r(R_y(\frac{\pi}{2}) \cdot P(\phi + \delta))$ where we allow δ to be any non-negative real number such that $\max(K_1, K_2)\|(\delta, \delta, 0)\|_2 \leq \epsilon$. Then recall that for any finite set $\{0, \ldots, n-1\}$ we have the operations increment $\oplus\ 1$ and decrement $\ominus\ 1$ modulo n. This gives rise to the isometry $S : 2 \otimes n \to 2 \otimes n$,

$$S(|0\rangle \otimes |i\rangle) = |0\rangle \otimes |i \ominus 1\rangle \qquad S(|1\rangle \otimes |i\rangle) = |1\rangle \otimes |i \oplus 1\rangle$$

(it is an isometry because it is a permutation of the canonical basis of $\mathbb{C}^2 \otimes \mathbb{C}^n$). Intuitively, S uses the left qubit as control to either move to the left or to the right in the circle. We then interpret $[\![\mathtt{S}]\!] = \mathrm{Y}r(S)$. The only thing that remains to prove is that the axiom $x : \mathtt{qbit} \rhd \mathtt{H}(x) =_{\epsilon} \mathtt{H}^{\epsilon}(x) : \mathtt{qbit}$ propounded above is sound in this interpretation. So we reason,

$$
\begin{aligned}
& a\left(\llbracket \mathtt{H}(x) \rrbracket, \llbracket \mathtt{H}^{\epsilon}(x) \rrbracket\right) \\
& = a\left(\mathrm{Y}r(R_y(\tfrac{\pi}{2}) \cdot P(\phi)), \mathrm{Y}r(R_y(\tfrac{\pi}{2}) \cdot P(\phi + \delta))\right) \\
& = a\left(r(R_y(\tfrac{\pi}{2})) \cdot r(P(\phi)), r(R_y(\tfrac{\pi}{2})) \cdot r(P(\phi + \delta))\right) && \{\text{Enriched Yoneda}\} \\
& \leq a\left(r(P(\phi)), r(P(\phi + \delta))\right) && \{\mathsf{CPTP} \text{ is } \mathsf{Met}\text{-enriched}\} \\
& \leq \max(K_1, K_2) \|(\delta, \delta, 0)\|_2 \\
& \leq \epsilon
\end{aligned}
$$

We have therefore obtained a concrete higher-order model for our idealised quantum walk, its approximation, and the fact that when ϵ tends to 0 the distance between both walks tends to 0 as well when bounded by a specific number of steps.

5. Concluding notes

We end the paper with three brief notes concerning our work. The first one discusses the extension of our results from the linear to the so-called *affine* setting. The second one establishes a functorial connection between the categorical semantics of $\mathcal{V}\lambda$-calculus (presented in §3) and previous work on *algebraic semantics* of linear logic [DP99]. Finally the third provides a brief exposition of future work.

5.1. **From linear to affine.** The results we have obtained in § 3 incide on the *linear* version of λ-calculus, which does not admit the weakening rule and thus renders impossible to freely discard variables when building a judgement. In this subsection we introduce an *affine* variant of λ-calculus, *i.e.* an extension of the linear version that admits a weakening rule. We will see that all results that we have established thus far for linear λ-calculus and its $\mathcal{V}$-equational system can be easily extended to the affine variant.

First, the grammar of types for affine λ-calculus is defined as in the linear case. Then regarding judgement derivation, it is tempting to formalise the affine nature of the calculus merely by adding the weakening rule,

$$\frac{\Gamma \rhd v : \mathbb{B}}{\Gamma, x : \mathbb{A} \rhd v : \mathbb{B}}\ (\textbf{weakening})$$

to those presented in Fig. 1. This indeed allows to discard x in the construction of v, but it also breaks the 'unique derivation' principle that was previously discussed (recall §2). For example, the simple judgement $x : \mathbb{A}, y : \mathbb{B} \rhd f(x) : \mathbb{C}$ with $f : \mathbb{A} \to \mathbb{C}$ can now be derived in at least two different ways:

$$\frac{\dfrac{x : \mathbb{A} \rhd x : \mathbb{A}}{x : \mathbb{A} \rhd f(x) : \mathbb{C}}\ (\textbf{ax})}{x : \mathbb{A}, y : \mathbb{B} \rhd f(x) : \mathbb{C}}\ (\textbf{weakening}) \qquad\qquad \frac{\dfrac{x : \mathbb{A} \rhd x : \mathbb{A}}{x : \mathbb{A}, y : \mathbb{B} \rhd x : \mathbb{A}}\ (\textbf{weakening})}{x : \mathbb{A}, y : \mathbb{B} \rhd f(x) : \mathbb{C}}\ (\textbf{ax})$$

Our approach instead is to add the term formation rule in Fig. 5 to those in Fig. 1. This rule marks a term v as *discardable*, which can then be effectively discarded via Rule ($\mathbb{I}_{\mathbf{e}}$). We thus obtain a weakening rule,

$$\frac{\Gamma \rhd v : \mathbb{B}}{\Gamma, x : \mathbb{A} \rhd \mathtt{dis}(x)\ \mathtt{to}\ *\, .\ v : \mathbb{B}}$$

Substitution is defined in the expected way and like in §2 we obtain the following result.

Term formation rule	Semantics rule
$\dfrac{\Gamma \rhd v : \mathbb{A}}{\Gamma \rhd \mathtt{dis}(v) : \mathbb{I}}$ (**discardable**)	$\dfrac{[\![\Gamma \rhd v : \mathbb{A}]\!] = f}{[\![\Gamma \rhd \mathtt{dis}(v) : \mathbb{I}]\!] = !_{[\![\mathbb{A}]\!]} \cdot f}$
Equations	
$\mathtt{dis}(v) = \mathtt{dis}(x_1)\ \mathtt{to}\ *.\ \ldots\ \mathtt{dis}(x_n)\ \mathtt{to}\ *.\ *$ $v\ \mathtt{to}\ *.\ * = \mathtt{dis}(v)\ \mathtt{to}\ *.\ *$	

FIGURE 5. Additional data for affine λ-calculus.

Theorem 5.1. *The affine λ-calculus defined above enjoys the following properties:*
(1) *(Unique typing) For any two judgements $\Gamma \rhd v : \mathbb{A}$ and $\Gamma \rhd v : \mathbb{A}'$, we have $\mathbb{A} = \mathbb{A}'$;*
(2) *(Unique derivation) Every judgement $\Gamma \rhd v : \mathbb{A}$ has a unique derivation;*
(3) *(Exchange) For every judgement $\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \rhd v : \mathbb{C}$ we can derive $\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \rhd v : \mathbb{C}$;*
(4) *(Substitution) For all judgements $\Gamma, x : \mathbb{A} \rhd v : \mathbb{B}$ and $\Delta \rhd w : \mathbb{A}$ we can derive $\Gamma, \Delta \rhd v[w/x] : \mathbb{B}$.*

Proof. The proof is analogous to that of Thm. 2.1. □

Our next step is to interpret the rule (**discardable**) in a sensible way. To that effect we move from the setting of autonomous categories to that of *semi-Cartesian autonomous categories*, or more specifically autonomous categories whose unit object I is terminal. In order to not overburden nomenclature, let us call such categories affine. Next, given an affine category C and a C-object X denote by $!_X = X \to I$ the terminal map to the unit object. We then add to the rules of Fig. 2 (which define the interpretation of linear λ-calculus on autonomous categories) the rule for the interpretation of the `dis` construct, given in the second column of Fig. 5.

We now focus on equipping affine λ-calculus with an equational system. Recall the axiomatics of autonomous categories which was provided in Fig. 3. We extend it with the equations in the bottom line of Fig. 5. Before proving that this extension is sound w.r.t. ($\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-)affine categories, let us analyse the two equational schema just introduced. Intuitively, the first equation states that making a judgement $x_1 : \mathbb{A}_1, \ldots, x_n : \mathbb{A}_n \rhd v : \mathbb{A}$ discardable is equivalent to making no use of any variable in its context. The second equation states that if a judgement $\Gamma \rhd v : \mathbb{I}$ is already discardable (*i.e.* it has type $\mathbb{I}$) then marking it as such is redundant. It is clear that both equations must be sound in affine categories, for they only involve judgements of type $\mathbb{I}$ which is always interpreted as a terminal object.

Definition 5.2 (Affine $\mathcal{V}\lambda$-theories and their models)**.** Consider a tuple (G, Σ) consisting of a class G of ground types and a class of sorted operation symbols $f : \mathbb{A}_1, \ldots, \mathbb{A}_n \to \mathbb{A}$ with $n \geq 1$. An affine $\mathcal{V}\lambda$-theory $((G, \Sigma), Ax)$ is a tuple such that Ax is a class of $\mathcal{V}$-equations-in-context over *affine* λ-terms built from (G, Σ).

Consider an affine $\mathcal{V}\lambda$-theory $((G, \Sigma), Ax)$ and a $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-enriched *affine* category C. Suppose that for each $X \in G$ we have an interpretation $[\![X]\!]$ as a C-object and analogously for the operation symbols. This interpretation structure is a model of the theory if all axioms in Ax are satisfied by the interpretation.

Let $Th(Ax)$ be the smallest class that contains Ax, that is closed under the rules of Fig. 3, bottom line of Fig. 5, and of Fig. 4. The elements of $Th(Ax)$ are called theorems of the theory. Note that we do not include a *compatibility rule*,

$$\frac{v =_q w}{\mathtt{dis}(v) =_q \mathtt{dis}(w)}$$

in the definition of $Th(Ax)$. The reason is that this rule is already derivable from the present $\mathcal{V}$-equational system, as we show next. For any two judgements $\Gamma \rhd v : \mathbb{A}$ and $\Gamma \rhd w : \mathbb{A}$ both cases $\mathtt{dis}(v)$ and $\mathtt{dis}(w)$ can be equated to $\mathtt{dis}(x_1)\ \mathtt{to}\ *.\ \ldots\ \mathtt{dis}(x_n)\ \mathtt{to}\ *.\ *$. This gives us the $\mathcal{V}$-equation $\mathtt{dis}(v) =_\top \mathtt{dis}(w)$. Then since $q \leq \top$ we obtain $\mathtt{dis}(v) =_q \mathtt{dis}(w)$ via Rule (**weak**).

Theorem 5.3 (Soundness and Completeness)**.** *Consider an affine $\mathcal{V}\lambda$-theory $\mathscr{T}$. An equation in context $\Gamma \rhd v =_q w : \mathbb{A}$ is a theorem of $\mathscr{T}$ iff it is satisfied by all models of the theory.*

Proof. The proof of soundness is analogous to that of Thm. 3.16. The proof of completeness is also analogous to that of Thm. 3.16, but with the important difference that we need to show that the generated syntactic category $\mathrm{Syn}(\mathscr{T})$ is affine. So let us consider two judgements $x : \mathbb{A} \rhd v : \mathbb{I}$ and $x : \mathbb{A} \rhd w : \mathbb{I}$ and reason in the following way:

$$\begin{aligned}
&v \\
&= v\ \mathtt{to}\ *.\ * && \{v\ \mathtt{to}\ *.\ w[*/z] = w[v/z]\ \text{(Fig. 3)}\} \\
&= \mathtt{dis}(v)\ \mathtt{to}\ *.\ * \\
&= \mathtt{dis}(w)\ \mathtt{to}\ *.\ * && \{\text{Previous remark on compatibility of } \mathtt{dis}\} \\
&= w\ \mathtt{to}\ *.\ * \\
&= w
\end{aligned}$$

This entails that $[v] = [w]$ by the definition of a syntatic category. $\square$

All other results that we have established for linear $\mathcal{V}\lambda$-theories (most notably, the equivalence theorem presented in Thm. 3.21) can be obtained for the affine case as well, simply by retracing the path walked in §3.5. So we omit the repetition of such steps here.

To conclude this subsection, let us briefly illustrate the affine $\mathcal{V}$-equational system at work in relation to the linear case. Observe that the equations $w =_q w'$ and $u =_r u'$ between *linear* λ-terms give rise to,

$$(\lambda x.\ \lambda y.\ v)\ w\ u =_{q \otimes r} (\lambda x.\ \lambda y.\ v)\ w'\ u'$$

which intuitively states that the 'differences' q and r between the input terms w, w' and u, u' compound when applied to the same operation. Now let us see what happens when the term $(\lambda x.\ \lambda y.\ v)$ does not use one of the arguments, for instance it discards x (which should not happen in the linear case). By taking advantage of Rule **(trans)**, the fact that $k = \top$, and of equation $\mathtt{dis}(w) = \mathtt{dis}(w')$ (discussed above), we logically deduce:

$$\begin{aligned}
&(\lambda x.\ \lambda y.\ \mathtt{dis}(x)\ \mathtt{to}\ *.\ v)\ w\ u \\
&= (\lambda y.\ \mathtt{dis}(w)\ \mathtt{to}\ *.\ v)\ u \\
&= (\lambda y.\ \mathtt{dis}(w')\ \mathtt{to}\ *.\ v)\ u \\
&=_r (\lambda y.\ \mathtt{dis}(w')\ \mathtt{to}\ *.\ v)\ u' \\
&= (\lambda x.\ \lambda y.\ \mathtt{dis}(x)\ \mathtt{to}\ *.\ v)\ w'\ u'
\end{aligned}$$

Note that the value q is now *not* accounted for in the relation between the λ-terms $(\lambda x.\ \lambda y.\ \mathtt{dis}(x)\mathtt{to}*\ .\ v)\ w\ u$ and $(\lambda x.\ \lambda y.\ \mathtt{dis}(x)\ \mathtt{to}\ *\ .\ v)\ w'\ u'$; as a consequence of w and w' being discarded by the operation. This is a manifestation of the available interplay between discarding resources and $\mathcal{V}$-equations.

5.2. **Functorial connection to previous work.** Let us introduce a simple yet instructive functorial connection between (1) the categorical semantics of linear λ-calculus with the $\mathcal{V}$-equational system, (2) the categorical semantics of linear λ-calculus with the equational system of §2, and (3) the *algebraic semantics* of the exponential free, multiplicative fragment of linear logic. First we need to recall some well-known facts. As detailed before, typical categorical models of linear λ-calculus and its classical equational system are locally small autonomous categories, *i.e.* Set-enriched autonomous categories. The latter form the quasi-category $\mathsf{Set}\text{-}\mathsf{Aut}$ whose morphisms are autonomous functors. Then the usual algebraic models of the exponential free, multiplicative fragment of linear logic are the so-called *lineales* [DP99]. In a nutshell, a lineale is a poset $(X, \leq)$ paired with a commutative monoid operation $\otimes : X \times X \to X$ that satisfies certain conditions. Lineales are almost quantales: the only difference is that they do not require X to be cocomplete. The key idea in algebraic semantics is that the order $\leq$ in the lineale encodes the logic's entailment relation.

A functorial connection between Set-autonomous categories and lineales (*i.e.* between (2) and (3)) is already stated in [DP99] and is based on the following two observations. First, (possibly large) lineales can be seen as *thin* autonomous categories, which are equivalently $\{0 \leq 1\}$-enriched autonomous categories. Let us use $\{0 \leq 1\}\text{-}\mathsf{Aut}$ denote the quasi-category of $\{0 \leq 1\}$-enriched autonomous categories and $\{0 \leq 1\}$-enriched autonomous functors. Second, the inclusion functor $\{0 \leq 1\}\text{-}\mathsf{Aut} \hookrightarrow \mathsf{Set}\text{-}\mathsf{Aut}$ has a left adjoint which *collapses all morphisms* of a given autonomous category C (intuitively, it eliminates the ability of C to differentiate different terms between two types). This provides an adjoint situation between (2) and (3).

We can then expand the connection just described to our categorical semantics of linear λ-calculus and corresponding $\mathcal{V}$-equational system (*i.e.* (1)) in the following way: the forgetful functor $\mathcal{V}\text{-}\mathsf{Cat} \to \mathsf{Set}$ has a left adjoint $D : \mathsf{Set} \to \mathcal{V}\text{-}\mathsf{Cat}$ which sends a set X to $DX = (X, d)$, where $d(x_1, x_2) = k$ if $x_1 = x_2$ and $d(x_1, x_2) = \perp$ otherwise. This left adjoint is (strict) monoidal, specifically we have $D(X_1 \times X_2) = DX_1 \otimes DX_2$ and $I = (1, (*, *) \mapsto k) = D1$. This gives rise to the change-of-base functors on the left side of the diagram,

$$\mathcal{V}\text{-}\mathsf{Cat}\text{-}\mathsf{Aut} \underset{\longrightarrow}{\overset{\hat{D}}{\underset{\perp}{\longleftarrow}}} \mathsf{Set}\text{-}\mathsf{Aut} \underset{\longleftarrow}{\overset{C}{\underset{\perp}{\longrightarrow}}} \{0 \leq 1\}\text{-}\mathsf{Aut}$$

The functor $\hat{D}$ equips the hom-sets of a Set-autonomous category with the corresponding discrete $\mathcal{V}$-category and C collapses all morphisms of a Set-autonomous category as described earlier. The right adjoint of $\hat{D}$ forgets the $\mathcal{V}$-categorical structure between terms (*i.e.* between morphisms) and the right adjoint of C is the inclusion functor mentioned earlier. Note that $\hat{D}$ restricts to $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sep}}\text{-}\mathsf{Aut}$ and $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym,sep}}\text{-}\mathsf{Aut}$, and thus we obtain a functorial connection between the categorical semantics of linear λ-calculus with the $\mathcal{V}$-equational system (*i.e.* (1)), (2), and (3). In essence, the connection formalises the fact that our categorical models admit a richer structure between terms (*i.e.* morphisms) than the categorical models of linear λ-calculus and its classical equational system. The latter in turn permits the existence of different terms between two types as opposed to the algebraic semantics of the exponential free, multiplicative fragment of linear logic. The connection also shows that models for (2) and (3) can be mapped into models of our categorical semantics by equipping the respective hom-sets with a trivial, discrete structure.

5.3. **Future work.** We introduced the notion of a $\mathcal{V}$-equation which generalises the notions of equation, inequation [KV17, AFMS20], and metric equation [MPP16, MPP17]. We then presented a sound and complete $\mathcal{V}$-equational system, illustrated with different examples concerning real-time, probabilistic, and quantum computing. We additionally showed that linear $\mathcal{V}\lambda$-theories are the syntactic counterpart of $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sep}}$-enriched autonomous categories, a connection that allows to seamlessly invoke both logical and categorical constructs when studying any of these structures.

Linear λ-calculus is at the root of different ramifications of λ-calculus that relax resource-based conditions in different ways. Here we already presented the affine case which allows to discard resources. We are now studying the possibility of extending our results to *mixed linear-non-linear* calculus [Ben94].

Next, our main examples of $\mathcal{V}\lambda$-theories (see §4) used either the Boolean or the metric quantale. We would like to study linear $\mathcal{V}\lambda$-theories whose underlying quantales are neither the Boolean nor the metric one, for example the ultrametric quantale which is (tacitly) used to interpret Nakano's guarded λ-calculus [BSS10] and also to interpret a higher-order language for functional reactive programming [KB11]. Another interesting quantale is the Gödel one which is a basis for fuzzy logic [DEW13] and whose $\mathcal{V}$-equations give rise to what we call fuzzy inequations.

Finally we plan to further explore the connections between our work and different results on metric universal algebra [MPP16, MPP17, Ros20] and inequational universal algebra [KV17, AFMS20, Ros20]. For example, an interesting connection is that the monad construction presented in [MPP16] crucially relies on quotienting a pseudometric space into a metric space – this is a particular case of quotienting a $\mathcal{V}$-category into a separated $\mathcal{V}$-category which we also use in our work.

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