Paraconsistent Reactive Graphs

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Abstract. This paper introduces *Paraconsistent Reactive Graphs*, as an extension of Reactive graphs that incorporates paraconsistency into the ground edges to address vagueness and inconsistency within dynamic systems. By assigning pairs of truth values to ground edges, this framework captures the uncertainty and contradictions stemming from incomplete or conflicting information. We explore the semantics of these graphs and provide a practical example to illustrate the proposed approach.

Keywords: Reactive systems · Paraconsistency · Contradictions

1 Introduction

Dov Gabbay introduced *reactive graphs* as structures where the accessibility relation can be modified by prior transitions [15]. This dynamic behavior is achieved through higher-order edges, or hyper-edges, which update the accessibility relation when traversed. Hyper-edges connect one edge to another, acting as switches that either activate ($\twoheadrightarrow$) or deactivate ($\rightarrow\!\!\!\times$) the connected edge [14]. Although reactive graphs are a powerful tool, it is natural to seek extensions that allow the representation of information that may not always adhere to classical reasoning. Such extensions are particularly relevant in nuanced applications where certain factors may reinforce or contradict one another.

A fuzzy approach to reactive graphs, initially limited to activating higher-order edges, was proposed in [17], where both ground and higher-order edges could take values in the closed interval $[0, 1]$. This approach also introduced a modal logic to verify properties of such systems and demonstrated its potential application through an example in biology. Subsequently, this framework was extended in [4,5] to allow hyper-edges to act as switches, either activating or deactivating edges.

We propose a paraconsistent extension to this framework by incorporating inconsistency and vagueness into the ground edges of the graph, while hyper-edges remain bivalent—they either activate or deactivate other edges. Similarly to fuzzy reasoning [17], which uses a spectrum of values between 0 and 1 to measure the certainty of an event's occurrence, our paraconsistent framework assigns two truth values that gauge certainty in contrasting ways. Unlike fuzzy

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reasoning, which only addresses vagueness, our framework accommodates both vague and inconsistent information by using a pair of truth values to measure the certainty of occurrence and non-occurrence of an event. Vagueness arises when the sum of these values is less than the unit, indicating incomplete information. Inconsistency, on the other hand, is reflected when the sum exceeds the unit, suggesting conflicting information from multiple sources. For further details on this framework, see previous work [13].

While fuzzy approaches allow the representation of vague information, our paraconsistent approach extends this capability to encompass both vague and inconsistent information. Other paraconsistent approaches, such as [6], introduce paraconsistency into reactive graphs by employing four-valued propositional variables. These values represent truth, falsehood, absolute contradiction, and a complete lack of information. Furthermore, [6] proposes a paraconsistent logic and explores additional topics, such as measuring inconsistencies, as seen in [7,8]. In contrast, our work maintains the binary switching nature of hyperedges but incorporates paraconsistency into the ground edges, allowing them to exhibit both vague and inconsistent information.

Introducing reactivity to paraconsistent models is a natural approach, as contradictions often stem from different sources, beliefs, or even lies, leading to an ever-evolving model as more information is received. This paper introduces *Paraconsistent Reactive Graphs*, their semantics, and a concrete application example.

2 Paraconsistent Reactive Graphs

2.1 Paraconsistent Graphs

Before introducing Paraconsistent Graphs, we first establish the underlying framework. Similar to research on many-valued Kripke frames [2], we use a specific class of residuated lattices $\mathbf{A}$ over a set A representing the set of possible truth values.

We focus on Heyting algebras [1], structures $\mathbf{A} = \langle A, \wedge, \vee, 1, 0, \rightarrow\rangle$ where $\wedge$ has $\rightarrow$ as its residuum. Examples of Heyting algebras include the Boolean algebra $\mathbf{2} = \langle 0, 1, \wedge, \vee, 1, 0, \rightarrow\rangle$ and the Gödel algebra $\mathbf{G} = \langle [0, 1], \min, \max, 0, 1, \rightarrow_G\rangle$, with $a \rightarrow b = \left\{\begin{smallmatrix} 1 & \text{if } a \leq b \\ b & \text{otherwise} \end{smallmatrix}\right.$. In our paraconsistent framework, events are represented as pairs of weights $(tt, ff) \in A \times A$, with tt measuring the evidence of occurrence and ff the evidence of not occurrence.

The product of $\mathbf{A}$ and its order dual $\mathbf{A}^\partial$ forms a residuated lattice known as a twist-structure [3], which will be discussed in the concluding remarks (Sect. 3). This structure enables the joint computation of pairs $A \times A$, which is crucial for this framework (see [11,12]).

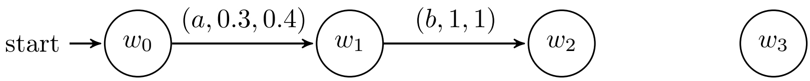
Definition 1. *[10] Given an Heyting algebra* $\mathbf{A}$ *over a set* A*, an* $\mathbf{A}$-twist structure $\mathcal{A} = \langle A^2, \hat{\wedge}, \check{\vee}, /\!\!/, (1, 0), (0, 1)\rangle$ *is defined for any pair* A^2 *as:*

$$/\!\!/\,(a, b) = (b, a) \qquad (a, b) \hat{\wedge} (c, d) = (a \wedge c, b \vee d) \qquad (a, b) \check{\vee} (c, d) = (a \vee c, b \wedge d)$$

Finally, we present the definition of a Paraconsistent Graph whose transitions are labeled by a pair of weights taken from a set A of possible truth values.

Definition 2. *Let* $\mathbf{A}$ *be a Heyting algebra over a set* A *and* Act *a set of actions. An* (A, Act)*-Paraconsistent transition system is a tuple* $M = (W, w_0, R)$ *where* W *is a non-empty set of states,* $w_0 \in W$ *is the initial state and* $R \subseteq W \times \text{Act} \times \mathbf{A}^2 \times W$ *is a* paraconsistent relation*, i.e., if* $(w, a, tt, ff, w') \in R$ *then,* tt *measures the evidence of the transition from* w *to* w' *occurring via action* a *and* ff *measures the evidence of the transition being prevented from occurring.*

Example 1. The structure $(\{w_0, w_1, w_3\}, w_0, R)$ depicted below is a $(\mathbf{G}, \{a, b\})$-paraconsistent graph.



The pair of weights $(0.3, 0.4)$ represents vague information, while $(1, 1)$ represents contradictory information. Additionally, transitions that are consistently known not to occur, such as $(w_2, a, 0, 1, w_3)$, are omitted from the diagram.

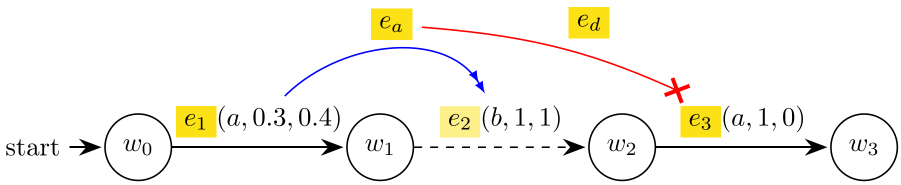
2.2 Paraconsistent Reactive Graphs

The definitions in this Section are adapted from recent work on reactive graphs [18]. We introduce *Paraconsistent Reactive Graphs*, which extend Paraconsistent Graphs by embedding hyper-edges that activate ($\twoheadrightarrow$) or deactivate ($\rightarrow\!\!\times$) other edges. Although not traversable, these hyper-edges capture the model's reactive behavior by showing how edges change when a ground edge ($\rightarrow$) is traversed. We then present the semantics and a concrete example of these graphs.

Definition 3. *Let* $\mathbf{A}$ *be an Heyting algebra and* Act *a set of actions. A* (A, Act)-Paraconsistent Reactive Graph *is a tuple* $M = (W, w_0, \alpha_0, E, \rightarrow, \twoheadrightarrow, \rightarrow\!\!\times, \bar{\cdot})$ *where:*

- W *is a set of states and* $w_0 \in W$ *is the* initial state*;*
- E *is the set of all edge names and* $\alpha_0 \subseteq E$ *is the set of* initially active edges*;*
- $\rightarrow \subseteq W \times Act \times \mathbf{A}^2 \times W$ *is the set of ground edges,* $\twoheadrightarrow \subseteq E \times E$ *is the set of activating edges and* $\rightarrow\!\!\times \subseteq E \times E$ *is the set of deactivating edges;*
- $\bar{\cdot} : E \longrightarrow (\rightarrow \cup \twoheadrightarrow \cup \rightarrow\!\!\times)$ *is a bijective function that maps edges in* E *to their internal details*

Example 2. The following model is a $([0, 1], \{a, b\})$-Paraconsistent Reactive graph with $E = \{e_1, e_2, e_3, e_a, e_d\}$ and initial active edges $\alpha_0 = E \setminus \{e_2\}$. To illustrate $\bar{\cdot}$, for the ground edge e_3 and the hyper edge e_d the function takes values $(w_2, a, 1, 0, w_3)$ and (e_a, e_3), respectively.



Definition 4. *Given a* (A, Act)*-Paraconsistent Reactive Graph with a set of active edges* $\alpha \subseteq E$ *and an edge* $e \in E$*. The set of edges activated by* e *(resp. deactivated by* e*), written* $\mathsf{on}(e, \alpha)$ *(resp.*$\mathsf{off}(e, \alpha)$*) are defined as follows.*

$$\begin{aligned}
\mathsf{from}(e_s) =& \{e \mid \exists e_t \cdot \overline{e} = (e_s, e_t)\} \\
\mathsf{from}^*(e, \alpha) =& \textstyle\bigcup_{r \in (\mathsf{from}(e) \cap \alpha)} \mathsf{from}^*(r, \alpha \backslash \{e\}) \cup \{r\} \\
\mathsf{on}(e, \alpha) =& \{e_t \mid e_{trg} \in \mathsf{from}^*(e, \alpha) \wedge \exists e_s \cdot \overline{e_{trg}} = (e_s, e_t) \in \twoheadrightarrow\} \\
\mathsf{off}(e, \alpha) =& \{e_t \mid e_{trg} \in \mathsf{from}^*(e, \alpha) \wedge \exists e_s \cdot \overline{e_{trg}} = (e_s, e_t) \in \rightarrow\!\!\times\}
\end{aligned}$$

Intuitively, $\mathsf{from}(e_s)$ returns the hyper-edges originating from e_s. from^* recursively traverses from to gather all (active) hyper-edges triggered from a given edge. Additionally, $\mathsf{on}(e, \alpha)$ (resp. $\mathsf{off}(e, \alpha)$) collect all the targets triggered from e by an activating (resp. deactivating) edge.

Example 3. Recall the Paraconsistent Reactive graph in Ex. 2. The edge e_1, represents a transition from state w_0 to w_1 via action a with occurrence and non-occurrence evidence of 0.3 and 0.4, respectively, that triggers hyper-edges e_a and e_d, which activate e_2 and deactivate e_3, respectively. Thus, $\mathsf{from}(e_1) = \{e_a\}$ returns the hyper edges that start from e_1, $\mathsf{from}^*(e_1, \alpha_0) = \{e_a, e_d\}$ returns the hyper edges triggered by e_1, $\mathsf{on}(e_1, \alpha_0) = \{e_2\}$ and $\mathsf{off}(e_1, \alpha_0) = \{e_3\}$ return the edges triggered from e_1 by activation and deactivation, respectively.

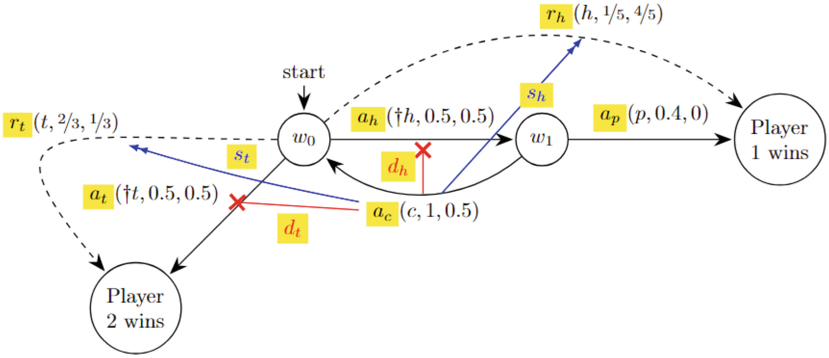
Definition 5. *The semantics of a* (A, Act)*-Paraconsistent Reactive Graph* P *is given by evolving a configuration* $\langle w, \alpha\rangle$*, with* $w \in W$ *and* $\alpha \subseteq E$*, starting from* $\langle w_0, E_0\rangle$*, as defined by the following rule.*

$$\frac{\exists e \in \alpha \;\cdot\; \overline{e} = (w, a, tt, ff, w') \;\wedge\; \alpha' = \big(\alpha \cup \mathsf{on}(e, \alpha)\big) \backslash \mathsf{off}(e, \alpha)}{\langle w, \alpha\rangle \xrightarrow{(a, tt, ff)}_P \langle w', \alpha'\rangle}$$

Applying the semantics above to Ex. 2 yields the Paraconsistent Graph in Ex. 1.

Example 4. (Liar's Coin) Two players are engaged in a game called *Liar's Coin*, which is adapted from work on epistemic logic [16]. Player 1 bets on heads, while Player 2 bets on tails. Player 1 tosses a coin and announces either heads or tails, but keeps the actual outcome hidden from Player 2. If Player 1 announces tails

($\dagger t$), Player 2 wins, and the game ends. If Player 1 announces heads ($\dagger h$), Player 2 has two options: pass (p)—in which case Player 1 wins—or challenge (c), forcing Player 1 to show the coin (h or t). The player with the correct bet then wins. From Player 2's perspective, the game can be represented by the following model.



Initially, Player 2's belief about the outcome (heads or tails) mirrors the probability of a fair coin toss (0.5 for each). When deciding whether to pass or challenge, Player 2 can rely on various vague indicators—such as Player 1's behavior when lying or knowledge from past rounds—to assess how confident they are about their decision. If Player 2 challenges, the game resets: the announcement edges (a_h and a_t) are deactivated, and the reveal edges (r_h and r_t) are activated, with their weights reflecting Player 1's likelihood of lying.

3 Conclusions

This work is part of a research agenda on paraconsistent transition structures [9,11–13]. We combine this agenda with Reactive Graphs [15] by allowing ground edges to represent consistent, vague, or inconsistent information. Consequently, we can model paraconsistent systems where beliefs, lies, incomplete information, and contradictions arise naturally and evolve as the system acquires more information.

Future research will focus on replacing bivalent hyper-edges with pairs of weights for each hyper-edge that convey evidence for activation or deactivation. These paraconsistent hyper-edges may represent different sources, with varying trust levels that may sometimes be even doubtful. As with fuzzy systems [5,17], we anticipate that this will require incorporating aggregation functions based on operators from the twist-structure. Potential applications include adjusting the impact of hyper-edges on the system based on the degree of vagueness or inconsistency and prioritizing hyper-edges with fewer inconsistencies or greater overall certainty.

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