Paraconsistent transition structures: compositional principles and a modal logic *

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Abstract

Often in Software Engineering a modelling formalism has to support scenarios of inconsistency in which several requirements either reinforce or contradict each other. Paraconsistent transition systems are proposed in this paper as one such formalism: states evolve through two accessibility relations capturing weighted evidence of a transition or its absence, respectively. Their weights come, parametrically, from a residuated lattice. This paper explores both i) a category of these systems, and the corresponding compositional operators, and ii) a modal logic to reason upon them. Futhermore, two notions of *crisp* and *graded* simulation and bisimulation are introduced in order to relate two paraconsistent transition systems. Finally, results of modal invariance, for specific subsets of formulas, are discussed across them.

1 Introduction

In classical bivalent logic, propositions are ascribed to exactly one truth value: true or false. While this binary framework has been indispensable in numerous mathematical and logical applications, the inherent simplicity and rigidity may fall short in capturing the intricacies present in real-world scenarios. For instance, in Software Engineering it is common to encounter application scenarios where requirements either reinforce or contradict each other. One such scenario comes from current practice in quantum computation in the context of NISQ (*Noisy Intermediate-Scale Quantum*) technology [Pre18] in which levels of decoherence of quantum memory need to be articulated with the length of the circuits to assess program quality. Similar challenges emerge in AI applications and data engineering.

The foundation of our work lies in a recent paper [CMB22a], where the authors introduced a weighted transition systems that records, for each transition,

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a pair of weights, a positive and a negative weight. Informally, the positive weight captures the degree of effectiveness (*'presence'*) and the negative weight captures the degree of impossibility (*'absence'*) of a transition. This allows the models to capture both *vagueness*, whenever both weights sum less than 1, as usual e.g. in fuzzy systems, and *inconsitency*, when their sum exceeds 1. This last feature motivates the qualifier *paraconsistent* borrowed from the work on paraconsistent logic [Jaś69, dCKB07], which accommodates inconsistency in a controlled way, treating inconsistent information as potentially informative. Such logics were originally developed in Latin America in the decades of 1950 and 1960, primarily through the influential works of F. Asenjo and Newton da Costa. Quickly, however the topic attracted attention in the international community and the original scope of mathematical applications broadened out as witnessed in a recent book emphasizing the engineering potential of paraconsistency [Aka16]. A number of other applications have emerged to themes from robotics, quantum mechanics and quantum information theory in works by J. Abe and his collaborators [ATL$^+$07], D. Chiara [CG00] and W. Carnielli and his collaborators [AC10, dCK14].

Paraconsistent transition structures and their composition were further studied in the preliminary conference communication [CMB22b] that this paper extends. The research program in [CMB22b] continued in two directions. First a suitable notion of morphism for paraconsistent labelled transition systems (PLTS) was introduced leading to the definition of the corresponding category and its algebra. Then, notions of trace for PLTS were discussed, as well as, notions of *crisp* simulation and bisimulation to relate two paraconsistent transition systems.

This paper starts by revising this work, namely lifting the residuated structure underlying the (parametric) domain of weights to the twisted structure which enables joint computation of positive and negative weights. We adopt the framework outlined in [BEGR09], focusing our exploration on many-valued (paraconsistent) modal logics defined over commutative integral residuated lattices over a set A of possible truth values. The motivation behind these systems is to combine modal and graded reasoning, with existing research on many-valued logic addressing problems related to decidability, complexity [CMRR17], and axiomatizability of a (many-valued) dynamic logic [MNM16]. Further contributions such as [BCN23] involve investigating frame definability in finitely valued modal logics and [MM14] delves into the expressivity of modal languages in a many-valued setting. Subsequent work in [BO20] demonstrated that no compact extension of the employed many-valued logic, displays both the Tarski Union Property and strong invariance for the proposed notion of *crisp* bisimulation introduced in [MM14].

While some of the previously cited works ([MM14, BCN23]) concentrate on crisp many-valued logics, where accessibility is a binary relation and proposition valuation is many-valued, parametric to a residuated lattice A. Other works ([CR15, JMM19]) delve into fuzzy Kripke models, where both propositions and the accessibility relation are many-valued, taking values in the standard Gödel algebra $[0, 1]$. Our approach aims to combine these perspectives by working with models that are parametric to a residuated lattice A, where the accessibility and valuation relation are simultaneously many-valued, taking values in $A \times A$. We believe that this approach not only admits different instances to better suit each modelling problem at hand but also provides a nuanced perspective, allowing transitions and propositions to represent consistent, vague, or inconsistent information.

The proposed work in this paper extends [CMB22b] with three main contributions:

1. A notion of *graded* simulation and bisimulation, which enables comparing PLTS in a (formal) paraconsistent way;

2. A modal (minimal) logic over the corresponding Kripke structures, in which, in the tradition of so-called Hennessy-Milner logics, modalities are indexed by transition labels, or actions;

3. A number of results on modal preservation by both crisp and graded simulations and bisimulations.

Rather than approaching the concept of simulation or bisimulation as a *crisp* relation, as in previous works on many-valued logics [MM14, JMM19, CMB22b]. We propose the notion of *paraconsistent bisimulation* defined by a pair of weights, such that, one represents the evidence for bisimilarity and the other for non-bisimilarity. This approach results in a less strict notion of bisimulation, aligning with the inherent uncertainty and inconsistencies possibly present in PLTS. Our work draws inspiration from [Ngu22], where a logical characterization of fuzzy bisimulations in fuzzy modal logics was provided. Thus, we extend this approach to a paraconsistent realm, exploring logical characterizations of paraconsistent bisimulation in a paraconsistent modal logic over a general residuated lattice.

Furthermore, it is worth noting that preliminary results on classical soundness and the introduction of graded soundness for paraconsistent Kripke structures are presented in Subsection 5.2. In contrast to classical soundness, graded soundness allows for premises and conclusions that may involve vagueness or contradictions. A similar motivation can be found in [CR10], where the authors consider a many-valued version of Kripke semantics in which both the propositions and accessibility relations are infinitely many-valued in the standard Gödel algebra $[0, 1]$. The authors define a graded notion of logical consequence $T \models_{\leq GK} \varphi$ if and only if for any model M and any world x in M, $\inf e(x, T) \leq e(x, \varphi)$ and prove the soundness of the $\Box$-fragment (and similarly the $\Diamond$-fragment). The soundness is stated as $T \vdash_{\mathcal{G}_\Box} \varphi$ implies $T \models_{\leq GK} \varphi$. Our approach to defining graded soundness for paraconsistent Kripke structures is similar. However, the main difference lies in studying theories where the classical notion of logical consequence is not equivalent to the graded notion.

To keep the paper within reasonable page limits, the application example in [CMB22b] dealing with optimization of quantum circuits is omitted. The interested reader may access further developments of this application in [MB23]. Furthermore, for those interested in specification theory and the stepwise implementation process *à la* Sanella and Tarlecki [DS12] applied to paraconsistent transition systems and their corresponding processes, we refer to the authors' work outlined in [CMB23b, CMB23a].

Paper structure. Section 2 revisits the concept of a PLTS and characterises the twisted structure for the joint computation of positive and negative weights. The whole constructions are parametric in a variant of a residuated lattice in which weights are specified. The compositional construction of (pointed) PLTS are characterised in section 3 by exploring the relevant category, following G. Winskel and M. Nielsen's 'recipe' [WN95]. In section 4 PTLS are extended to (paraconsistent) Kripke structures over which forms of crisp and graded simulation and bisimulation are studied. A modal logic for paraconsistent transition structures is introduced in section 5, alongside preliminary results of soundness and results of modal preservation for crisp and graded simulation and bisimulation. Finally, in section 6 we explore possible applications of PLTS. Section 7 concludes and points out a number of future research directions.

2 Paraconsistent labelled transition systems

In line with the work docummented in [BEGR09], we adopt a residuated lattice over a set A of possible truth values, where the classical modal logic corresponds to a Boolean algebra with two elements. Our work extends this framework to the paraconsistent realm, introducing the concept of a paraconsistent transition system where transitions are represented by pairs of weights $(a, b) \in A \times A$. We further introduce the notion of an $\boldsymbol{A}$-*twisted algebra* to manipulate pairs of weights, laying the foundation for the subsequent sections.

Formally, a *paraconsistent labelled transition system* (PLTS) incorporates two accessibility relations, classified as positive and negative, respectively, which characterise each transition in opposite ways: one represents the evidence of its presence and other the evidence of its absence. Both relations are weighted by elements of a residuated lattice $\boldsymbol{A} = \langle A, \wedge, \vee, \odot, \rightarrow, 1, 0\rangle$, where, $\langle A, \wedge, \vee, 1, 0\rangle$ is a lattice, $\langle A, \odot, 1\rangle$ is a monoid, and operation $\odot$ is the left adjoint to $\rightarrow$, its *residuum*, i.e. for all $a, b, c \in A$,

$$a \odot b \leq c \text{ iff } b \leq a \rightarrow c$$

We will focus on a particular class of residuated lattices that are bounded by a maximal 1 and a minimal element 0, respectively, and where lattice meet ($\wedge$) and monoidal composition ($\odot$) coincide. Hence, the adjunction above can be rephrased as

$$a \wedge b \leq c \text{ iff } b \leq a \rightarrow c \tag{1}$$

Furthermore, a pre-linearity condition is enforced

$$(a \rightarrow b) \vee (b \rightarrow a) = 1 \tag{2}$$

A residuated lattice obeying pre-linearity is known as a *MTL-algebra* [EG01]. With a slight abuse of nomenclature, the designation *iMTL-algebra*, from *integral MTL-algebra*, will be used in the sequel for the class of semantic structures considered, i.e. prelinear, residuated lattices such that $\wedge$ and $\odot$ coincide.

The Gödel algebra $G = \langle [0, 1], min, max, min, \rightarrow, 1, 0\rangle$ is an example of such a structure, that will be used in the sequel. Operators max and min retain the usual definitions, whereas implication is given by

$$a \rightarrow b = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{otherwise} \end{cases} .$$

The following Lemma builds upon [Ngu22, Lemma 2.1.] to introduce a number of properties of these structures that will be useful in the paper.

Lemma 1. *Let $\boldsymbol{A} = \langle A, \wedge, \vee, \odot, \rightarrow, 1, 0\rangle$ be a residuated lattice over a non empty set A. The following properties hold, for any a, a', b, b', $c \in A$*

$$a \leq a' \textit{ iff } (a \rightarrow a') = 1 \tag{3}$$
$$a \leq a' \textit{ and } b \leq b' \textit{ implies } a \wedge b \leq a' \wedge b' \tag{4}$$
$$a' \leq a \textit{ and } b \leq b' \textit{ implies } a \rightarrow b \leq a' \rightarrow b' \tag{5}$$
$$(a \leftrightarrow a') \wedge (b \leftrightarrow b') \;\leq\; (a \wedge b) \leftrightarrow (a' \wedge b') \tag{6}$$
$$(a \leftrightarrow a') \wedge (b \leftrightarrow b') \;\leq\; (a \vee b) \leftrightarrow (a' \vee b') \tag{7}$$
$$a \rightarrow (b \rightarrow c) \;=\; b \rightarrow (a \rightarrow c) \tag{8}$$
$$a \rightarrow (b \rightarrow c) \;=\; (a \wedge b) \rightarrow c \tag{9}$$
$$a \wedge (b \vee b') \;=\; (a \wedge b) \vee (a \wedge b') \tag{10}$$
$$a \vee (b \wedge b') \;=\; (a \vee b) \wedge (a \vee b') \tag{11}$$
$$a \wedge (a \rightarrow c) \;\leq\; c \tag{12}$$
$$(a \rightarrow c) \wedge (b \rightarrow c) \;=\; (a \vee b) \rightarrow c \tag{13}$$
$$(a \rightarrow b) \wedge (a \rightarrow c) \;=\; a \rightarrow (b \wedge c) \tag{14}$$
$$a \wedge (b \rightarrow c) \;\leq\; (a \rightarrow b) \rightarrow c \tag{15}$$
$$a \wedge (b \rightarrow c) \;\leq\; b \rightarrow (a \wedge c) \tag{16}$$

Proof. We focus on properties (12), (13), (14) and (16). All the others are proved in [Ngu22]. Thus,

- Property (12) is a direct consequence of (1) applied to the trivial assertion $a \rightarrow c \;\leq\; a \rightarrow c$
- To prove Property (13) we start by proving

$$(a \rightarrow c) \wedge (b \rightarrow c) \;\leq\; (a \vee b) \rightarrow c \tag{17}$$

  and then

$$(a \rightarrow c) \wedge (b \rightarrow c) \;\geq\; (a \vee b) \rightarrow c \tag{18}$$

  Trivially, $c \;\geq\; c \wedge (b \rightarrow c)$ and $c \;\geq\; c \wedge (a \rightarrow c)$ and since $\vee$ is monotone, $c \;\geq\; (c \wedge (b \rightarrow c)) \vee (c \wedge (a \rightarrow c))$. Then, Property (12) entails $c \;\geq\; a \wedge (a \rightarrow c)$ and $c \;\geq\; b \wedge (b \rightarrow c)$, leading to

$$c \;\geq\; (a \wedge (a \rightarrow c) \wedge (b \rightarrow c)) \vee (b \wedge (b \rightarrow c) \wedge (a \rightarrow c))$$

  as $\wedge$ and $\vee$ are monotone as well. Since $\wedge$ is associative, property (10) entails $c \;\geq\; (a \vee b) \wedge ((a \rightarrow c) \wedge (b \rightarrow c))$. Finally, using Property (1) we have (17). To prove (18) we start with the trivial assertions $a \leq a \vee b$ and $b \leq a \vee b$. By Property (5), $(a \vee b) \rightarrow c \;\leq\; a \rightarrow c$ and $(a \vee b) \rightarrow c \;\leq\; b \rightarrow c$. Finally, since $\wedge$ is monotone and idempotent we prove (18).
- To prove property (14) we first start by proving $(a \rightarrow b) \wedge (a \rightarrow c) \;\leq\; a \rightarrow (b \wedge c)$ and then $a \rightarrow (b \wedge c) \;\leq\; (a \rightarrow b) \wedge (a \rightarrow c)$. According to property (12), $a \wedge (a \rightarrow b) \;\leq\; b$ and $a \wedge (a \rightarrow c) \;\leq\; c$. Since $\wedge$ is monotone and idempotent, $a \wedge (a \rightarrow b) \wedge (a \rightarrow c) \;\leq\; b \wedge c$. Using (1), entails $(a \rightarrow b) \wedge (a \rightarrow c) \;\leq\; a \rightarrow (b \wedge c)$. Moreover, trivially $b \wedge c \leq b$ and $b \wedge c \leq c$ by Property (5) $a \rightarrow (b \wedge c) \;\leq\; a \rightarrow b$ and $a \rightarrow (b \wedge c) \;\leq\; a \rightarrow c$. Finally, given that $\wedge$ is monotone and idempotent $a \rightarrow (b \wedge c) \;\leq\; (a \rightarrow b) \wedge (a \rightarrow c)$

- To prove property (16), observe that, by (12), $(b \to c) \wedge b \leq c$. Since $\wedge$ is monotone and associative $b \wedge a \wedge (b \to c) \leq (c \wedge a)$. Using (1) it follows that $a \wedge (b \to c) \leq b \to (c \wedge a)$.

$\square$

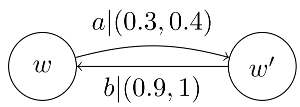
The following definition introduces the paper's basic semantic structure, which is to a large extent, independent of the particular residuated lattice chosen.

Definition 1. *A* ***paraconsistent labelled transition system*** *(PLTS) over a* iMTL-algebra $\boldsymbol{A}$*, and a set of atomic actions* Act *is a structure* $\langle W, R\rangle$ *where,*

- W *is a non-empty set of states*
- $R = (R_a \colon W \times W \to A \times A)_{a \in \mathrm{Act}}$ *is an* Act*-indexed family of functions. Given any pair of states* $(w_1, w_2) \in W \times W$ *and an atomic action* $a \in \mathrm{Act}$*, relation* R_a *assigns a pair* $(\alpha, \beta) \in A \times A$ *such that* α *weights the evidence of the transition from* w_1 *to* w_2 *occurring through action* a *and* β *weights the evidence of the transition being absent.*

For any pair of weights $\omega = (\alpha, \beta) \in A \times A$, ω^+ denotes α, referred to as the *positive weight* as it measures the possibility of occurrence, and ω^- denotes β, referred to as the *negative weight*.

Example 1. *A PLTS over the set* $\{a, b\}$ *of atomic actions, taking weights from the Gödel algebra* G *is depicted below*



Note that $R_a(w, w')$ *represents vague information, because the sum of positive and negative weights for the corresponding transition is less than 1. Such and* $R_b(w', w)$ *represents inconsistent information.*

Actually, the road from *vague* to *inconsistent* transitions is represented in Figure 1 which depicts all pairs of weights $(\alpha, \beta) \in [0, 1] \times [0, 1]$ in a Gödel algebra.

These pairs of weights jointly express different behaviours:

- *inconsistency*, when the positive and negative weights are contradictory, i.e. they sum to some value greater then 1; this corresponds to the upper triangle in Figure 1, filled in grey.
- *vagueness*, when the sum is less than 1, corresponding to the lower, periwinkle triangle in Figure 1.
- *consistency*, when the sum is exactly 1, which means that the measures of the factors enforcing or preventing a transition are complementary, corresponding to the red line in Figure 1.

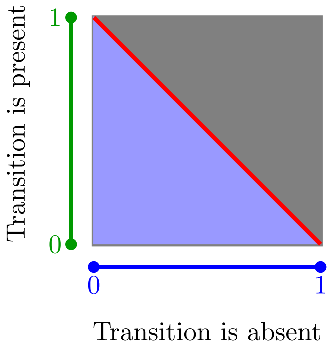


Figure 1: The vagueness-inconsistency square

This discussion can be lifted to arbitrary iMTL-algebras over A through the introduction of a metric space over A^2, as proposed in our previous work [CMB22a].

Jointly operating with pairs of (positive and negative) weights is essential for defining a (modal) logic over PLTS. The main conceptual tool for this is the notion of a *twisted algebra* [Kra98].

Definition 2. *Given an iMTL-algebra* $\boldsymbol{A} = \langle A, \wedge, \vee, 1, 0, \rightarrow\rangle$ *a* $\boldsymbol{A}$*-twisted algebra*

$$\mathcal{A} \;=\; \langle A \times A, \overset{\wedge}{\wedge}, \overset{\vee}{\vee}, \Rightarrow, /\!\!/\rangle$$

is defined as follows:

- $(a, b) \overset{\wedge}{\wedge} (c, d) = (a \wedge c, b \vee d)$
- $(a, b) \overset{\vee}{\vee} (c, d) = (a \vee c, b \wedge d)$
- $(a, b) \Rightarrow (c, d) = ((a \rightarrow c) \wedge (d \rightarrow b), a \wedge d)$
- $/\!\!/(a, b) = (b, a)$

Equivalence is defined by abbreviation.

$$(a, b) \Leftrightarrow (c, d) \;=\; ((a, b) \Rightarrow (c, d)) \overset{\wedge}{\wedge} ((c, d) \Rightarrow (a, b))$$

The order in $\boldsymbol{A}$ *is lifted to* $\mathcal{A}$ *as* $(a, b) \preccurlyeq (c, d)$ *iff* $a \leq c$ *and* $b \geq d$

The problem with the definition above is that the original $\rightarrow, \wedge$ adjunction in $\boldsymbol{A}$ does not lift to $\mathcal{A}$. A counterexample in the Gödel algebra falsifies the envisaged condition

$$(a, b) \overset{\wedge}{\wedge} (c, d) \;\preccurlyeq\; (e, f) \text{ iff } (a, b) \;\preccurlyeq\; (c, d) \Rightarrow (e, f)$$

Indeed, let $(a, b) = (0.9, 0.6)$, $(c, d) = (0.9, 0.9)$ and $(e, f) = (1, 0.8)$. Clearly,

$$(0.9, 0.6) \overset{\wedge}{\wedge} (0.9, 0.9) \;=\; (0.9 \wedge 0.9, 0.6 \vee 0.9) \;=\; (0.9, 0.9) \;\preccurlyeq\; (1, 0.8)$$

However,

$$(0.9, 0.6) \not\preccurlyeq (0.9, 0.9) \Rightarrow (1, 0.8) \;=\; (0.9 \rightarrow 0.8 \wedge 1 \rightarrow 0.9, 0.9 \wedge 0.8) \;=\; (0.8, 0.8)$$

The adjunction is recovered, however, replacing $\overset{\wedge}{\wedge}$ by

$$(a,b) \otimes (c,d) \; = \; (a \wedge c, a \to d \wedge c \to b)$$

Thus,

Lemma 2. *The operator $\otimes$ as defined above has $\Rightarrow$ as its residuum, i.e. as a right adjoint:*

$$(a,b) \otimes (c,d) \; \preccurlyeq \; (e,f) \text{ iff } (a,b) \; \preccurlyeq \; (c,d) \Rightarrow (e,f) \tag{19}$$

Proof. We start by simplifying both sides of equivalence (19):

$$\begin{array}{ll}(a,b) \otimes (c,d) \; \preccurlyeq \; (e,f) & (a,b) \; \preccurlyeq \; (c,d) \Rightarrow (e,f) \\ \text{iff} \;\; (a \wedge c, (a \to d) \wedge (c \to b)) \; \preccurlyeq \; (e,f) & \text{iff} \;\; (a,b) \; \preccurlyeq \; ((c \to e) \wedge (f \to d), c \wedge f) \\ \text{iff} \;\; a \wedge c \; \leq \; e \text{ and } (a \to d) \wedge (c \to b) \; \geq \; f & \text{iff} \;\; a \; \leq \; (c \to e) \wedge (f \to d) \text{ and } b \; \geq \; c \wedge f\end{array}$$

Let us now prove that, if $(a,b) \otimes (c,d) \; \preccurlyeq \; (e,f)$ then $(a,b) \; \preccurlyeq \; (c,d) \Rightarrow (e,f)$. By hypothesis $c \wedge a \; \leq \; e$ using (1), $a \; \leq \; c \to e$. Also by hypothesis, $a \to d \; \geq \; f$, using (1) $a \wedge f \; \leq \; d$, that is the same as writing $f \wedge a \; \leq \; d$, using (1) again $a \; \leq \; f \to d$. Since $\wedge$ is monotone we prove that $a \; \leq \; (c \to e) \wedge (f \to d)$. Finally, by hypothesis $c \to b \; \geq \; f$ by (1), $c \wedge f \; \leq \; b$.

For the converse implication, if $(a,b) \; \preccurlyeq \; (c,d) \Rightarrow (e,f)$ then $(a,b) \otimes (c,d) \; \preccurlyeq \; (e,f)$, note that, by hypothesis $a \; \leq \; c \; \to \; e$ using (1), $c \wedge a \; \leq \; e$. Again, $b \; \geq \; c \wedge f$ using (1), $f \; \leq \; c \to b$ also by hypothesis $a \; \leq \; f \; \to \; d$ using (1), $f \wedge a \; \leq \; d$ which is the same as $a \wedge f \; \leq \; d$, using (1), $f \; \leq \; a \to d$. Since $\wedge$ is monotone, $f \; \leq \; a \to d \wedge c \to b$. □

The following crucial lemma introduces a number of properties relating pairs of weights that will be used in the sequel.

Lemma 3. *Let $\mathcal{A} \; = \; \langle A \times A, \overset{\wedge}{\wedge}, \overset{\vee}{\vee}, \Rightarrow, /\!\!/ \rangle$ be a $\boldsymbol{A}$-twisted algebra over $\boldsymbol{A} \; = \; \langle A, \wedge, \vee, 1, 0, \to \rangle$. Thus,*

$$\text{if } (a',b') \; \preccurlyeq \; (a,b) \text{ and } (c,d) \; \preccurlyeq \; (c',d') \text{ then } (a,b) \Rightarrow (c,d) \; \preccurlyeq \; (a',b') \Rightarrow (c',d') \tag{20}$$

$$\text{if } (a',b') \; \preccurlyeq \; (a,b) \text{ and } (c,d) \; \preccurlyeq \; (c',d') \text{ then } (a,b) \otimes (c,d) \; \preccurlyeq \; (a',b') \otimes (c',d') \tag{21}$$

$$((a,b) \Leftrightarrow (c,d)) \; = \; (/\!\!/(a,b) \Leftrightarrow /\!\!/(c,d)) \tag{22}$$

$$(a,b) \otimes \Big((c,d) \overset{\vee}{\vee} (e,f)\Big) \; = \; \Big((a,b) \otimes (c,d)\Big) \overset{\vee}{\vee} \Big((a,b) \otimes (e,f)\Big) \tag{23}$$

$$(a,b) \otimes \Big((c,d) \overset{\wedge}{\wedge} (e,f)\Big) \; \preccurlyeq \; \Big((a,b) \otimes (c,d)\Big) \overset{\wedge}{\wedge} \Big((a,b) \otimes (e,f)\Big) \tag{24}$$

$$(a,b) \otimes \Big((c,d) \Rightarrow (e,f)\Big) \; \preccurlyeq \; (c,d) \Rightarrow \Big((a,b) \otimes (e,f)\Big) \tag{25}$$

$$(a,b) \Rightarrow \Big((c,d) \Rightarrow (e,f)\Big) \; \preccurlyeq \; \Big((a,b) \otimes (c,d)\Big) \Rightarrow (e,f) \tag{26}$$

$$(a,b) \otimes \Big((c,d) \Rightarrow (e,f)\Big) \preccurlyeq \Big((a,b) \Rightarrow (c,d)\Big) \Rightarrow (e,f) \tag{27}$$

$$(a,b) \Rightarrow \Big((c,d) \Rightarrow (e,f)\Big) \preccurlyeq (c,d) \Rightarrow \Big((a,b) \Rightarrow (e,f)\Big) \tag{28}$$

$$\Big((a,b) \Leftrightarrow (a',b')\Big) \overset{\wedge}{\wedge} \Big((c,d) \Leftrightarrow (c',d')\Big) \preccurlyeq \Big((a,b) \overset{\wedge}{\wedge} (c,d)\Big) \Leftrightarrow \Big((a',b') \overset{\wedge}{\wedge} (c',d')\Big) \tag{29}$$

$$\Big((a,b) \Leftrightarrow (a',b')\Big) \overset{\wedge}{\wedge} \Big((c,d) \Leftrightarrow (c',d')\Big) \preccurlyeq \Big((a,b) \overset{\vee}{\vee} (c,d)\Big) \Leftrightarrow \Big((a',b') \overset{\vee}{\vee} (c',d')\Big) \tag{30}$$

$$\Big((a,b) \Rightarrow (a',b')\Big) \overset{\wedge}{\wedge} \Big((c,d) \Rightarrow (c',d')\Big) \preccurlyeq \Big((a,b) \overset{\vee}{\vee} (c,d)\Big) \Rightarrow \Big((a',b') \overset{\vee}{\vee} (c',d')\Big) \tag{31}$$

$$\Big((a,b) \Rightarrow (a',b')\Big) \overset{\wedge}{\wedge} \Big((c,d) \Rightarrow (c',d')\Big) \preccurlyeq \Big((a,b) \overset{\wedge}{\wedge} (c,d)\Big) \Rightarrow \Big((a',b') \overset{\wedge}{\wedge} (c',d')\Big) \tag{32}$$

Proof.

- To prove Property (20) we need to prove $(a \to c) \wedge (d \to b) \leq (a' \to c') \wedge (d' \to b')$ and $a \wedge d \geq a' \wedge d'$. Let us assume that $a' \leq a$, $c \leq c'$, $b' \geq b$ and $d \geq d'$. By Property (5), $(a \to c) \leq (a' \to c')$ and $(d \to b) \leq (d' \to b')$. Since $\wedge$ is monotone, $(a \to c) \wedge (d \to b) \leq (a' \to c') \wedge (d' \to b')$. Finally, by hypothesis and the fact that $\wedge$ is monotone, $a \wedge d \geq a' \wedge d'$.

- To prove Property (21) we need to prove that $a \wedge c \leq a' \wedge c'$ and that $(a \to d) \wedge (c \to b) \geq (a' \to d') \wedge (c' \to b')$. By hypothesis $a \leq a'$ and $c \leq c'$, since $\wedge$ is monotone $a \wedge c \leq a' \wedge c'$. Also by hypothesis, $b \geq b'$ and $d \geq d'$, by Property (5), $(a \to d) \wedge (c \to b) \geq (a' \to d') \wedge (c' \to b')$.

- To prove Property (22) recall that

  $$\begin{aligned} (a,b) \Leftrightarrow (c,d) &= ((a \leftrightarrow c) \wedge (d \leftrightarrow b), (a \wedge d) \vee (c \wedge b)) \\ /\!\!/(a,b) \Leftrightarrow /\!\!/(c,d) = (b,a) \Leftrightarrow (d,c) &= ((b \leftrightarrow d) \wedge (a \leftrightarrow c), (c \wedge b) \vee (a \wedge d)) \end{aligned}$$

  Since $\wedge$ and $\vee$ are associative, the pair $(a,b) \Leftrightarrow (c,d)$ is equal to the pair $/\!\!/(a,b) \Leftrightarrow /\!\!/(c,d)$.

- To prove Property (23) recall that

  $$(a,b) \otimes ((c,d) \overset{\vee}{\vee} (e,f)) = (a \wedge (c \vee e), a \to (d \wedge f) \wedge (c \vee e) \to b)$$
  $$((a,b) \otimes (c,d)) \overset{\vee}{\vee} ((a,b) \otimes (e,f)) = ((a \wedge c) \vee (a \wedge e), (a \to d) \wedge (c \to b) \wedge (a \to f) \wedge (e \to b))$$

  By Property (10), we prove that $a \wedge (c \vee e) = (a \wedge c) \vee (a \wedge e)$ and by using Property (13) and (14) we prove $a \to (d \wedge f) \wedge (c \vee e) \to b = (a \to d) \wedge (c \to b) \wedge (a \to f) \wedge (e \to b)$.

- To prove Property (24) note that $(c,d) \overset{\wedge}{\wedge} (e,f) \preccurlyeq (c,d)$ and $(c,d) \overset{\wedge}{\wedge} (e,f) \preccurlyeq (e,f)$. Using Property (21),

  $$(a,b) \otimes \Big((c,d) \overset{\wedge}{\wedge} (e,f)\Big) \preccurlyeq (a,b) \otimes (c,d)$$
  $$(a,b) \otimes \Big((c,d) \overset{\wedge}{\wedge} (e,f)\Big) \preccurlyeq (a,b) \otimes (e,f)$$

Finally, since the operator $\overset{\wedge}{\wedge}$ is monotone it implies that,

$$(a,b)\otimes\Big((c,d)\overset{\wedge}{\wedge}(e,f)\Big)\preccurlyeq\Big((a,b)\otimes(c,d)\Big)\overset{\wedge}{\wedge}\Big((a,b)\otimes(e,f)\Big)$$

- To prove Property (25) note that,

$$\begin{aligned}(a,b)\otimes((c,d)\Rightarrow(e,f)) &= (a,b)\otimes((c\to e)\wedge(f\to d),c\wedge f)\\ &= (a\wedge(c\to e)\wedge(f\to d),(a\to(c\wedge f))\wedge((c\to e\wedge f\to d)\to b))\\ (c,d)\Rightarrow((a,b)\otimes(e,f)) &= (c,d)\Rightarrow(a\wedge e,(a\to f)\wedge(e\to b))\\ &= ((c\to(a\wedge e))\wedge(((a\to f)\wedge(e\to b))\to d),c\wedge(a\to f)\wedge(e\to b))\end{aligned}$$

Thus, we want to prove

$$a\wedge(c\to e)\wedge(f\to d)\ \le\ (c\to(a\wedge e))\wedge((a\to f\wedge e\to b)\to d) \tag{33}$$

$$(a\to(c\wedge f))\wedge(((c\to e)\wedge(f\to d))\to b)\ \ge\ c\wedge(a\to f)\wedge(e\to b) \tag{34}$$

The proof of inequality (33) follows as

$$\begin{aligned}&a\wedge(c\to e)\wedge(f\to d)\\ =&\ \{\wedge\text{ is idempotent and associative}\}\\ &a\wedge(c\to e)\wedge a\wedge(f\to d)\\ \le&\ \{\ (16)\}\\ &(c\to(a\wedge e))\wedge a\wedge(f\to d)\\ \le&\ \{\ (15)\}\\ &(c\to(a\wedge e))\wedge((a\to f)\to d)\\ \le&\ \{\text{since }(a\to f)\ \ge\ (a\to f)\wedge(e\to b)\text{ and (5)}\}\\ &(c\to(a\wedge e))\wedge(((a\to f)\wedge(e\to b))\to d)\end{aligned}$$

For inequality (34) consider

$$\begin{aligned}&(a\to(c\wedge f))\wedge(((c\to e)\wedge(f\to d))\to b)\\ \ge&\ \{(16)\}\\ &c\wedge(a\to f)\wedge(((c\to e)\wedge(f\to d))\to b)\\ \ge&\ \{\text{since }c\to e\ \ge\ (c\to e)\wedge(f\to d)\text{ by (5)}\}\\ &c\wedge(a\to f)\wedge((c\to e)\to b)\\ \ge&\ \{\ (15)\}\\ &c\wedge(a\to f)\wedge c\wedge(e\to b)\\ =&\ \{\wedge\text{ is idempotent}\}\\ &c\wedge(a\to f)\wedge(e\to b)\end{aligned}$$

- To prove Property (26) note that,

$$(a,b)\Rightarrow((c,d)\Rightarrow(e,f))=(a,b)\Rightarrow((c\to e)\wedge(f\to d),c\wedge f)$$

$$
\begin{aligned}
&= ((a \to ((c \to e) \wedge (f \to d))) \wedge ((c \wedge f) \to b), a \wedge c \wedge f) \\
((a,b) \otimes (c,d)) \Rightarrow (e,f) &= (a \wedge c, (a \to d) \wedge (c \to b)) \Rightarrow (e,f) \\
&= (((a \wedge c) \to e) \wedge (f \to ((a \to d) \wedge (c \to b))), a \wedge c \wedge f)
\end{aligned}
$$

Thus, we want to prove that,

$$(a \to ((c \to e) \wedge (f \to d))) \wedge ((c \wedge f) \to b) \leq ((a \wedge c) \to e) \wedge (f \to ((a \to d) \wedge (c \to b)))$$

which proceeds as follows:

$$
\begin{aligned}
& (a \to ((c \to e) \wedge (f \to d))) \wedge ((c \wedge f) \to b) \\
\leq & \{ \text{(5), and } \wedge \text{ is monotone} \} \\
& (a \to (c \to e)) \wedge (a \to (f \to d)) \wedge ((c \wedge f) \to b) \\
= & \{(9)\} \\
& ((a \wedge c) \to e) \wedge (a \to (f \to d)) \wedge ((c \wedge f) \to b) \\
= & \{(8)\} \\
& ((a \wedge c) \to e) \wedge (f \to (a \to d)) \wedge ((c \wedge f) \to b) \\
= & \{ (9)\} \\
& ((a \wedge c) \to e) \wedge (f \to (a \to d)) \wedge (f \to (c \to b)) \\
\leq & \{ (14)\} \\
& ((a \wedge c) \to e) \wedge (f \to ((a \to d) \wedge (c \to b)))
\end{aligned}
$$

- To prove Property (27) let us start by noting that,

$$
\begin{aligned}
(a,b) \otimes ((c,d) \Rightarrow (e.f)) &= (a,b) \otimes ((c \to e) \wedge (f \to d), c \wedge f) \\
&= (a \wedge (c \to e) \wedge (f \to d), (a \to (c \wedge f)) \wedge (((c \to e) \wedge (f \to d)) \to b)) \\
((a,b) \Rightarrow (c,d)) \Rightarrow (e,f) &= ((a \to c) \wedge (d \to b), a \wedge d) \Rightarrow (e,f) \\
&= ((((a \to c) \wedge (d \to b)) \to e) \wedge (f \to (a \wedge d)), (a \to c) \wedge (d \to b) \wedge f)
\end{aligned}
$$

Thus, we need to prove the following inequalities,

$$a \wedge (c \to e) \wedge (f \to d) \leq (((a \to c) \wedge (d \to b)) \to e) \wedge (f \to (a \wedge d)) \tag{35}$$

$$(a \to (c \wedge f)) \wedge (((c \to e) \wedge (f \to d)) \to b) \geq (a \to c) \wedge (d \to b) \wedge f \tag{36}$$

Thus, for (35),

$$
\begin{aligned}
& a \wedge (c \to e) \wedge (f \to d) \\
= & \{\wedge \text{ is idempotent and associative}\} \\
& a \wedge (c \to e) \wedge a \wedge (f \to d) \\
\leq & \{(15)\} \\
& ((a \to c) \to e) \wedge a \wedge (f \to d) \\
\leq & \{ (5)\} \\
& (((a \to c) \wedge (d \to b)) \to e) \wedge a \wedge (f \to d)
\end{aligned}
$$

$$\begin{aligned}&\leq \{(16)\}\\&\quad (((a \to c) \wedge (d \to b)) \to e) \wedge (f \to (a \wedge d))\end{aligned}$$

and (36),

$$\begin{aligned}&\quad (a \to (c \wedge f)) \wedge (((c \to e) \wedge (f \to d)) \to b)\\&\geq \{(16)\}\\&\quad f \wedge (a \to c) \wedge (((c \to e) \wedge (f \to d)) \to b)\\&\geq \{(5)\}\\&\quad f \wedge (a \to c) \wedge ((f \to d) \to b)\\&\geq \{\text{ (16) and } \wedge \text{ idempotent}\}\\&\quad f \wedge (a \to c) \wedge (d \to b)\end{aligned}$$

- To prove (28) notice that,

$$\begin{aligned}(a,b) \Rightarrow ((c,d) \Rightarrow (e,f)) &= (a,b) \Rightarrow ((c \to e) \wedge (f \to d), c \wedge f)\\&= ((a \to ((c \to e) \wedge (f \to d))) \wedge ((c \wedge f) \to b), a \wedge c \wedge f)\\(c,d) \Rightarrow ((a,b) \Rightarrow (e,f)) &= (c,d) \Rightarrow ((a \to e) \wedge (f \to b), a \wedge f)\\&= ((c \to ((a \to e) \wedge (f \to b))) \wedge ((a \wedge f) \to d), a \wedge c \wedge f)\end{aligned}$$

Thus, we prove

$$(a \to ((c \to e) \wedge (f \to d))) \wedge ((c \wedge f) \to b) \;\leq\; (c \to ((a \to e) \wedge (f \to b))) \wedge ((a \wedge f) \to d)$$

as follows,

$$\begin{aligned}&\quad (c \to ((a \to e) \wedge (f \to b))) \wedge ((a \wedge f) \to d)\\&= \{\text{ (9)}\}\\&\quad (c \to ((a \to e) \wedge (f \to b))) \wedge (a \to (f \to d))\\&\geq \{\text{ (14)}\}\\&\quad (c \to (a \to e)) \wedge (c \to (f \to b)) \wedge (a \to (f \to d))\\&= \{(8)\}\\&\quad (a \to (c \to e)) \wedge (c \to (f \to b)) \wedge (a \to (f \to d))\\&= \{(9)\}\\&\quad (a \to (c \to e)) \wedge ((c \wedge f) \to b) \wedge (a \to (f \to d))\\&\geq \{+(5)\}\\&\quad (a \to ((c \to e) \wedge (f \to d))) \wedge ((c \wedge f) \to b) \wedge (a \to ((c \to e) \wedge (f \to d)))\\&= \{\wedge \text{ idempotent}\}\\&\quad (a \to ((c \to e) \wedge (f \to d))) \wedge ((c \wedge f) \to b)\end{aligned}$$

- To prove Properties (29) and (30) observe that,

$$\begin{aligned}&\quad ((a,b) \Leftrightarrow (a',b')) \;\overset{\wedge}{\wedge}\; ((c,d) \Leftrightarrow (c',d'))\\&= ((a \leftrightarrow a') \wedge (b \leftrightarrow b'), (a \wedge b') \vee (a' \wedge b)) \;\overset{\wedge}{\wedge}\; ((c \leftrightarrow c') \wedge (d \leftrightarrow d'), (c \wedge d') \vee (c' \wedge d))\\&= ((a \leftrightarrow a') \wedge (b \leftrightarrow b') \wedge (c \leftrightarrow c') \wedge (d \leftrightarrow d'), (a \wedge b') \vee (a' \wedge b) \vee (c \wedge d') \vee (c' \wedge d))\end{aligned}$$

$$
\begin{aligned}
&((a,b) \overset{\wedge}{\wedge} (c,d)) \Leftrightarrow ((a',b') \overset{\wedge}{\wedge} (c',d')) \\
=\ & (a \wedge c, b \vee d) \Leftrightarrow (a' \wedge c', b' \vee d') \\
=\ & ((a \wedge c) \leftrightarrow (a' \wedge c') \wedge (b \vee d) \leftrightarrow (b' \vee d'), (a \wedge c \wedge (b' \vee d')) \vee (a' \wedge c' \wedge (b \vee d)))
\end{aligned}
$$

and,

$$
\begin{aligned}
&((a,b) \overset{\vee}{\vee} (c,d)) \Leftrightarrow ((a',b') \overset{\vee}{\vee} (c',d')) \\
=\ & (a \vee c, b \wedge d) \Leftrightarrow (a' \vee c', b' \wedge d') \\
=\ & (((a \vee c) \leftrightarrow (a' \vee c')) \wedge ((b \wedge d) \leftrightarrow (b' \wedge d')), ((a \vee c) \wedge b' \wedge d') \vee ((a' \vee c') \wedge b \wedge d))
\end{aligned}
$$

Therefore, to prove (29) it is enough to verify the following inequalities,

$$
(a \leftrightarrow a') \wedge (b \leftrightarrow b') \wedge (c \leftrightarrow c') \wedge (d \leftrightarrow d') \ \leq \ ((a \wedge c) \leftrightarrow (a' \wedge c')) \wedge ((b \vee d) \leftrightarrow (b' \vee d')) \tag{37}
$$

$$
(a \wedge b') \vee (a' \wedge b) \vee (c \wedge d') \vee (c' \wedge d) \ \geq \ (a \wedge c \wedge (b' \vee d')) \vee (a' \wedge c' \wedge (b \vee d)) \tag{38}
$$

The inequality (37) follows from applying (6) twice

$$
\begin{aligned}
&(a \leftrightarrow a') \wedge (b \leftrightarrow b') \wedge (c \leftrightarrow c') \wedge (d \leftrightarrow d') \\
\leq\ & \{ \text{ (6) twice and } \wedge \text{ monotone} \} \\
&((a \wedge c) \leftrightarrow (a' \wedge c')) \wedge ((b \wedge d) \leftrightarrow (b' \wedge d'))
\end{aligned}
$$

For (38)

$$
\begin{aligned}
&((a \vee c) \wedge b' \wedge d') \vee ((a' \vee c') \wedge b \wedge d) \\
=\ & \{ \text{ (10)} \} \\
&(a \wedge b' \wedge d') \vee (c \wedge b' \wedge d') \vee (a' \wedge b \wedge d) \vee (c' \wedge b \wedge d) \\
\leq\ & \{ \vee \text{ and } \wedge \text{ monotone} \} \\
&(a \wedge b') \vee (c \wedge d') \vee (a' \wedge b) \vee (c' \wedge d)
\end{aligned}
$$

- To prove (30) the following inequalities have to be shown

$$
(a \leftrightarrow a') \wedge (b \leftrightarrow b') \wedge (c \leftrightarrow c') \wedge (d \leftrightarrow d') \ \leq \ ((a \vee c) \leftrightarrow (a' \vee c')) \wedge ((b \wedge d) \leftrightarrow (b' \wedge d')) \tag{39}
$$

$$
(a \wedge b') \vee (a' \wedge b) \vee (c \wedge d') \vee (c' \wedge d) \ \geq \ ((a \vee c) \wedge b' \wedge d') \vee ((a' \vee c') \wedge b \wedge d) \tag{40}
$$

Iinequality (39) follows from applying (7) twice and the fact that $\wedge$ is monotone. The inequality (2) follows by

$$
\begin{aligned}
&((a \vee c) \wedge b' \wedge d') \vee ((a' \vee c') \wedge b \wedge d) \\
=\ & \{ \text{ (10)} \} \\
&(a \wedge b' \wedge d') \vee (c \wedge b' \wedge d') \vee (a' \wedge b \wedge d) \vee (c' \wedge b \wedge d) \\
\leq\ & \{ \vee \text{ monotone} \} \\
&(a \wedge b') \vee (c \wedge d') \vee (a' \wedge b) \vee (c' \wedge d)
\end{aligned}
$$

To prove Properties (31) and (32) start by noting that,

$$\begin{aligned}
&((a,b) \Rightarrow (a',b')) \overset{\wedge}{\wedge} ((c,d) \Rightarrow (c',d')) \\
&= ((a \to a') \wedge (b' \to b), a \wedge b') \overset{\wedge}{\wedge} ((c \to c') \wedge (d' \to d), c \wedge d') \\
&= ((a \to a') \wedge (b' \to b) \wedge (c \to c') \wedge (d' \to d), (a \wedge b') \vee (c \wedge d'))
\end{aligned}$$

For (31) reason

$$\begin{aligned}
((a,b) \overset{\vee}{\vee} (c,d)) \Rightarrow ((a',b') \overset{\vee}{\vee} (c',d')) &= (a \vee c, b \wedge d) \Rightarrow (a' \vee c', b' \wedge d') \\
&= (((a \vee c) \to (a' \vee c')) \wedge ((b' \wedge d') \to (b \wedge d)), (a \vee c) \wedge b' \wedge d')
\end{aligned}$$

Thus, proving (31) amounts to show the following inequalities

$$(a \to a') \wedge (b' \to b) \wedge (c \to c') \wedge (d' \to d) \;\le\; ((a \vee c) \to (a' \vee c')) \wedge ((b' \wedge d') \to (b \wedge d)) \tag{41}$$

$$(a \wedge b') \vee (c \wedge d') \;\ge\; (a \vee c) \wedge b' \wedge d' \tag{42}$$

Let us start by proving (41)

$$\begin{aligned}
&(a \to a') \wedge (b' \to b) \wedge (c \to c') \wedge (d' \to d) \\
\le\;& \{\text{ (5)}\} \\
&(a \to (a' \vee c')) \wedge (c \to (a' \vee c')) \wedge ((b' \wedge d') \to b) \wedge ((b' \wedge d') \to d) \\
\le\;& \{\text{ (13) and (14)}\} \\
&((a \vee c) \to (a' \vee c')) \wedge ((b' \wedge d') \to (b \wedge d))
\end{aligned}$$

The prove of inequality (42) follows as,

$$\begin{aligned}
&(a \vee c) \wedge b' \wedge d' \\
=\;& \{\text{ (10)}\} \\
&(a \wedge b' \wedge d') \vee (c \wedge b' \wedge d') \\
\le\;& \{\vee \text{ monotone}\} \\
&(a \wedge b') \vee (c \wedge d')
\end{aligned}$$

- To prove (32) note that

$$\begin{aligned}
((a,b) \overset{\wedge}{\wedge} (c,d)) \Rightarrow ((a',b') \overset{\wedge}{\wedge} (c',d')) &= (a \wedge c, b \vee d) \Rightarrow (a' \wedge c', b' \vee d') \\
&= (((a \wedge c) \to (a' \wedge c')) \wedge ((b' \vee d') \to (b \vee d)), a \wedge c \wedge (b' \vee d'))
\end{aligned}$$

Thus, we need to prove the following inequalities,

$$(a \to a') \wedge (b' \to b) \wedge (c \to c') \wedge (d' \to d) \;\le\; ((a \wedge c) \to (a' \wedge c')) \wedge ((b' \vee d') \to (b \vee d)) \tag{43}$$

$$(a \wedge b') \vee (c \wedge d') \;\ge\; a \wedge c \wedge (b' \vee d') \tag{44}$$

Thus, to prove (43) note that

$$\begin{aligned}
&(a \to a') \wedge (b' \to b) \wedge (c \to c') \wedge (d' \to d) \\
\le\;& \{\text{Property (5)}\} \\
&((a \wedge c) \to a') \wedge ((a \wedge c) \to c') \wedge (b' \to (b \vee d)) \wedge (d' \to (b \vee d))
\end{aligned}$$

$\leq$ {Property (13) and (14)}
$((a \wedge c) \to (a' \wedge c')) \wedge ((b' \vee d') \to (b \vee d))$

Finally, (43) follows from

$$
\begin{aligned}
& a \wedge c \wedge (b' \vee d') \\
= & \{\text{Property (10)}\} \\
& (a \wedge c \wedge b') \vee (a \wedge c \wedge d') \\
\leq & \{\vee \text{ is monotone}\} \\
& (a \wedge b') \vee (c \wedge d')
\end{aligned}
$$

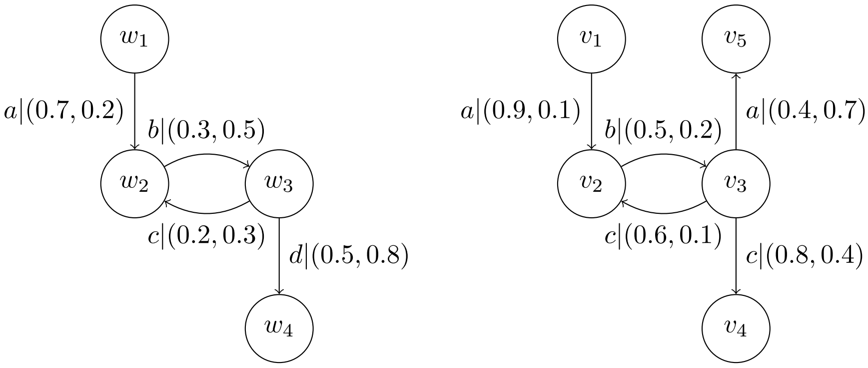
□

3 New PLTS from old

As any other mathematical, or computational structure, PLTS over a iMTL-algebra $\mathcal{A}$ live in a category whose morphisms respect, as one would expect, the structure of both accessibility relations. I.e., given two PLTS $T_1 = \langle W, R, V \rangle$, $T_2 = \langle W', R', V' \rangle$ over $\mathcal{A}$ and the same set of actions symbols Act, a **morphism** from T_1 to T_2 is a function $h : W \to W'$ such that, for any $a \in \text{Act}$, $R_a(w, v) \preccurlyeq R'_a(h(w), h(v))$.

It is worth noting that PLTSs and their morphisms form a category, with composition and identities borrowed from Set.

Example 2. *Consider the two PLTS over the Gödel algebra $M_1 = \langle W_1, R_1, V_1 \rangle$ and $M_2 = \langle W_2, R_2, V_2 \rangle$ depicted bellow over the set $\{a, b, c, d\}$ of atomic actions. Function $h = \{w_1 \mapsto v_1, w_2 \mapsto v_2, w_3 \mapsto v_3\}$ is a morphism from M_1 to M_2.*



In this setting, new PLTS can be built compositionally. This section introduces the relevant operators, always parametric on a underlying iMTL-algebra $\mathcal{A}$, by exploring the structure of the category of $\mathsf{Pt}_{\mathcal{A}}$ of *pointed* PLTS over $\mathcal{A}$, i.e. whose objects are PLTS with a distinguished initial state, i.e. $\langle W, i, R \rangle$, where $\langle W, R \rangle$ is a PLTS and $i \in W$. Arrows in $\mathsf{Pt}_{\mathcal{A}}$ are allowed between PLTS with different sets of labels, therefore generalizing the informal definition above as follows:

Definition 3. *Let $T = \langle W, i, R \rangle$ and $T' = \langle W', i', R' \rangle$ be two pointed PLTS over the set of atomic actions* Act *and* Act′*, respectively. A morphism in* $\mathsf{Pt}_{\mathcal{A}}$

from T to T' consists on a pair of functions ($\sigma : W \to W'$, $\lambda : \mathrm{Act} \to_\perp \mathrm{Act}'$) such that[1] *$\sigma(i) = i'$ and for any $a \in \mathrm{Act}$,*

$$R_a(w, w') \preccurlyeq R'^{\perp}_{\lambda(a)}(\sigma(w), \sigma(w'))$$

For an accessibility relation R, $R^\perp = R \cup R_\perp$ with $R_\perp(w, w) = (1, 0)$ for any state $w, w \in W$, denotes R enriched with idle *transitions in each state.*

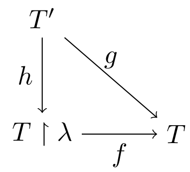
Clearly $\mathsf{Pt}_{\mathcal{A}}$ forms a category, with composition inherited from Set and $\mathsf{Set}_\perp$, the later standing for the category of sets and partial functions, with $T_{nil} = \langle\{*\}, *, \emptyset, \emptyset\rangle$ as both the initial and final object. The corresponding unique morphisms are $! : T \to T_{\mathrm{nil}}$, given by $\langle \underline{*}, ()\rangle$, and $? : T_{nil} \to T$, given by $\langle \underline{i}, ()\rangle$, where $()$ is the empty map and notation $\underline{x}$ stands for the constant, everywhere x, function.

An algebra of PLTS typically includes some form of parallel composition, disjoint union, restriction, relabelling and prefixing, as one is used to from the process algebra literature [BBR10]. Accordingly, these operators are defined along the lines proposed by G. Winskel and M. Mielsen [WN95], for the standard, more usual case.

Restriction. The restriction operator is intended to control the interface of a transition system, preserving, in the case of a PLTS, the corresponding positive and negative weights. Formally,

Definition 4. *Let $T = \langle W, i, R\rangle$ be pointed PLTS over the set of action symbols Act and $\lambda : \mathrm{Act}' \to \mathrm{Act}$ be an inclusion. The **restriction** of T to λ, $T \restriction \lambda$, is a PLTS $\langle W, i, R'\rangle$ over Act' such that, for any $w, v \in W$ and $a \in \mathrm{Act}'$, $R'_a(w, v) = R_a(w, v)$*

There is a morphism $f = (1_W, \lambda)$ from $T \restriction \lambda$ to T, and a functor $P : \mathsf{Pt}_{\mathcal{A}} \to \mathsf{Set}_\perp$ which sends a morphism $(\sigma, \lambda) : T \to T'$ to the partial function $\lambda : \mathrm{Act}' \to \mathrm{Act}$. Clearly, f is the Cartesian lifting of morphism $P(f) = \lambda$ in $\mathsf{Set}_\perp$. Being Cartesian means that for any $g : T' \to T$ in $\mathsf{Pt}_{\mathcal{A}}$ such that $P(g) = \lambda$ there is a unique morphism h such that $P(h) = 1_{\mathrm{Act}'}$ making the following diagram to commute:



Relabelling. In the same group of *interface-modifier* operators, is *relabelling*, which renames the labels of a PLTS according to the total functin $\lambda : \mathrm{Act} \to \mathrm{Act}'$.

Definition 5. *Let $T = \langle W, i, R\rangle$ be a pointed PLTS over Act and $\lambda : \mathrm{Act} \to \mathrm{Act}'$ be a total function. The **relabelling** of T according to λ is denoted by $T\{\lambda\}$ and it is the PLTS $\langle W, i, R'\rangle$ over Act' where for any $w, v \in W$ and $a \in \mathrm{Act}$, $R'_{\lambda(a)}(w, v) = R_a(w, v)$.*

[1] Notation $\lambda : \mathrm{Act} \to_\perp \mathrm{Act}'$ stands for the totalization of a partial function by mapping to $\perp$ all elements of Act for which the function is undefined.

Dually to the previous case, there is a morphism $f = (1_W, \lambda)$ from T to $T\{\lambda\}$ which is the cocartesian lifting of λ $(= P(f))$.

Parallel composition. The product of two PLTSs combines their state spaces and includes all *synchronous* transitions, triggered by the simultaneous occurrence of an action of each component, as well as *asynchronous* ones in which a transition in one component is paired with an *idle* transition and valuations, labelled by $\perp$, in the other. Formally,

Definition 6. *Let $T_1 = \langle W_1, i_1, R_1\rangle$ and $T_2 = \langle W_2, i_2, R_2\rangle$ be two pointed PLTS over the set Act_1 and Act_2, respectively. Their* ***parallel composition*** *$T_1 \times T_2$ is the PLTS $\langle W_1 \times W_2, (i_1, i_2), R\rangle$ over the set of atomic actions,*

$$\mathrm{Act}_1 \times_\perp \mathrm{Act}_2 = \{(a, \perp) \mid a \in \mathrm{Act}_1\} \cup \{(\perp, b) \mid b \in \mathrm{Act}_2\} \cup \{(a, b) \mid a \in \mathrm{Act}_1, b \in \mathrm{Act}_2\}$$

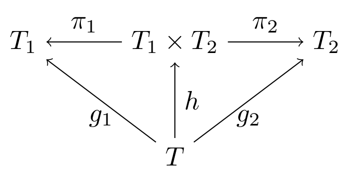
Such that, $R_{(a,b)}((w_1, w_2), (v_1, v_2)) = (\alpha, \beta)$ if and only if, $(R_1)^\perp_a(w_1, v_1) = (\alpha_1, \beta_1)$ and $(R_2)^\perp_b(w_2, v_2) = (\alpha_2, \beta_2)$ and $(\alpha, \beta) = (\alpha_1, \alpha_2) \curlywedge (\beta_1, \beta_2)$.

Lemma 4. *Parallel composition is the product construction in $\mathsf{Pt}_{\mathcal{A}}$.*

Proof. In the diagram below let $g_i = (\sigma_i, \lambda_i)$, for $i = 1, 2$, and define h as $h = (\langle\sigma_1, \sigma_2\rangle, \langle\lambda_1, \lambda_2\rangle)$, where $\langle f_1, f_2\rangle(x) = (f_1(x), f_2(x))$ is the universal arrow in a product diagram in Set. Clearly, h lifts universality to $\mathsf{Pt}_{\mathcal{A}}$, as the unique arrow making the diagram to commute. It remains show it is indeed an arrow in the category. Indeed, let $T = \langle W, i, R\rangle$ over the set Act, $T_1 = \langle W_1, i_1, R_1\rangle$ over the set Act_1, and define $T_1 \times T_2 = \langle W_1 \times W_2, (i_1, i_2), R'\rangle$ over Act' according to definition 6. Thus, for each $R_a(w, w') = (\alpha, \beta)$, there is a transition $(R_1)^\perp_{\lambda_1(a)}(\sigma_1(w), \sigma_1(w')) = (\alpha_1, \beta_1)$ such that $\alpha \leq \alpha_1$ and $\beta \geq \beta_1$; and also a transition $R_2{}^\perp_{\lambda_2(a)}(\sigma_2(w), \sigma_2(w')) = (\alpha_2, \beta_2)$ such that $\alpha \leq \alpha_1$ and $\beta \geq \beta_2$. Moreover, there is a transition

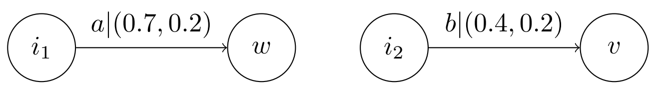
$$R'_{\langle\lambda_1,\lambda_2\rangle(a)}(\langle\sigma_1, \sigma_2\rangle(w), \langle\sigma_1, \sigma_2\rangle(w')) = (\alpha_1, \beta_1) \curlywedge (\alpha_2, \beta_2)$$

Thus, there is a transition $R'_{\langle\lambda_1,\lambda_2\rangle(a)}(\langle\sigma_1, \sigma_2\rangle(w), \langle\sigma_1, \sigma_2\rangle(w')) = (\alpha', \beta')$, for any $R_a(w, w') = (\alpha, \beta)$, such that $\alpha \leq \alpha'$ and $\beta \geq \beta'$. Furthermore, $\langle\sigma_1, \sigma_2\rangle(i) = (\sigma_1(i), \sigma_2(i)) = (i_1, i_2)$. This establishes h as a $\mathsf{Pt}_{\mathcal{A}}$ morphism.

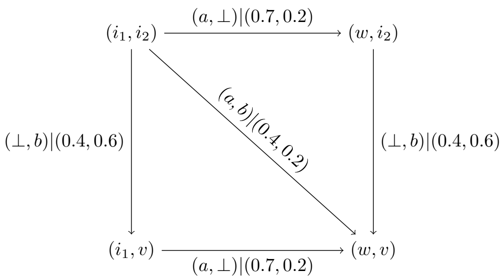


□

Example 3. *Consider the two PLTSs, T_1 and T_2 depicted below.*



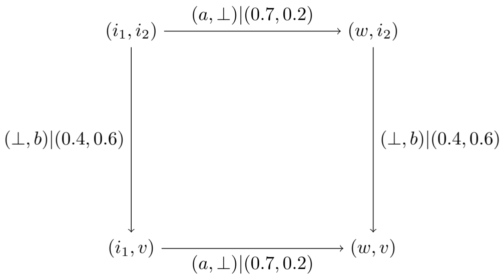
Their product T is the PLTS



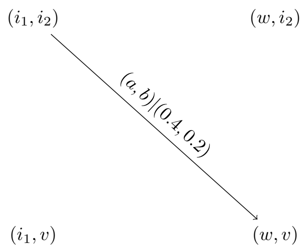
A suitable combination of parallel composition and restriction may enforce different synchronization disciplines. For example, *interleaving* or *asynchronous product* $T_1 \mathbin{|||} T_2$ is defined as $(T_1 \times T_2) \upharpoonright \lambda$ with the inclusion $\lambda : \mathrm{Act} \to \mathrm{Act}_1 \times_\perp \mathrm{Act}_2$ for $\mathrm{Act} = \{(a, \perp) \mid a \in \mathrm{Act}_1\} \cup \{(\perp, b) \mid b \in \mathrm{Act}_2\}$. This results in a PLTS $\langle W_1 \times W_2, (i_1, i_2), R, V\rangle$ over Act such that for any $a \in \mathrm{Act}$, if $R'_a(w, w') = (\alpha, \beta)$ then $R_a(w, w') = (\alpha, \beta)$.

Similarly, the *synchronous product* $T_1 \otimes T_2$ is also defined as $(T_1 \times T_2) \upharpoonright \lambda$, taking now $\mathrm{Act} = \{(a, b) \mid a \in \mathrm{Act}_1 \text{ and } b \in \mathrm{Act}_2\}$ as the domain of λ.

Example 4. *Interleaving and synchronous product of T_1 and T_2 as in Example 3, are depicted below.*



$T_1 \mathbin{|||} T_2$



$T_1 \otimes T_2$

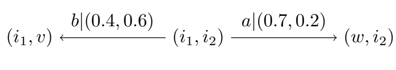
Sum. The sum of two PLTSs corresponds to their non-determinisitic composition: the resulting PLTS behaves as either of its components. Formally,

Definition 7. *Let $T_1 = \langle W_1, i_1, R_1\rangle$ and $T_2 = \langle W_2, i_2, R_2\rangle$ be two pointed PLTS over the set* Act_1 *and* Act_2*, respectively. Their sum $T_1 + T_2$ is the PLTS $\langle W, (i_1, i_2), R\rangle$ over* $\mathrm{Act} = \mathrm{Act}_1 \cup \mathrm{Act}_2$*, where*

- $W = (W_1 \times \{i_2\}) \cup (\{i_1\} \times W_2)$
- $R_a((w_1, w_2), (v_1, v_2)) = (\alpha, \beta)$ *if and only if there is a transition* $(R_1)_a(w_1, v_1) = (\alpha, \beta)$ *or there is a transition* $(R_2)_a(w_2, v_2) = (\alpha, \beta)$

Sum is actually a coproduct in $\mathsf{Pt}_{\mathcal{A}}$ (the proof follows the argument used for the product case), making $T_1 + T_2$ dual to $T_1 \times T_2$.

Example 5. *The sum $T_1 + T_2$, for T_1, T_2 defined as in Example 3 is given by*

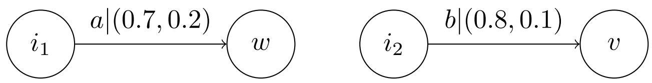


Prefixing. As a limited form of sequential composition, prefix appends to a pointed PLTS a new initial state and a new transition to the previous initial state, after which the system behaves as the original one.

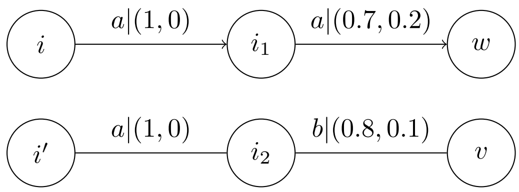
Definition 8. *Let $T = \langle W, i, R\rangle$ be a pointed PLTS over the set* Act. *Consider w_{new} a fresh state identifier not in W. Given an action $a \notin$ Act, and $\alpha, \beta \in A$, the prefix $a|(\alpha, \beta)T$ is defined as $\langle W \cup \{w_{new}\}, w_{new}, R'\rangle$ over the set of actions* $\mathrm{Act} \cup \{a\}$, *where $R'_b(w, v) = R_b(w, v)$ if $b \in$ Act and $R'_a(w_{new}, i) = (\alpha, \beta)$.*

Since it is not required that the prefixing label is distinct from the ones in the original system, prefixing does not extend to a functor in $\mathsf{Pt}_{\mathcal{A}}$, as illustrated in the counterexample below. This is obviously the case for a category of classical labelled transition systems as well. In both cases, however, prefix extens to a functor if the corresponding categories are restricted to action-preserving morphisms, i.e. in which the action component of a morphism is always an inclusion

Example 6. *Consider two pointed PLTS T_1 and T_2*



connected by a morphism $(\sigma, \lambda) : T_1 \to T_2$ such that $\sigma(i_1) = i_2$, $\sigma(w) = v$ and $\lambda(a) = b$. Now consider the prefixes $a|(1, 0)T_1$ and $a|(1, 0)T_2$ depicted below.



Clearly, a mapping from the actions in $a|(1, 0)T_1$ to the actions in $a|(1, 0)T_1$ does not exist so neither exists a morphism between the two systems.

Functorial extensions. Other useful operations between PLTSs, typically acting on transitions' positive and negative weights, and often restricted to PLTSs over a specific residuated lattice, can be defined functorially in $\mathsf{Pt}_{\mathcal{A}}$. An example involving a PLTS defined over a Gödel algebra is an operation that uniformly increases or decreases the value of the positive (or the negative, or both) weight in all transitions. Let

$$a \oplus b = \begin{cases} 1 \text{ if } a + b \geq 1 \\ 0 \text{ if } a + b \leq 0 \\ a + b \text{ otherwise} \end{cases}$$

Thus,

Definition 9. *Let $T = \langle W, i, R\rangle$ be a pointed PLTS over the set of actions* Act. *Taking $v \in [-1, 1]$, the* ***positive v-approximation*** *$T_{\oplus_v^+}$ is a pointed PLTS $\langle W, i, R'\rangle$ where, if $R_a(w, w') = (\alpha, \beta)$ then $R'_a(w, w') = (\alpha \oplus v, \beta)$*

The definition extends to a functor in $\mathsf{Pt}_{\mathcal{A}}$ which is the identity in morphisms. Similar operations can be defined to act on the negative accessibility relation or both.

Another useful operation removes all transitions in a pointed PLTS for which the positive accessibility relation is below a certain value and the negative accessibility relation is above a certain value. Formally,

Definition 10. *Let $T = \langle W, i, R\rangle$ be a pointed PLTS over the set of actions* Act*. For $p, n \in [0,1]$, the* ***purged*** *PLTS $T_{p\uparrow\downarrow n}$ is defined as $\langle W, i, R'\rangle$ where, if $R_a(w, w') = (\alpha, \beta)$ and $(p, m) \preccurlyeq (\alpha, \beta)$ then $R'_a(w, w') = (\alpha, \beta)$*

Clearly, the operation extends to a functor in $\mathsf{Pt}_{\mathcal{A}}$, mapping morphisms to themselves.

4 From PLTS to paraconsistent Kripke structures: crisp and graded bisimulations

Having introducing PLTS and their algebra, this section goes another step ahead. Stating and proving properties of such systems entails the need for a suitable modal logic to explore their (double) transition structures. We start in this section by endowing PLTS with propositions that take values over $A\times A$, therefore providing a corresponding notion of a (paraconsistent) Kripke model structure (PKS). Two approaches to defining simulation and bisimulation over them, one crips, as in the original paper [CMB22b], and another graded. We start with the following definition:

Definition 11. *A* ***paraconsistent Kripke structure*** *over a iMTL-algebra* $\boldsymbol{A}$*, a set of atomic actions* Act *and a set of proposition symbols* Prop *is a structure $\langle W, R, V\rangle$ where $\langle W, R\rangle$ is a paraconsistent transition system and $V : W \times \mathrm{Prop} \to A \times A$ is a valuation function that given any proposition $p \in \mathrm{Prop}$ and state $w \in W$, $V(w, p) = (\alpha, \beta) \in A \times A$ such that α weights the evidence of p holding at state w and β weights the evidence of p not holding at w.*

Paraconsistent Kripke structures can be related by simulation or bisimulation relations, either crisp or graded, as discussed in the sequel. As before, all constructions are parametric in the underlying iMTL-algebra $\boldsymbol{A}$, whose explicit reference is now often omitted.

4.1 Crisp simulation and bisimulation

Definition 12. *Let $M = \langle W, R, V\rangle$ and $M' = \langle W', R', V'\rangle$ be two PKS over a set of action symbols* Act *and another of propositions* Prop*. A relation $S \subseteq W \times W'$ is a* ***simulation*** *provided that, for all $(w, w') \in S$, $p \in \mathrm{Prop}$ and $a \in \mathrm{Act}$*

$$V(w,p) \preccurlyeq V'(w',p) \quad and$$

$$w \xrightarrow{a|(\alpha,\beta)}_M v \;\; then \;\; \langle \exists_{v'\in W'} .\, \exists_{\gamma,\delta\in A} .\, w' \xrightarrow{a|(\gamma,\delta)}_{M'} v' \wedge (v,v') \in S \wedge (\alpha,\beta) \preccurlyeq (\gamma,\delta)\rangle$$

The latter condition can be abbreviated to

$$w \xrightarrow{a|(\alpha,\beta)}_M v \;\; then \;\; \langle \exists_{v'\in W'} .\, w' \xrightarrow{a|(\;\gamma:\;\gamma\geq\alpha\;,\;\delta:\;\delta\leq\beta)}_{M'} v' \;\wedge\; (v,v') \in S\rangle$$

Whenever one restricts in the definition above to the existence of values γ (resp. δ) such that $\gamma \geq \alpha$ (resp. $\delta \leq \beta$), the corresponding simulation is called *positive* (resp. *negative*).

Example 7. *Recall the PLTS depicted above over the empty set of proposition symbols. The following relation is a crisp simulation,*

$$S = \{(w_1, v_1), (w_2, v_2), (w_3, v_2), (w_4, v_3), (w_5, v_4)\}$$

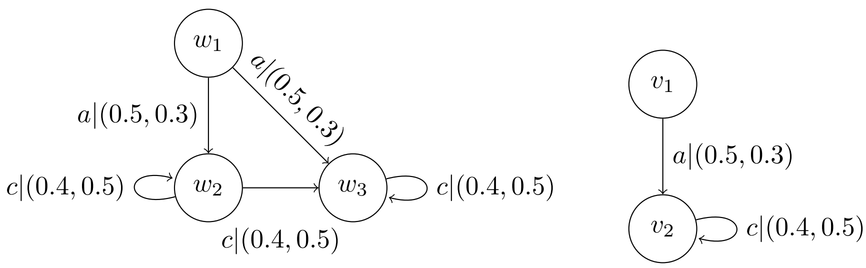
Definition 13. *Let* $M = \langle W, R, V\rangle$ *and* $M' = \langle W', R', V'\rangle$ *be two PKS over the set of actions* Act *and the set of propositions* Prop*. A relation* $B \subseteq W \times W'$ *is a* ***bisimulation*** *if for* $(w, w') \in B$, $p \in \text{Prop}$ *and* $a \in \text{Act}$ *the following conditions hold:*

$$V(w, p) = V'(w', p)$$

$$w \xrightarrow{a|(\alpha,\beta)}_M v \;\; \textit{then} \;\; \langle \exists v' \in W' : w' \xrightarrow{a|(\alpha,\beta)}_{M'} v' \wedge (v, v') \in B\rangle$$

$$w' \xrightarrow{a|(\alpha,\beta)}_{M'} v' \;\; \textit{then} \;\; \langle \exists v \in W : w \xrightarrow{a|(\alpha,\beta)}_M v \wedge (v, v') \in B\rangle$$

Example 8. *Consider the two PKS depicted below.*



with $V_1(w, p) = V_2(v_1, p)$ *and* $V_1(w_2, p) = V_1(w_3, p) = V_2(v_2, p)$ *for any proposition symbol* p*. The relation* $B = \{(w_1, v_1), (w_2, v_2), (w_3, v_3)\}$ *is a crisp bisimulation.*

4.2 Graded simulation and bisimulation

In the context of comparing paraconsistent transition structures, a more intuitive approach involves treating bisimulation as a graded relation, encompassing both positive and negative weights. This relation we refer to as *paraconsistent bisimulation* follows the lines of work documented in [Ngu22] for fuzzy modal logic. This subsection generalizes [Ngu22] approach for paraconsistent modal logic and discusses its, slightly weird, expressivity.

Definition 14. *Let* $M = \langle W, R, V\rangle$ *and* $M' = \langle W', R', V'\rangle$ *be two PKS over a iMTL-algebra* $\boldsymbol{A}$ *and a set Act of actions. A relation* $S^G : W \times W' \to A \times A$ *is a* ***graded simulation*** *if the following conditions hold for all* $p \in \text{Prop}$, $a \in \text{Act}$ *and all possible values of free variables:*

$$S^G(w, w') \preccurlyeq (V(w, p) \Rightarrow V'(w', p)) \tag{45}$$

$$\exists_{v' \in W'} \; (S^G(w, w') \otimes R_a(w, v)) \preccurlyeq (R'_a(w', v') \otimes S^G(v, v')) \tag{46}$$

It is worth noticing that the crisp simulation is not a particular case of the graded simulation. The following two examples discuss this issue.

Example 9. *Consider the following PKS with one proposition p:*

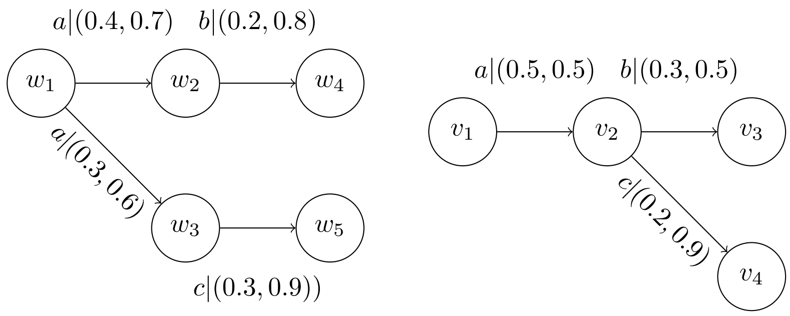
with $V(w,p) = V'(w',p) = (1,0)$. *Since* $V(w,p) \preccurlyeq V'(w',p)$ *and* $R_a(w,w) \preccurlyeq R'_a(w',w')$ *relation* $S = \{(w,w')\}$ *is a crisp simulation, i.e. it is possible to informally write* $S(w,w') = (1,0)$ *since there is complete evidence that states* w *and* w' *are similar and absolute minimal evidence that they are not. Notice that* $S(w,w') = (1,0)$ *satisfies conditions* (45) *and* (46), *that is,*

$$(1,0) \preccurlyeq V(w,p) \Rightarrow V'(w',p) = (1,0)$$

$$(1,0) \otimes (1,1) \preccurlyeq (1,0) \otimes (1,0)$$

Thus, $S(w,w') = (1,0)$ *is also a graded simulation. Since any graded simulation is bounded by the pair* $(1,0)$ *we have that* $S(w,w') = (1,0)$ *is the biggest graded simulation between the two PKS. Actually, while the notion of a crisp simulation is bivalent, that is, either states are or are not similar, for the notion of a graded simulation that is not the case. It is often possible to define other graded simulations that satisfy all conditions. For example, the relation* $S'(w,w') = (0.5, 0.5)$ *is also a graded simulation between the models that satisfies* (45) *and* (46).

Example 10. *Consider now the following PKS assuming* $(1,0)$ *as the value of proposition* p *in all states.*



The following relation is a graded simulation

$$S^G(w_1,v_1) = S^G(w_2,v_2) = S^G(w_4,v_3) = S^G(w_5,v_4) = (1,0)$$
$$S^G(w_3,v_2) = (0.2,1)$$

Transitions from state w_3 *to* w_5 *and from state* v_2 *to* v_4 *are similar with equal negative weights, i.e.* $R_c^-(w_3,w_5) = R'^-_c(v_2,v_4)$. *However, despite the transitions being similar, as per the definition of a crisp simulation, there is no crisp simulation* S *such that* $(w_3,v_2) \in S$. *However, when considering a graded simulation, a different perspective emerges. There is complete evidence that the states are not similar, as witnessed by the crisp relation, while simultaneously indicating some evidence,* 0.2, *of their similarity.*

Let us take this question a little further and investigate under which conditions graded simulation coincides with the crisp simulation. Consider two PKS

$M = \langle W, R, V\rangle$, $M' = \langle W', R', V'\rangle$ and a crisp simulation S between them. Consider any pair of states $w \in W$, $w' \in W'$ such that $(w, w') \in S$. Is there a graded simulation $S^G : W \times W' \to A \times A$ such that $S^G(w, w') = (1, 0)$? In other words, if two states are related by a crisp simulation, is there a graded simulation relating them with weight $(1, 0)$, i.e. is there complete evidence that the states are similar and absolute minimal evidence that they are not similar?

For some proposition symbol p and action symbol a, consider $V(w, p) = (a, b)$, $V'(w', p) = (a', b')$, $R_a(w, v) = (\alpha, \beta)$ and $R'_a(w', v') = (\alpha', \beta')$. Since $(w, w') \in S$, we have that $(a, b) \preccurlyeq (a', b')$ and $(\alpha, \beta) \preccurlyeq (\alpha', \beta')$, as well as, $(v, v') \in S$. Let us now evaluate if it is possible to have a graded simulation such that $S^G(w, w') = (1, 0)$, or without loss of generality, $S^G(w, w') \preccurlyeq (1, 0)$. Relation S^G must satisfy condition (45),

$$
\begin{aligned}
S^G(w, w') &\preccurlyeq V(w, p) \Rightarrow V'(w', p) \\
&\quad \{\text{defn of } \Rightarrow\} \\
&= ((a \to a') \wedge (b' \to b), a \wedge d) \\
&\quad \{\text{Hypothesis: } a \le a' \text{ and } b \ge b'\} \\
&= (1, a \wedge d)
\end{aligned}
$$

Thus, from condition (45) $S^G \preccurlyeq (1, 0)$ implies $a = 0$ or $d = 0$. Relation S^G must also satisfy condition (46),

$$S^G(w, w') \otimes R_a(w, v) \preccurlyeq R'_a(w', v') \otimes S^G(v, v')$$

Using adjunction (19), $S^G(w, w') \preccurlyeq R_a(w, v) \Rightarrow (R'_a(w', v') \otimes S^G(v, v'))$. Thus,

$$
\begin{aligned}
S^G(w, w') &\preccurlyeq R_a(w, v) \Rightarrow (R'_a(w', v') \otimes S^G(v, v')) \\
&\quad \{S^G(v, v') \preccurlyeq (1, 0) \text{ by (21) and (20)}\} \\
&\preccurlyeq R_a(w, v) \Rightarrow (R'_a(w', v') \otimes (1, 0)) \\
&\quad \{\text{defn of } \otimes\} \\
&= R_a(w, v) \Rightarrow (\alpha', (\alpha' \to 0) \wedge \beta') \\
&\quad \{\text{defn of } \Rightarrow\} \\
&= ((\alpha \to \alpha') \wedge (((\alpha' \to 0) \wedge \beta') \to \beta), \alpha \wedge \alpha' \to 0 \wedge \beta') \\
&\quad \{\text{Hypothesis: } \alpha \le \alpha' \text{ and } \beta' \le \beta\} \\
&= (1, \alpha \wedge (\alpha' \to 0) \wedge \beta')
\end{aligned}
$$

Thus, from condition (46), $S^G \preccurlyeq (1, 0)$ implies $\alpha = 0$ or $\beta' = 0$ or $\alpha' > 0$.

In conclusion, if $(w, w') \in S$ there is a graded simulation S^G such that $S^G(w, w') = (1, 0)$ only when $V^+(w, p) = 0$ or $V'^-(w', p) = 0$ and either $R^+_a(w, v) = 0$ or $R'^-_a(w', v') = 0$ or $R'^+_a(w', v') > 0$.

This discussion sheds light on a somehow surprising conclusion: that identity fails to be a simulation. Actually, consider the following PKSs,

with $V(w, p) = V'(w', p) = (0.2, 0.2)$. The relation $S^G(w, w') = (1, 0)$ is not a graded simulation because $(1, 0) \not\preccurlyeq (0.2, 0.2) \Rightarrow (0.2, 0.2) = (1, 0.2)$

Definition 15. *Let $M = \langle W, R, V\rangle$ and $M' = \langle W', R', V'\rangle$ be two PKS over a iMTL-algebra $\boldsymbol{A}$ and a set Act of actions. A relation $B^G : W \times W' \to A \times A$ is a* ***graded bisimulation*** *if the following conditions hold for all $p \in \mathrm{Prop}$, $a \in \mathrm{Act}$ and all possible values of free variables:*

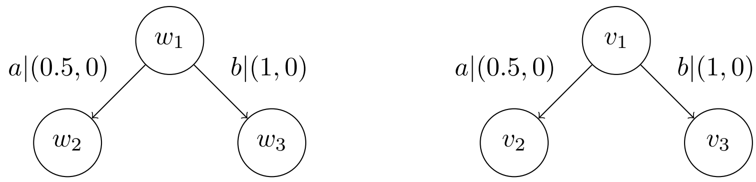
$$B^G(w, w') \preccurlyeq (V(w, p) \Leftrightarrow V'(w', p)) \tag{47}$$

$$\exists v' \in W' \; (B^G(w, w') \otimes R_a(w, v)) \preccurlyeq (R'_a(w', v') \otimes B^G(v, v')) \tag{48}$$

$$\exists v \in W \; (B^G(w, w') \otimes R'_a(w', v')) \preccurlyeq (R_a(w, v) \otimes B^G(v, v')) \tag{49}$$

Again a crisp bisimulation is not, in general, a particular case of a graded one, as illustrated in the following examples.

Example 11. *Consider the PKS depicted below where $V(w_1, p) = V'(v_1, p) = (0, 1)$, $V(w_2, p) = V'(v_2, p) = (1, 0)$ and $V(w_3, p) = V'(v_3, p) = (0, 0)$.*



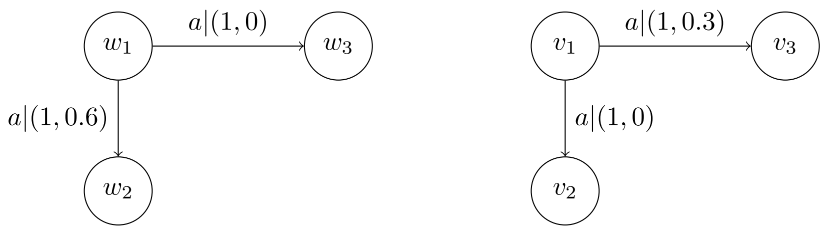
Clearly, $B = \{(w_1, v_1), (w_2, v_2), (w_3, v3)\}$ is a crisp bisimulation. Informally, B can be written as a graded bisimulation $B^G : W \times W \to A \times A$ where $B^G(w_1, v_1) = B^G(w_2, v_2) = B^G(w_3, v_3) = (1, 0)$. Notice that, B^G satisfies condition (47)*,*

$$\begin{aligned}(1, 0) &\preccurlyeq (0, 1) \Leftrightarrow (0, 1) = (1, 0)\\ (1, 0) &\preccurlyeq (1, 0) \Leftrightarrow (1, 0) = (1, 0)\\ (1, 0) &\preccurlyeq (0.5, 0) \Leftrightarrow (0.5, 0) = (1, 0)\end{aligned}$$

and conditions (48) *and* (49)*,*

$$\begin{aligned}(1, 0) \otimes (0.5, 0) = (0.5, 0) &\preccurlyeq (0.5, 0) \otimes (1, 0) = (0.5, 0)\\ (1, 0) \otimes (1, 0) = (1, 0) &\preccurlyeq (1, 0) \otimes (1, 0) = (1, 0)\end{aligned}$$

Example 12. *For an example in the opposite direction, consider*



where $V(w_1, p) = V'(v_1, p) = (1, 1)$, $V(w_2, p) = V'(v_3, p) = (1, 0.5)$ and $V(w_3, p) = V'(v_2, p) = (0.8, 0.8)$. A graded bisimulation B^G between the above PKS can be computed as follows:

- *From condition* (47)*,*

$$B^G(w_1, v_1) \preccurlyeq (1, 1) \Leftrightarrow (1, 1) = (1, 1)$$

$$\begin{aligned}
B^G(w_2, v_2) &\preccurlyeq (1, 0.5) \Leftrightarrow (0.8, 0.8) = (0.5, 0.8) \\
B^G(w_3, v_3) &\preccurlyeq (0.5, 0.8) \\
B^G(w_2, v_3) &\preccurlyeq (1, 0.5) \\
B^G(w_3, v_2) &\preccurlyeq (1, 0.8)
\end{aligned}$$

- *From condition* (48), *on the other hand,*

  $$B^G(w_1, v_1) \otimes R_a(w_1, w_2) \preccurlyeq R'_a(v_1, v_2) \otimes B^G(w_2, v_2) \preccurlyeq (1, 0) \otimes (0.5, 0.8) = (0.5, 0)$$

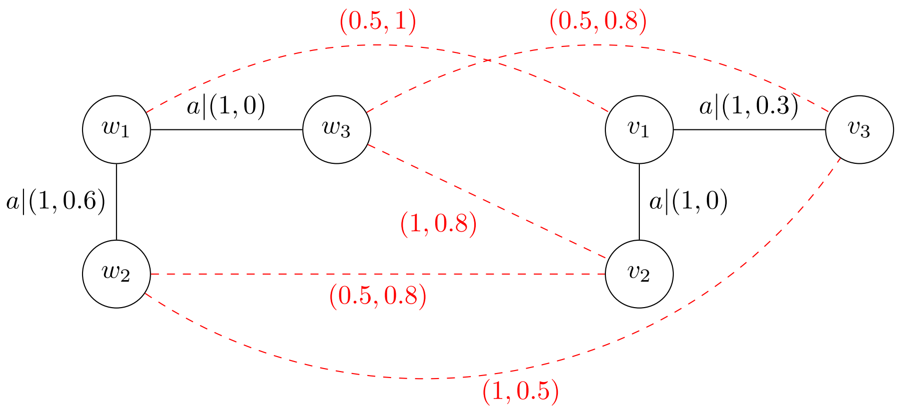
  Thus, $B^{G^+}(w_1, v_1) \wedge 1 = B^{G^+}(w_1, v_1) \leq 0.5$. *Similarly,*

  $$B^G(w_1, v_2) \otimes R_a(w_1, w_2) \preccurlyeq R'_a(v_2, v_2) \otimes B^G(w_2, v_2) \preccurlyeq (0, 1) \otimes (0.5, 0.8) = (0, 1)$$

  Thus, $B^G(w_1, v_2) = (0, 1)$. *A similar reasoning yealds* $B^G(w_1, v_3) = (0, 1)$.

- *Finally, from condition* (49), *we conclude that* $B^G(w_2, v_1) = B^G(w_3, v_1) = (0, 1)$.

Therefore, one possible graded bisimulation between the two PKS is $B^G(w_1, v_1) = (0.5, 1)$, $B^G(w_2, v_2) = (0.5, 0.8)$, $B^G(w_2, v_3) = (1, 0.5)$, $B^G(w_3, v_2) = (1, 0.8)$ *and* $B^G(w_3, v_3) = (0.5, 0.8)$ *since it satisfies conditions* (47)-(49). *The figure below depicts this graded bisimulation through red dashed lines.*



Once again, we may ask which conditions are necessary for crisp and graded bisimulations to match. Consider two PKS $M = \langle W, R, V \rangle$, $M' = \langle W', R', V' \rangle$ and a crisp bisimulation B between them. Consider any pair of states $w \in W$, $w' \in W'$ such that $(w, w') \in B$. Is there a graded bisimulation $B^G : W \times W' \to A \times A$ such that $B^G(w, w') = (1, 0)$?

If $(w, w') \in B$ there is a graded bisimulation B^G such that $B^G(w, w') = (1, 0)$ only if $V^+(w, p) = 0$ or $V^-(w, p) = 0$ and either $R_a^+(w, v) = 0$ or $R'^-_a(w', v') = 0$ or $R'^+_a(w', v') > 0$ and either $R'^+_a(w', v') = 0$ or $R_a^-(w, v) = 0$ or $R_a^+(w, v) > 0$.

5 A modal logic for paraconsistent structures

5.1 The logic

We are now able to introduce a modal logic $\mathrm{P}(\mathcal{A})$ to specify properties of paraconsistent structures parametric on an iMTL-algebra.

Definition 16. *Given an iMTL-algebra* $\boldsymbol{A}$*, a set of proposition symbols* Prop *and a set of action symbols* Act*, the sentences* $\mathsf{Sen}(\mathrm{Prop}, \mathrm{Act})$ *of* $\mathrm{P}(\mathcal{A})$ *are generated by the following grammar:*

$$\varphi \coloneqq \bot \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle a \rangle \varphi$$

where $p \in \mathrm{Prop}$ *and* $a \in \mathrm{Act}$*. As usual, the following abbreviations are considered:* $\top = \neg\bot$*,* $\varphi \vee \varphi' = \neg(\neg\varphi \wedge \neg\varphi')$*,* $\varphi \rhd \varphi' = \neg\varphi \vee \varphi' = \neg(\varphi \wedge \neg\varphi')$*,* $\varphi \bowtie \varphi' = (\varphi \rhd \varphi') \wedge (\varphi' \rhd \varphi)$ *and* $[a]\varphi = \neg\langle a \rangle \neg\varphi$*.*

The satisfaction relation in $\mathrm{P}(\mathcal{A})$ evaluates the satisfaction of a sentence φ by a pair of weights $(a, b) \in A \times A$ of a PKS M over $\mathcal{A}$. As before, the positive weight a stands for the evidence that φ holds in M while the negative weight b stands for the evidence of the opposite fact.

Definition 17. *Let* $\boldsymbol{A} = \langle A, \wedge, \vee, 1, 0, \rightarrow \rangle$ *be an iMTL-algebra and* $M = \langle W, R, V \rangle$ *a PKS. The satisfaction relation,*

$$\models : M \times \mathsf{Sen}(\mathrm{Prop}, \mathrm{Act}) \rightarrow A \times A$$

is given by

$$(M \models \varphi) = \underset{w \in W}{\hat{\wedge}} (M, w \models \varphi)$$

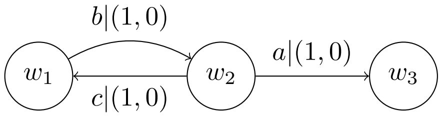
where relation $\models$ *is recursively defined as follows,*

- $(M, w \models \bot) = (0, 1)$
- $(M, w \models p) = V(w, p)$
- $(M, w \models \neg\varphi) = /\!\!/(M, w \models \varphi)$
- $(M, w \models \varphi \wedge \varphi') = (M, w \models \varphi) \mathbin{\hat{\wedge}} (M, w \models \varphi')$
- $(M, w \models \langle a \rangle \varphi) = \underset{v \in W}{\check{\vee}} \Big(R_a(w, v) \otimes (M, v \models \varphi) \Big)$

Informally, sentence φ is evaluated at each state of the model M and (a, b) is the result of the conjunction of each of these evaluations. Hence, a sentence φ is said to be *valid* if, for any state $w \in W$, $(M, w \models \varphi) = (1, 0)$, i.e. there is complete evidence that φ holds at state w and absolute minimal evidence that it does not hold. Note how the operators characterising the twist-structure introduced earlier in the paper reappear in the satisfaction relation to take care of pairs of weights being computed.

The next example emphasizes the fact that PKS generalize transition structures in which the information is consistent, i.e. in which the transition relation and the valuation function are either $(1, 0)$ or $(0, 1)$.

Example 13. *Consider the set* $\text{Act} = \{a, b, c\}$ *of action symbols and the set* $\text{Prop} = \{p\}$ *of propositions. Let* M *be the following PKS over the Gödel algebra*

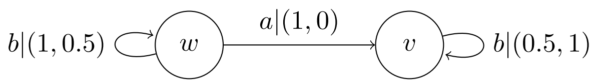


with $V(w_1, p) = V(w_3, p) = (1, 0)$ *and* $V(w_2, p) = (0, 1)$*. Then,*

$$
\begin{aligned}
(M, w_2 \models \langle c\rangle p \wedge [a]\neg p) &= \Big(M, w_2 \models \langle c\rangle p\Big) \mathbin{\hat{\wedge}} \Big(M, w_2 \models [a]\neg p\Big) \\
&= \Big((R_c(w_2, w_1) \otimes (M, w_1 \models p)\Big) \mathbin{\hat{\wedge}} \Big(M, w_2 \models \neg\langle a\rangle\neg\neg p\Big) \\
&= \Big((1,0) \otimes (1,0)\Big) \mathbin{\hat{\wedge}} /\!\!/\Big(M, w_2 \models \langle a\rangle p\Big) \\
&= (1 \wedge 1, 1 \to 0 \wedge 1 \to 0) \mathbin{\hat{\wedge}} /\!\!/\Big(R_a(w_2, w_3) \otimes (M, w_3 \models p)\Big) \\
&= (1,0) \mathbin{\hat{\wedge}} /\!\!/\Big((1,0) \otimes (0,1)\Big) \;=\; (1,0) \mathbin{\hat{\wedge}} /\!\!/(0,1) \\
&= (1 \wedge 1, 0 \vee 0) \\
&= (1,0)
\end{aligned}
$$

Informally, this can be read as follows: at state w_2 *there is a transition trough action* c *to a state where* p *holds and all transitions trough action* a *reach a state where* p *does not hold. As expected, the result is a consistent pair of weights* $(1, 0)$*, that is, there is complete evidence,* 1*, that this sentence holds at state* w_2 *and absolute non evidence,* 0*, evidence that it does not hold.*

Example 14. *Consider another PKS over the Gödel algebra, with* $\text{Act} = \{a, b\}$ *and* $Prop = \{p, q, r\}$



with, $V(w, p) = (1, 1)$*,* $V(w, q) = V(v, p) = (0, 0.5)$*,* $V(w, r) = V(v, r) = (0.5, 0.5)$ *and* $V(v, q) = (0, 0)$*. Then,*

$$
\begin{aligned}
(M, w \models r \rhd (p \vee q)) &= (M, w \models \neg r \vee (p \vee q)) \\
&= (M, w \models \neg(\neg\neg r \wedge \neg(p \vee q))) \\
&= /\!\!/\Big(M, w \models r \wedge \neg(p \vee q)\Big) \\
&= /\!\!/\Big((M, w \models r) \mathbin{\hat{\wedge}} (M, w \models \neg(p \vee q))\Big) \\
&= /\!\!/\Big(V(w, r) \mathbin{\hat{\wedge}} /\!\!/(M, w \models p \vee q)\Big) \\
&= /\!\!/\Big(V(w, r) \mathbin{\hat{\wedge}} (M, w \models \neg p) \mathbin{\hat{\wedge}} (M, w \models \neg q)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= /\!\!/ \left(V(w,r) \overset{\wedge}{\wedge} /\!\!/ V(w,p) \overset{\wedge}{\wedge} /\!\!/ V(w,q) \right) \\
&= /\!\!/ \left((0.5, 0.5) \overset{\wedge}{\wedge} (1,1) \overset{\wedge}{\wedge} (0.5, 0) \right) \\
&= /\!\!/ \, (0.5 \wedge 1 \wedge 0.5, 0.5 \vee 1 \vee 0) \; = \; /\!\!/ (0.5, 1) \\
&= (1, 0.5)
\end{aligned}
$$

Similarly, $(M, v \models r \rhd (p \vee q)) \; = \; (0.5, 0)$*. Therefore,*

$$M \models r \rhd (p \vee q) \; = \; (1, 0.5) \overset{\wedge}{\wedge} (0.5, 0) \; = \; (0.5, 0.5)$$

giving 0.5 *evidence that the sentence* $r \rhd (p \vee q)$ *holds and does not hold in this PKS.*

5.2 Properties

This subsection presents results concerning soundness of paraconsistent Kripke structures. In this initial exploration, we adjust the classical notion of soundness, where both premises and conclusions are consistently true, and instead, introduce a graded notion of soundness, where both premises and conclusions may encompass vagueness and contradictions.

A similar motivation can be found in [CR10] where the authors adopt a classical notion of logical consequence. Specifically, $T \models_{GK} \varphi$ if and only if for any GK-model M and any world x of M, $M \models_x T$ implies $M \models_x \varphi$. Later, they introduce another version of logical consequence denoted as $T \models_{\leq GK} \varphi$, where for any model M and any world x in M, inf$e(x, T), \leq, e(x, \varphi)$. They then proceed to prove the soundness of the $\Box$-fragment (and similarly the $\diamond$-fragment), stating soundness as $T \vdash_{\mathcal{G}\Box} \varphi$ implies $T \models \leq GK\varphi$. However, it's important to note that the main results highlighted in [CR10] pertain to countable theories T, where the notions of $\models_{GK}$ and $\models_{\leq GK}$ are equivalent. Our approach for paraconsistent Kripke structures is similar, as we also consider classical and graded notions of soundness, the latter of which we term as *g-soundness*. It is evident that *g-soundness* is a more relaxed notion of classical soundness, thus implying that g-soundness entails classical soundness. However, the converse is not true, leading us to explore soundness results in Kripke structures where g-soundness and classical soundness are not equivalent.

Let us start by revisiting the classical notion of soundness and then introduce a graded notion of soundness, which extends upon the classical understanding. Following that, we will conduct a preliminary investigation into the properties of soundness for paraconsistent Kripke structures.

Definition 18. *A rule*

$$\frac{\varphi_1 \;\cdots\; \varphi_n}{\varphi}$$

is sound *if for any paraconsistent Kripke structure* $M = \langle W, R, V \rangle$*,*

$$\textit{if } (M \models \varphi_i) \; = \; (1, 0) \textit{ for } i \in \{1, \ldots, n\}, \textit{ then } (M \models \varphi) \; = \; (1, 0)$$

The following notion of soundness accommodates vague and inconsistent evaluations, implying that from the evidence of truth and falsity of premises, it is possible to draw conclusions which evidence of truth is higher and evidence of falsity is lower.

Definition 19. *A rule*

$$\frac{\varphi_1 \ \cdots \ \varphi_n}{\varphi}$$

is g-sound *if for any paraconsistent Kripke structure* $M = \langle W, R, V \rangle$,

$$\overset{n}{\underset{i=1}{\stackrel{\wedge}{\wedge}}} \ (M \models \varphi_i) \preccurlyeq (M \models \varphi)$$

The following theorem is highly significant for subsequent findings because it assures us that all results of g-soundness generalize classical soundness. Specifically, for any paraconistent Kripke strucure $M = \langle W, R, V \rangle$ a rule is g-sound whenever

$$\overset{n}{\underset{i=1}{\stackrel{\wedge}{\wedge}}} \ (M \models \varphi_i) \preccurlyeq (M \models \varphi)$$

Consequently, if all premises $\varphi_1, \ldots, \varphi_n$ are consistently true, then the conclusion φ is also consistently true. Thus, *g-soundness* implies classical *soundness*. However, the converse is not necessarily true, and we will provide counterexamples in future remarks.

Theorem 1. *If a rule is* g-sound*, then it is* sound*.*

Proof. Consider a g-sound rule whose premises are $\varphi_1 \ \ldots \ \varphi_n$ and conclusion φ. We must prove that this rule is sound. Let us fix a paraconsistent Kripke structure $M = \langle W, R, V \rangle$ such that the premises are consistently true, that is, $(M \models \varphi_i) = (1, 0)$ for any $i \in \{1, \ldots, n\}$. By hypothesis

$$(M \models \varphi_1) \stackrel{\wedge}{\wedge} (M \models \varphi_2) \stackrel{\wedge}{\wedge} \cdots \stackrel{\wedge}{\wedge} (M \models \varphi_n) \preccurlyeq (M \models \varphi)$$

Since the premises are consistently true it follows, $(1, 0) \preccurlyeq (M \models \varphi)$. Trivially, this implies that $(M \models \varphi) = (1, 0)$. Therefore, any rule that is sound is a particular case of a rule that is g-sound whenever the premises are consistently true. □

We proceed by examining classical and graded soundness results in PKS and present counterexamples for rules that are not sound.

Theorem 2. *The following rules*

$$\frac{\varphi, \ \varphi \triangleright \varphi'}{\varphi'} \quad (\mathbf{MP}) \qquad \frac{\varphi}{[a]\varphi} \quad (\mathbf{Gen})$$

are sound.

Proof. Let $M = \langle W, R, V \rangle$ be any PKS.

- For (**MP**) let us assume that for any state $w \in W$,

  $$(M, w \models \varphi) = (1, 0) \text{ and } (M, w \models \varphi \triangleright \varphi') = (1, 0)$$

  By definition of $\models$,

  $$(M, w \models \varphi \triangleright \varphi') = (M, w \models \neg\varphi) \stackrel{\vee}{\vee} (M, w \models \varphi') = (1, 0)$$

  By hypothesis $(M, w \models \neg\varphi) = /\!\!/(M, w \models \varphi) = (0, 1)$. Finally, using the definition of $\stackrel{\vee}{\vee}$, it follows that $(M, w \models \varphi') = (1, 0)$.

- For (**Gen**) let us assume that $(M, w \models \varphi) = (1, 0)$ for any $w \in W$. Then,

$$
\begin{aligned}
& M, w \models [a]\varphi \\
=& \{\text{defn. } [a]\varphi = \neg\langle a\rangle\neg\varphi\} \\
& M, w \models \neg\langle a\rangle\neg\varphi \\
=& \{\text{defn. of } \models\} \\
& /\!\!/ \left(\bigvee\!\!\!\!\bigvee_{v\in W} \Big(R_a(w, v) \otimes (M, v \models \neg\varphi) \Big) \right) \\
=& \{\text{Hypothesis}\} \\
& /\!\!/ \left(\bigvee\!\!\!\!\bigvee_{v\in W} \Big(R_a(w, v) \otimes (0, 1) \Big) \right) \\
=& \{\text{defn. of } \otimes\} \\
& /\!\!/ \, (0, 1) \\
=& \{\text{defn. of } /\!\!/\} \\
& (1, 0)
\end{aligned}
$$

□

Theorem 3. *The following rules*

$$\frac{\langle a\rangle(\varphi \vee \varphi')}{\langle a\rangle\varphi \vee \langle a\rangle\varphi'} \tag{50}$$

$$\frac{\langle a\rangle(\varphi \wedge \varphi')}{\langle a\rangle\varphi \wedge \langle a\rangle\varphi'} \tag{51}$$

are g-sound.

Proof. Let $M = \langle W, R, V\rangle$ be any PKS.

- For (50) note that for any $w \in W$,

$$
\begin{aligned}
& M, w \models \langle a\rangle(\varphi \vee \varphi') \\
=& \{\text{defn. of } \models\} \\
& \bigvee\!\!\!\!\bigvee_{v\in W} \Big(R_a(w, v) \otimes (M, v \models \varphi \vee \varphi') \Big) \\
=& \{\text{defn. of } \models\} \\
& \bigvee\!\!\!\!\bigvee_{v\in W} \Big(R_a(w, v) \otimes \big((M, v \models \varphi) \curlyvee (M, v \models \varphi') \big) \Big) \\
=& \{\text{Property (23)}\} \\
& \bigvee\!\!\!\!\bigvee_{v\in W} \Big(\big(R_a(w, v) \otimes (M, v \models \varphi) \big) \curlyvee \big(R_a(w, v) \otimes (M, v \models \varphi') \big) \Big) \\
=& \{\text{Distribution of } \curlyvee\} \\
& \bigvee\!\!\!\!\bigvee_{v\in W} \Big(R_a(w, v) \otimes (M, v \models \varphi) \Big) \curlyvee \bigvee\!\!\!\!\bigvee_{v\in W} \Big(R_a(w, v) \otimes (M, v \models \varphi') \Big) \\
=& \{\text{defn. of } \models\} \\
& M, w \models \langle a\rangle\varphi \vee \langle a\rangle\varphi'
\end{aligned}
$$

Hence, $(M \models \langle a\rangle(\varphi \vee \varphi')) \preccurlyeq (M \models \langle a\rangle\varphi \vee \langle a\rangle\varphi')$.

- For (51) note that for any $w \in W$,

$$
\begin{aligned}
& M, w \models \langle a \rangle (\varphi \wedge \varphi') \\
= & \{\text{defn. of } \models\} \\
& \underset{v \in W}{\overset{\vee}{\vee}} \Big(R_a(w, v) \otimes (M, v \models \varphi \wedge \varphi') \Big) \\
= & \{\text{defn. of } \models\} \\
& \underset{v \in W}{\overset{\vee}{\vee}} \Big(R_a(w, v) \otimes \Big((M, v \models \varphi) \overset{\wedge}{\wedge} (M, v \models \varphi') \Big) \Big) \\
\preccurlyeq & \{\text{Property (24)}\} \\
& \underset{v \in W}{\overset{\vee}{\vee}} \Big(\Big(R_a(w, v) \otimes (M, v \models \varphi) \Big) \overset{\wedge}{\wedge} \Big(R_a(w, v) \otimes (M, v \models \varphi') \Big) \Big) \\
= & \{\text{Distribution of } \overset{\vee}{\vee} \text{ over } \overset{\wedge}{\wedge}\} \\
& \underset{v \in W}{\overset{\vee}{\vee}} \Big(R_a(w, v) \otimes (M, v \models \varphi) \Big) \overset{\wedge}{\wedge} \underset{v \in W}{\overset{\vee}{\vee}} \Big(R_a(w, v) \otimes (M, v \models \varphi') \Big) \\
= & \{\text{defn. of } \models\} \\
& (M, w \models \langle a \rangle \varphi) \overset{\wedge}{\wedge} (M, w \models \langle a \rangle \varphi') \\
= & \{\text{defn. of } \models\} \\
& M, w \models \langle a \rangle \varphi \wedge \langle a \rangle \varphi'
\end{aligned}
$$

  Hence, $(M \models \langle a \rangle (\varphi \wedge \varphi') \preccurlyeq (M \models \langle a \rangle \varphi \wedge \langle a \rangle \varphi')$.

□

Corollary 1. *The following rules*

$$\frac{\langle a \rangle (\varphi \vee \varphi')}{\langle a \rangle \varphi \vee \langle a \rangle \varphi'} \qquad\qquad \frac{\langle a \rangle (\varphi \wedge \varphi')}{\langle a \rangle \varphi \wedge \langle a \rangle \varphi'}$$

are sound.

Proof. This corollary is a direct consequence of Theorem 3 and Theorem 1. □

We conclude this subsection by revisiting the rules outlined in Theorem 2 alongside axiom (**K**) and show how these rules are not g-sound.

Lemma 5. *The following rules*

$$\frac{\varphi, \ \varphi \rhd \varphi'}{\varphi'} \tag{MP}$$

$$\frac{\varphi}{[a]\varphi} \tag{Gen}$$

$$\frac{[a](\varphi \rhd \varphi')}{[a]\varphi \rhd [a]\varphi'} \tag{K}$$

are not g-sound.

Proof.

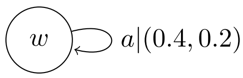
- Even though by Theorem 2, modus ponens is sound, this rule is not g-sound. For instance, consider a PKS $M = \langle W, R, V\rangle$ such that at some state $w \in W$, $(M, w \models \varphi) = (0.3, 0.4)$ and $(M, w \models \varphi') = (0.2, 0)$. Then,

$$(M \models \varphi) \mathrel{\hat{\wedge}} (M, w \models \varphi \rhd \varphi') = (0.3, 0.4) \mathrel{\hat{\wedge}} \Big((0.4, 0.3) \mathrel{\check{\vee}} (0.2, 0)\Big) = (0.3, 0.4)$$

However, $(0.3, 0.4) \not\preccurlyeq (0.2, 0)$. Hence the following assertion does not always hold.

$$\Big((M \models \varphi) \mathrel{\hat{\wedge}} (M, w \models \varphi \rhd \varphi')\Big) \preccurlyeq (M, w \models \varphi')$$

- Similarly, even though by Theorem 2 modal generalization is sound, this rule is not g-sound. For instance, consider the following PKS $M = \langle W, R, V\rangle$ depicted below.

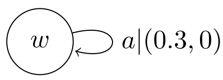


with $(M, w \models \varphi) = (0.3, 0.5)$, for some sentence φ. Then,

$$\begin{aligned}
&M, w \models [a]\varphi \\
=&M, w \models \neg\langle a\rangle\neg\varphi \\
=&\mathbin{/\!\!/} (R_a(w, w) \otimes \mathbin{/\!\!/}(M, w \models \varphi)) \\
=&\mathbin{/\!\!/} ((0.4, 0.2) \otimes (0.5, 0.3)) \\
=&(0.2, 0.4)
\end{aligned}$$

However, $(M, w \models \varphi) \not\preccurlyeq (M, w \models [a]\varphi)$.

- Finally, we explore the soudness of axiom (**K**). We will prove that this rule is not classically sound by providing a counterexample such that $(M, w \models [a](\varphi \rhd \varphi')) = (1, 0)$ however $(M, w \models [a]\varphi \rhd [a]\varphi') \neq (1, 0)$. For instance, consider the PKS $M = \langle W, R, V\rangle$ depicted below.



with $(M, w \models \varphi') = (0, 0.5)$ and $(M, w \models \varphi') = (0, 0)$, for some sentences φ and φ'. Then,

$$\begin{aligned}
&M, w \models [a](\varphi \rhd \varphi') \\
=&M, w \models \neg\langle a\rangle(\varphi \wedge \neg\varphi') \\
=&\mathbin{/\!\!/} \Big(R_a(w, w) \otimes ((M, w \models \varphi) \mathrel{\hat{\wedge}} \mathbin{/\!\!/}(M, w \models \varphi'))\Big) \\
=&\mathbin{/\!\!/} ((0.3, 0) \otimes (0 \wedge 0, 0.5 \vee 0)) \\
=&(1, 0)
\end{aligned}$$

However,

$$\begin{aligned}
&M, w \models [a]\varphi \rhd [a]\varphi' \\
=&M, w \models \langle a\rangle\neg\varphi \vee \neg\langle a\rangle\neg\varphi' \\
=&(M, w \models \langle a\rangle\neg\varphi) \mathrel{\substack{\vee\\\vee}} (M, w \models \neg\langle a\rangle\neg\varphi') \\
=&\Big(R_a(w,w) \otimes /\!\!/(M, w \models \varphi)\Big) \mathrel{\substack{\vee\\\vee}} /\!\!/\Big(R_a(w,w) \otimes /\!\!/(M, w \models \varphi')\Big) \\
=&((0.3,0) \otimes (0.5,0)) \mathrel{\substack{\vee\\\vee}} /\!\!/((0.3,0) \otimes (0,0)) \\
=&(0.3,0) \mathrel{\substack{\vee\\\vee}} (0,0) \\
=&(0.3,0)
\end{aligned}$$

Hence, the rule

$$\frac{[a](\varphi \rhd \varphi')}{[a]\varphi \rhd [a]\varphi'}$$

is not sound and by the contrapositive of Theorem 1 it is also not g-sound.

□

5.3 Validity preservation over simulation

Our last question addresses preservation of modal validity over different forms of simulation in PKS. We start with the crisp case.

Theorem 4. *Let $M = \langle W, R, V\rangle$ and $M' = \langle W', R', V'\rangle$ be two PKS over a iMTL-algebra $\mathcal{A}$, and $S \subseteq W \times W'$ be a crisp simulation. Then, for every $(w, w') \in S$ and any sentence $\varphi \in \mathsf{Sen}^+(\mathrm{Prop}, \mathrm{Act})$:*

$$(M, w \models \varphi) \preccurlyeq (M', w' \models \varphi) \tag{52}$$

where $\mathsf{Sen}^+(\mathrm{Prop}, \mathrm{Act})$ is the positive fragment of $\mathrm{P}(\mathcal{A})$.

Proof. The proof is by induction over the structure of sentences.

- Case $\perp$ is trivial, using the definition of $\models$ $(M, w \models \perp) = (0,1) = (M', w' \models \perp)$. Thus,

$$(M, w \models \perp) \preccurlyeq (M', w' \models \perp)$$

- For $p \in \mathrm{Prop}$ the result follows from the fact that S is a simulation hence, $V(w,p) \preccurlyeq V'(w',p)$, using the definition of $\models$, $(M, w \models p) \preccurlyeq (M', w' \models p)$.

- For $\varphi_1 \wedge \varphi_2$ we reason,

$$\begin{aligned}
&(M, w \models \varphi_1 \wedge \varphi_2) \\
=&\ \{\text{defn of} \models\} \\
&(M, w \models \varphi_1) \mathrel{\substack{\wedge\\\wedge}} (M, w \models \varphi_2) \\
\preccurlyeq&\ \{\text{Induction Hypothesis twice}\} \\
&(M', w' \models \varphi_1) \mathrel{\substack{\wedge\\\wedge}} (M', w' \models \varphi_2) \\
=&\ \{\text{defn of} \models\} \\
&(M', w' \models \varphi_1 \wedge \varphi_2)
\end{aligned}$$

- For $\langle a\rangle\varphi$,

$$
\begin{aligned}
& M, w \models \langle a\rangle\varphi \\
= & \{\text{defn of } \models\} \\
& \bigvee_{v\in W} \Big(R_a(w,v) \otimes (M, v \models \varphi) \Big) \\
\preccurlyeq & \{\star\} \\
& \bigvee_{v'\in W'} \Big(R'_a(w',v') \otimes (M', v' \models \varphi) \Big) \\
= & \{\text{defn of } \models\} \\
& M', w' \models \langle a\rangle\varphi
\end{aligned}
$$

The step labeled by $*$ in the last proof, is justied as follows: since S is a simulation, there exists $v' \in W'$ such that $R_a(w,v) \preccurlyeq R'_a(w',v')$ and by hypothesis $(M, v \models \varphi) \preccurlyeq (M', v' \models \varphi)$. Using (21),

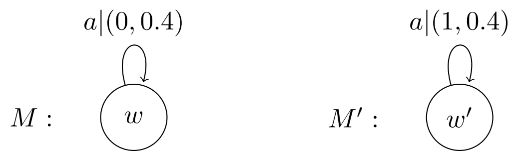
$$R_a(w,v) \otimes (M, v \models \varphi) \preccurlyeq R'_a(w',v') \otimes (M', v' \models \varphi)$$

Since $\bigvee$ is monotone, we cnclude that,

$$\bigvee_{v\in W} \Big(R_a(w,v) \otimes (M, v \models \varphi) \Big) \preccurlyeq \bigvee_{v'\in W'} \Big(R'_a(w',v') \otimes (M', v' \models \varphi) \Big)$$

□

Not all sentences are, however, preserved by a crisp simulation. Typical examples are $\neg\varphi$, $\varphi_1 \rhd \varphi_2$ and $[a]\varphi$, as shown by the following counterexamples. Consider the following PKS $M = \langle W, R, V\rangle$.



Note that, $R_a(w,w) = (0, 0.4) \preccurlyeq (1, 0.4) = R'_a(w',w')$ and $V(w,p) = (0.3, 0.8) \preccurlyeq (0.6, 0.7) = V'(w',p)$ and $V(w,q) = (0.1, 0.4) \preccurlyeq (0.2, 0.4) = V'(w',q)$. Hence, $S = \{(w,w')\}$ is a simulation.

- For $\neg p$ we have that $(M, w \models \neg p) = /\!\!/ V(w,p) = (0.8, 0.3)$ and $(M', w' \models \neg p) = /\!\!/ V'(w',p) = (0.7, 0.6)$. That is,

$$(M, w \models \neg p) \not\preccurlyeq (M', w' \models \neg p)$$

- For $p \rhd q$ we have that $(M, w \models p \rhd q)^- = 0.3$ and $(M', w' \models p \rhd q)^- = 0.4$. Thus,

$$(M, w \models p \rhd q) \not\preccurlyeq (M', w' \models p \to q)$$

- For $[a]p$, we have that $(M, w \models [a]p)^- = 0$ and $(M', w' \models [a]p)^- = 0.7$. Thus,

$$(M, w \models [a]p) \not\preccurlyeq (M', w' \models [a]p)$$

Theorem 5. *Let $M = \langle W, R, V\rangle$ and $M' = \langle W', R', V'\rangle$ be two PKS over a iMTL-algebra $\mathcal{A}$ and $S^G : W \times W' \to A \times A$ be a graded simulation. Then, the following property holds for every $w \in W$, $w' \in W'$ and sentence $\varphi \in \mathsf{Sen}^+(\mathrm{Prop}, \mathrm{Act})$:*

$$S^G(w, w') \preccurlyeq \Big((M, w \models \varphi) \Rightarrow (M', w' \models \varphi)\Big) \tag{53}$$

where $\mathsf{Sen}^+(\mathrm{Prop}, \mathrm{Act})$ is the positive fragment of $\mathrm{P}(\mathcal{A})$.

Proof. The proof is by induction over the structure of sentences.

- Case $\perp$ is trivial. By definition of $\models$ and $\Rightarrow$

  $$(M, w \models \perp) \Rightarrow (M', w' \models \perp) = (0, 1) \Rightarrow (0, 1) = (1, 0)$$

  and, since $S^G(w, w') \in A \times A$ we have that $S^G(w, w') \preccurlyeq (1, 0)$.

- For $p \in \mathrm{Prop}$, recall that S^G is a graded simulation and therefore satisfies condition (45). That is,

  $$S^G(w, w') \preccurlyeq V(w, p) \Rightarrow V'(w', p)$$

  Using the definition of $\models$

  $$S^G \preccurlyeq (M, w \models p) \Rightarrow (M', w' \models p)$$

- For $\varphi_1 \wedge \varphi_2$ (where φ_1, $\varphi_2 \in \mathsf{Sen}^+(\mathrm{Prop}, \mathrm{Act})$) the proof is follows as,

  $$\begin{aligned}
  & (M, w \models \varphi_1 \wedge \varphi_2) \Rightarrow (M', w' \models \varphi_1 \wedge \varphi_2) \\
  = & \{\text{defn} \models\} \\
  & (M, w \models \varphi_1 \curlywedge M, w \models \varphi_2) \Rightarrow (M', w' \models \varphi_1 \curlywedge M', w' \models \varphi_2) \\
  \succcurlyeq & \{\text{Property (32)}\} \\
  & (M, w \models \varphi_1 \Rightarrow M', w' \models \varphi_1) \curlywedge (M, w \models \varphi_2 \Rightarrow M', w' \models \varphi_2) \\
  \succcurlyeq & \{\text{Induction hypothesis and } \curlywedge \text{ is idempotent}\} \\
  & S^G(w, w')
  \end{aligned}$$

- For $\langle a\rangle\varphi$, consider, without loss of generality, a state $v \in W$ such that

  $$(M, w \models \langle a\rangle\varphi) = R_a(w, v) \otimes M, v \models \varphi \tag{54}$$

  Since S^G is a graded simulation, by (46) there is $v' \in W'$ such that the following inequality is satisfied.

  $$S^G(w, w') \otimes R_a(w, v) \preccurlyeq R'_a(w', v') \otimes S^G(v, v') \tag{55}$$

  We will prove that,

  $$S^G(w, w') \preccurlyeq (M, w \models \langle a\rangle\varphi) \Rightarrow (R'_a(w', v') \otimes M', v' \models \varphi) \tag{56}$$

  Observe that,

  $$(R'_a(w', v') \otimes M', v' \models \varphi) \preccurlyeq (M', w' \models \langle a\rangle\varphi) \tag{57}$$

Thus, by proving (56) resorting to property (20), we prove the case $\langle a\rangle\varphi$. Since S^G is a graded simulation, $S^G(v,v') \preccurlyeq (M,v \models \varphi) \Rightarrow (M',v' \models \varphi)$ through (20) in (55),

$$S^G(w,w') \otimes R_a(w,v) \preccurlyeq R'_a(w',v') \otimes ((M,v \models \varphi) \Rightarrow (M',v' \models \varphi))$$

Using Property (19),

$$S^G(w,w') \preccurlyeq R_a(w,v) \Rightarrow \Big(R'_a(w',v') \otimes ((M,v \models \varphi) \Rightarrow (M',v' \models \varphi))\Big)$$

Using Property (25),

$$S^G(w,w') \preccurlyeq R_a(w,v) \Rightarrow \Big((M,v \models \varphi) \Rightarrow (R'_a(w',v') \otimes (M',v' \models \varphi))\Big)$$

Using Property (26)

$$S^G(w,w') \preccurlyeq \Big(R_a(w,v) \otimes (M,v \models \varphi)\Big) \Rightarrow \Big(R'_a(w',v') \otimes (M',v' \models \varphi)\Big)$$

Given (54), (57) and (20), we conclude that

$$S^G(w,w') \preccurlyeq (M,w \models \langle a\rangle\varphi) \Rightarrow (M',w' \models \langle a\rangle\varphi)$$

□

Again some formulas are not preserved by a graded simulation. Such are the cases of $\neg\varphi$, $\varphi_1 \rhd \varphi_2$, $[a]\varphi$ counterexamples being provided below for each case.

- For $\neg\varphi$ consider the following PKSs.

  with $V(w,p) = (0,0.3)$ and $V'(w',p) = (0.3,0)$. Relation $S^G(w,w') = (1,0)$ is a graded simulation, since it satisfies conditions (45) and (46). Note that,

$$(M,w \models \neg p) \Rightarrow (M',w' \models \neg p) = (0.3,0) \Rightarrow (0,0.3) = (0,0.3)$$

  However, $S(w,w') = (1,0) \not\preccurlyeq (0,0.3)$.

- For $\varphi_1 \rhd \varphi_2$ consider again the above PKS now with $V(w,p) = V(w,q) = (0.5,0.5)$ and $V'(w',p) = (1,0.5)$ and $V'(w',q) = (0.5,1)$. Relation $S^G(w,w') = (1,0.5)$ is a graded simulation. Note that,

$$\begin{aligned}
&(M,w \models p \rhd q) \Rightarrow (M',w' \models p \rhd q)\\
&= /\!\!/ \Big((M,w \models p) \overset{\wedge}{\wedge} /\!\!/(M,w \models q)\Big) \Rightarrow /\!\!/\Big((M',w' \models p) \overset{\wedge}{\wedge} /\!\!/(M',w' \models q)\Big)\\
&= /\!\!/ \Big((0.5,0.5) \overset{\wedge}{\wedge} (0.5,0.5)\Big) \Rightarrow /\!\!/\Big((1,0.5) \overset{\wedge}{\wedge} (1,0.5)\Big)\\
&= (0.5,0.5) \Rightarrow (0.5,1)\\
&= (0.5,0.5)
\end{aligned}$$

  However, $S^G(w,w') = (1,0.5) \not\preccurlyeq (0.5,0.5)$.

- Finally, for $[a]\varphi$ consider the following two PKS.

with $V(w,p) = V'(w',p) = (1,0)$, $V(v,p) = (0.7,0.6)$ and $V'(v',p) = (0.1,0.1)$. Relation $S^G(w,w') = (1,0.1)$ and $S^G(v,v') = (0.1,0.1)$ is a graded simulation. Note that,

$$\begin{aligned}(M,w \models [a]p) \Rightarrow (M',w' \models [a]p) &= (M,w \models \neg\langle a\rangle\neg p) \Rightarrow (M',w' \models \neg\langle a\rangle\neg p)\\ &= \mathbin{/\!\!/}(M,w \models \langle a\rangle\neg p) \Rightarrow \mathbin{/\!\!/}(M',w' \models \langle a\rangle\neg p)\\ &= \mathbin{/\!\!/}(R_a(w,v) \otimes \mathbin{/\!\!/}V(v,p)) \Rightarrow \mathbin{/\!\!/}(R'_a(w',v') \otimes \mathbin{/\!\!/}V'(v',p))\\ &= (1,0.1) \Rightarrow (0.1,0.1)\\ &= (0.1,0.1)\end{aligned}$$

However, $S(w,w') = (1,0.1) \not\preccurlyeq (0.1,0.1)$.

Another interesting observation is that a disjunctive sentence $\varphi_1 \vee \varphi_2$ is preserved by graded simulations, even tough it is defined in terms of $\neg$ and $\wedge$, with $\neg$ is not preserved. Let us start by showing that,

$$(M,w \models \varphi_1 \vee \varphi_2) = (M,w \models \varphi_1) \overset{\vee}{\vee} (M,w \models \varphi_2) \tag{58}$$

Let $(M,w \models \varphi_1) = (\alpha,\beta)$ and $(M',w' \models \varphi_2) = (\alpha',\beta')$ with α, α', β, $\beta' \in A$.

$$\begin{aligned}M,w \models \varphi_1 \vee \varphi_2 &= M,w \models \neg(\neg\varphi_1 \wedge \neg\varphi_2)\\ &= \mathbin{/\!\!/}(M,w \models \neg\varphi_1 \wedge \neg\varphi_2)\\ &= \mathbin{/\!\!/}(\mathbin{/\!\!/}(M,w \models \varphi_1) \overset{\wedge}{\wedge} \mathbin{/\!\!/}(M,w \models \varphi_2))\\ &= \mathbin{/\!\!/}((\beta,\alpha) \overset{\wedge}{\wedge} (\beta',\alpha'))\\ &= \mathbin{/\!\!/}(\beta \wedge \beta', \alpha \vee \alpha')\\ &= (\alpha \vee \alpha', \beta \wedge \beta')\\ &= (M,w \models \varphi_1) \overset{\vee}{\vee} (M,w \models \varphi_2)\end{aligned}$$

Thus, the proof of Theorem 5 for $\varphi_1 \vee \varphi_2$ can be written as follows

$$\begin{aligned}&(M,w \models \varphi_1 \vee \varphi_2) \Rightarrow (M',w' \models \varphi_1 \vee \varphi_2)\\ =\ &\{(58)\}\\ &\Big((M,w \models \varphi_1) \overset{\vee}{\vee} (M,w \models \varphi_2)\Big) \Leftrightarrow \Big((M',w' \models \varphi_1) \overset{\vee}{\vee} (M',w' \models \varphi_2)\Big)\\ \succcurlyeq\ &\{(30)\}\\ &\Big((M,w \models \varphi_1) \Leftrightarrow (M',w' \models \varphi_1)\Big) \overset{\wedge}{\wedge} \Big((M,w \models \varphi_2) \Leftrightarrow (M',w' \models \varphi_2)\Big)\\ \succcurlyeq\ &\{\text{Induction hypothesis and } \overset{\wedge}{\wedge} \text{ idempotent}\}\\ &S^G(w,w')\end{aligned}$$

5.4 Modal invariance

Finally, modal invariance is discussed with respect to both crisp and graded bisimulation.

Theorem 6. *Let $M = \langle W, R, V\rangle$, $M' = \langle W', R', V'\rangle$ be two PKS over a iMTL-algebra $\mathcal{A}$, and $B \subseteq W \times W'$ be a crisp bisimulation. Then, for every $(w, w') \in B$ and any sentence $\varphi \in \mathsf{Sen}(\mathrm{Prop}, \mathrm{Act})$,*

$$(M, w \models \varphi) \;=\; (M', w' \models \varphi) \tag{59}$$

Proof. The proof is by induction over the structure of sentences.

- The case $\perp$ is trivial from the definition of $\models$, $(M, w \models \perp) \;=\; (0, 1) \;=\; (M', w' \models \perp)$.
- For $p \in \mathrm{Prop}$, since B is a crisp bisimulation, it follows that $V(w, p) \;=\; V'(w', p)$, and by definition of $\models$, conclude $(M, w \models p) \;=\; (M', w' \models p)$.
- For $\varphi_1 \wedge \varphi_2$ the proof is as follows,

$$\begin{aligned}
& M, w \models \varphi_1 \wedge \varphi_2 \\
= & \;\{\text{defn of} \models\} \\
& (M, w \models \varphi_1) \mathbin{\overset{\wedge}{\wedge}} (M, w \models \varphi_2) \\
= & \;\{\text{Induction Hypothesis twice}\} \\
& (M', w' \models \varphi_1) \mathbin{\overset{\wedge}{\wedge}} (M', w' \models \varphi_2) \\
= & \;\{\text{defn of} \models\} \\
& M', w' \models \varphi_1 \wedge \varphi_2
\end{aligned}$$

- For $\langle a\rangle\varphi$,

$$\begin{aligned}
& M, w \models \langle a\rangle\varphi \\
= & \;\{\text{defn of} \models\} \\
& \underset{v \in W}{\overset{\vee}{\vee}} \Big(R_a(w, v) \otimes (M, v \models \varphi) \Big) \\
= & \;\{(\star\star)\} \\
& \underset{v' \in W'}{\overset{\vee}{\vee}} \Big(R'_a(w', v') \otimes (M', v' \models \varphi) \Big) \\
= & \;\{\text{defn of} \models\} \\
& M', w' \models \langle a\rangle\varphi
\end{aligned}$$

  Step $(\star\star)$ is proved as follows: B being a bisimulation, there exists $v \in W$ such that $R'_a(w', v') \;=\; R_a(w, v)$ and, by hypothesis, $(M', v' \models \varphi) \;=\; (M, v \models \varphi)$. Therefore,

$$R'_a(w', v') \otimes (M', v' \models \varphi) \;=\; R_a(w, v) \otimes (M, v \models \varphi)$$

Since $\curlyvee$ is monotone, $\underset{v' \in W'}{\curlyvee} \Big(R'_a(w', v') \otimes (M', v' \models \varphi) \Big) = \underset{v \in W}{\curlyvee} \Big(R_a(w, v) \otimes (M, v \models \varphi) \Big)$

□

Theorem 7. *Let $M = \langle W, R, V \rangle$ and $M' = \langle W', R', V' \rangle$ be two PKS over a iMTL-algebra $\mathcal{A}$, and $B^G : W \times W' \to A \times A$ be a graded bisimulation. Then, for every $w \in W$, $w' \in W'$ and any sentence $\varphi \in \mathsf{Sen}(\mathrm{Prop}, \mathrm{Act})$,*

$$B^G(w, w') \preccurlyeq \Big((M, w \models \varphi) \Leftrightarrow (M', w' \models \varphi) \Big) \tag{60}$$

Proof. The proof is again by induction over the structure of sentences.

- Case $\perp$ is trivial,

$$B^G(w, w') \preccurlyeq (M, w \models \perp) \Leftrightarrow (M', w' \models \perp) = (0, 1) \Leftrightarrow (0, 1) = (1, 0)$$

- For $p \in \mathrm{Prop}$, since B^G is a graded bisimulation it satisfies condition (47), i.e.

$$B^G(w, w') \preccurlyeq V(w, p) \Leftrightarrow V'(w', p)$$

  Using the definition of $\models$, we conclude

$$B^G \preccurlyeq (M, w \models p) \Leftrightarrow (M', w' \models p)$$

- For $\neg\varphi$,

$$\begin{aligned}
& (M, w \models \neg\varphi) \Leftrightarrow (M, w \models \neg\varphi) \\
= & \{\text{defn of} \models\} \\
& /\!\!/ (M, w \models \varphi) \Leftrightarrow /\!\!/ (M, w \models \varphi) \\
= & \{\text{Property (22)}\} \\
& (M, w \models \varphi) \Leftrightarrow (M, w \models \varphi) \\
\succcurlyeq & \{\text{Induction hypothesis}\} \\
& B^G(w, w')
\end{aligned}$$

- For $\varphi_1 \wedge \varphi_2$,

$$\begin{aligned}
& \Big((M, w \models \varphi_1) \curlywedge (M, w \models \varphi_2) \Big) \Leftrightarrow \Big((M', w' \models \varphi_1) \curlywedge (M', w' \models \varphi_2) \Big) \\
\succcurlyeq & \{(29)\} \\
& \Big((M, w \models \varphi_1) \Leftrightarrow (M', w' \models \varphi_1) \Big) \curlywedge \Big((M, w \models \varphi_2) \Leftrightarrow (M', w' \models \varphi_2) \Big) \\
\succcurlyeq & \{\text{induction hypothesis and} \curlywedge \text{idempotent}\} \\
& B^G(w, w')
\end{aligned}$$

- For $\langle a\rangle\varphi$ we need to verify the following two inequalities,

$$B^G(w,w') \preccurlyeq (M,w \models \langle a\rangle\varphi) \Rightarrow (M',w' \models \langle a\rangle\varphi) \tag{61}$$
$$B^G(w,w') \preccurlyeq (M',w' \models \langle a\rangle\varphi) \Rightarrow (M,w \models \langle a\rangle\varphi) \tag{62}$$

The proof for (61) is equal to the one presented for case $\langle a\rangle\varphi$ in Theorem 5. Similarly, one can prove (62). Since B^G is a graded bisimulation, there is $v \in W$ such that the following inequality is satisfied

$$B^G(w,w') \otimes R'_a(w',v') \preccurlyeq R_a(w,v) \otimes B^G(v,v') \tag{63}$$

We will prove that for $v \in W$,

$$B^G(w,w') \preccurlyeq (M',w' \models \langle a\rangle\varphi) \Rightarrow (R_a(w,v) \otimes M,v \models \varphi) \tag{64}$$

Observe that,

$$(R_a(w,v) \otimes M,v \models \varphi) \preccurlyeq (M,w \models \langle a\rangle\varphi) \tag{65}$$

Thus, by proving (64) using property (20), we verify (62). Since B^G is a graded simulation, $B^G(v,v') \preccurlyeq (M',v' \models \varphi) \Rightarrow (M,v \models \varphi)$. Using (20) in (63), entails

$$B^G(w,w') \otimes R'_a(w',v') \preccurlyeq R_a(w,v) \otimes ((M',v' \models \varphi) \Rightarrow (M,v \models \varphi))$$

Then, by (19),

$$B^G(w,w') \preccurlyeq R'_a(w',v') \Rightarrow \Big(R_a(w,v) \otimes ((M',v' \models \varphi) \Rightarrow (M,v \models \varphi))\Big)$$

Using (25),

$$B^G(w,w') \preccurlyeq R'_a(w',v') \Rightarrow \Big((M',v' \models \varphi) \Rightarrow (R_a(w,v) \otimes (M,v \models \varphi))\Big)$$

Using (26),

$$B^G(w,w') \preccurlyeq \Big(R'_a(w',v') \otimes (M',v' \models \varphi)\Big) \Rightarrow \Big(R_a(w,v) \otimes (M,v \models \varphi)\Big)$$

By (20),

$$B^G(w,w') \preccurlyeq (M',w' \models \langle a\rangle\varphi) \Rightarrow (M,w \models \langle a\rangle\varphi)$$

Finally, by (61) and (62) and the fact that $\curlywedge$ is monotone,

$$B^G(w,w') \preccurlyeq (M,w \models \langle a\rangle\varphi) \Leftrightarrow (M',w' \models \langle a\rangle\varphi)$$

□

6 An application to . . .

6.1 Quantum circuits

As previously mentioned, from past research on paraconsistent transition systems it seems appropriate to model decahorence in quantum circuits we refer the interested reader to [CMB22b, MB23]. +++

6.2 Robotics

In this section, we take the initial steps towards exploring a potential application of paraconsistent transition systems in robotics. This inspiration is drawn from the work documented in [ATL+07] and its adaptation in [CM16], where paraconsistent logics are employed to determine the movements of a robot. The main goal is to investigate how the robot, which may receive contradictory information regarding the presence (or absence) of objects, navigates along a specific path.

While sensors for object detection remain internally consistent, discrepancies may arise between them, particularly if their methods for object detection are different. Furthermore, sensors are susceptible to contradictions due to various factors, such as hardware limitations that restrict the accuracy in measuring to a certain degree, and the fact that measurements themselves can be affected by outside conditions. Consequently, it is not uncommon for information collected by one sensor to contradict that collected by another. It comes as no surprise that combining various types of sensors is a common practice to enhance reliability and accuracy in obstacle detection and avoidance.

These challenges motivate our approach to incorporate various types of sensors, consider external conditions and use each sensor's characteristics in determining confidence in object detection as we navigate different paths. To illustrate, let us work with 4 sensors: two *ultrasonic sensors* that measure distance to an obstacle by using ultrasonic waves, and two *infrared reflective sensors* that detect the presence of an obstacle by measuring the amount of reflected infrared light. There will be two sets of sensors: one called the set of "positive sensors" that comprises one sensor of each type (one ultrasonic and one infrared), while the other is called the set "of negative sensors" and includes the remaining sensors of each type. Both sets of sensors evaluate if a path is free of obstacles or not then each set provides a value in the interval $[0, 1]$. Such value represents, for the "positive sensors", its certainty that there is no obstacle and the path is *clear*, while for the "negative sensors" it represents its certainty that there is an obstacle and the path is *obstructed*.

Certainty is a crucial aspect in object detection, where the system not only identifies objects but also quantifies its confidence in the accuracy of identification. Various approaches can be adopted to measure certainty, such as considering limitations in the precision of measurements, disregarding a sensor's output in the presence of environmental conditions that affect measurements, and setting different threshold levels for activation and deactivation helps maintaining stability in the output due to noise or minor fluctuations.It is also possible to take in consideration different placements of the sensors. For instance, in [CM16] the approach involved a robot with two sensors covering a total amplitude of 180º: one on the left covering 90º to the left, and the other on the right covering 90º to the right, as illustrated in Figure 2. Thus, at specific points along the path certain sensors may be temporarily disregarded or have default values adopted to increase confidence in detection.

In our example, the robot is capable of movement in four directions (left, right, up, and down) and is equipped with two sets of different types of sensors. To simplify the problem, as the robot navigates a map with four checkpoints it follows the condition that it is prohibited to return to a previously visited checkpoint (resulting in paths with maximum length of 3). We will also assume

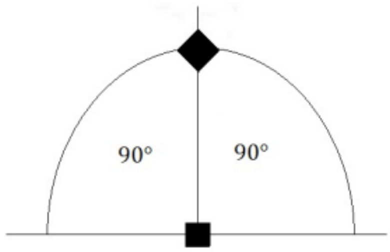


Figure 2: The angles of vision of each sensor of the robot [CM16]

an non adaptive approach since we will only focus on walks that have one or two moves. However, it is possible to consider that at each state the robot has more information, and makes decisions based on them. The checkpoints are represented by states in a PLTS and transitions between these states represent the robot's movement from one checkpoint to another. The transition labels indicate the information given by the set of "positive sensors" and the "negative sensors" to move in that direction. Thus, each path will entail a pair (α, β), where α indicates the certainty degree that the "positive sensors" detected the path is clear, and β indicates the certainty degree that the "negative sensors" detected the path is obstructed.

As before, for any given pair (α, β), if $\alpha + \beta = 1$, it represents a scenario where all sensors are consistent with each other, if $\alpha + \beta < 1$, the sensors are consistent but lack sufficient information together to draw a conclusion and if $\alpha + \beta > 1$, the sensors disagree with each other. Hence, a label

i $(1, 0)$ indicates complete agreement between all of the sensors that the path is clear, allowing the robot to move freely in that direction;

ii $(0, 1)$ indicates unanimous consensus between all of the sensors that the path is not clear, restricting the robot from moving in that direction;

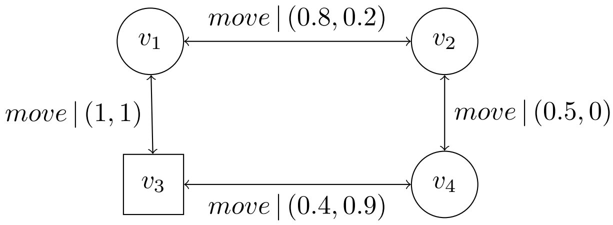
iii $(1, 1)$ suggests complete disagreement between both sensors, leading to the paradoxical situation where they are simultaneously certain that the path is clear and not clear.

iv $(0, 0)$ reflects minimal certainty from both sensors regarding the path's status as free or obstructed.

Scenarios of inconsistency, as in iii, or vagueness, as in iv, may be more challenging to picture than others. However, it is conceivable that under external unfavorable conditions, such as adverse weather, sensors could either demonstrate complete disagreement or lack confidence in their measurements.

Additionally, the proposition *Obst* serves as a crisp valuation indicating the presence of an obstacle at a given state, with a valuation of $(1, 0)$, or the absence of an obstacle, with a valuation of $(0, 1)$.

Let us consider the following map:



If the robot is initially positioned in state v_1, the path to v_2 is reportedly clear by the "positive sensors" with a certainty of 0.8, and it is reported as not clear by the "negative sensors" with a certainty of 0.2. The only state that contains an obstacle is state v_3 and therefore is denoted differently in the PLTS.

Let us evaluate a few sentences and explore the robot's behaviour in the PLTS.

	v_1	v_2	v_3	v_4
$Obst \to [move]\bot$	$(1, 0)$	$(1, 0)$	$(0, 1)$	$(1, 0)$
$\neg Obst \wedge \langle move \rangle \neg Obst$	$(0.8, 0)$	$(0.8, 0)$	$(0, 1)$	$(0.5, 0)$
$\neg Obst \wedge \langle move \rangle \neg Obst \wedge \langle move \rangle \langle move \rangle \neg Obst$	$(0.5, 0)$	$(0, 1)$	$(0, 1)$	$(0.5, 0)$

- The sentence $Obst \to [move]\bot$ indicates that whenever a state has an obstacle, the robot should not be able to move. As expected, this sentence at states v_1, v_2 and v_4 is consistently true since they do not have obstacles. However, at state v_3 the evaluation is consistently false. Even tough, state v_3 has an obstacle, there is positive evidence that it is possible to move to another state.

$$
\begin{aligned}
& M, v_3 \models Obst \to [move]\bot \\
=& M, v_3 \models Obst \to \neg \langle move \rangle \top \\
=& M, v_3 \models \neg \left(Obst \wedge \langle move \rangle \top \right) \\
=& /\!\!/ \left((M, v_3 \models Obst) \overset{\wedge}{\wedge} (M, v_3 \models \langle move \rangle \top) \right) \\
=& /\!\!/ \left(V(v_3, Obst) \overset{\wedge}{\wedge} ((R_{move}(v_3, v_1) \otimes (1,0)) \overset{\vee}{\vee} (R_{move}(v_3, v_4) \otimes (1,0)) \right) \\
=& /\!\!/ \left((1,0) \overset{\wedge}{\wedge} ((1,0) \overset{\vee}{\vee} (0.4, 0) \right) \\
=& (0, 1)
\end{aligned}
$$

The problem arises because both $R^{+}_{move}(v_3, v_1)$ and $R^{+}_{move}(v_3, v_4)$ are both different from 0. If, at that state, external and internal conditions for the sensors were improved to the extent that the positive sensors indicate the path is clear with 0 certainty, and the negative sensors report the same certainty of the path being obstructed, that is, $R_{move}(v_3, v_1) = (0, 1)$ and $R_{move}(v_3, v_4) = (0, 0.9)$. Then,

$$
\begin{aligned}
& M, v_3 \models Obst \to [move]\bot \\
=& /\!\!/ \left((M, v_3 \models Obst) \overset{\wedge}{\wedge} (M, v_3 \models \langle move \rangle \top) \right)
\end{aligned}
$$

$$
\begin{aligned}
&= /\!\!/ \left(V(v_3, Obst) \overset{\wedge}{\wedge} ((R_{move}(v_3, v_1) \otimes (1,0)) \overset{\vee}{\vee} (R_{move}(v_3, v_4) \otimes (1,0)) \right) \\
&= /\!\!/ \left(V(v_3, Obst) \overset{\wedge}{\wedge} (((0,1) \otimes (1,0)) \overset{\vee}{\vee} ((0,0.9) \otimes (1,0)) \right) \\
&= /\!\!/ \left((1,0) \overset{\wedge}{\wedge} ((0,1) \overset{\vee}{\vee} (0,0.9) \right) \\
&=(0.9,0)
\end{aligned}
$$

Only with such changes would the robot's behavior at that state align with the expected result, that is, the robot at state v_3 has (almost) complete certainty that in the presence of an obstacle, it cannot move.

- The sentence $\neg Obst \wedge \langle move \rangle \neg Obs$ indicates that at a state with no obstacle it is possible to move to another state where there is still no obstacle. At state v_4, there is less positive certainty of movement than at states v_1 and v_2. Thus, the evaluation of the sentence at v_4 is $(0.5, 0)$, while at v_1 and v_2 it is $(0.8, 0)$. Given that $(0.5, 0) \preccurlyeq (0.8, 0)$ when planning the robot's path, we might take the starting position at state v_1 instead of v_4. This is because there is overall more certainty that the first move from state v_1 will not encounter an obstacle.

- The sentence $\neg Obst \wedge \langle move \rangle \neg Obst \wedge \langle move \rangle \langle move \rangle \neg Obst$ indicates that at a state with no obstacle, it is possible to move two steps without ever encountering an obstacle. This sentence evaluation is consistently false at state v_3, which has an obstacle, and at state v_2 because every walk that starts at v_2 and takes two steps will always lead to v_3. Finally, at states v_1 and v_4, there is a 0.5 certainty that taking two steps will lead to a path without obstacles, and there is a 0 certainty that taking two steps will encounter an obstacle. This situation represents a *vague* scenario where the sum of the weights is less than 1. This is because transitions from v_1 to v_2 to v_4 and vice versa always present positive evidence smaller than 1, so they do not have complete certainty that it is possible to move. However, they present 0 or close to 0 negative evidence that the transitions will be prevented from occurring due to obstacle detection.

There is still much to be done in this application scenario. However, it is worth noting that this example can be enhanced by incorporating dynamic operators from dynamic logic extended to the paraconsistent setting, as introduced in prior work [CMB23a].

7 Conclusions and future work

The paper studied a new kind of labelled transition structures able to capture both *vagueness* and *inconsistency* in software modelling scenarios. Two main research directions were pursued. First, the structure of a category of (pointed) PLTS was explored to define a number of useful operators to build such systems in a compositional way. Then, a minimal modal logic, whose modalities are indexed by transition labels, was proposed. Preliminary results of classical and graded soundness, which we term *g-soundness*, were explored. This graded notion entails adjusting the classical definition to accommodate vague and inconsistent evaluations. Thus, a rule of the form:

$$\frac{\varphi_1 \ \cdots \ \varphi_n}{\varphi}$$

is *g-sound* if for any paraconsistent Kripke structure $M = \langle W, R, V \rangle$,

$$\bigwedge_{i=1}^{n} (M \models \varphi_i) \preccurlyeq (M \models \varphi)$$

This implies that from the evidence of truth and falsity of premises, it is possible to draw conclusions which evidence of truth is higher and evidence of falsity is lower. Remarkably, while g-soundness entails classical soundness, the reverse is not true. This lack of equivalence prompts further investigation in Subsection 5.2 of classical and graded soundness in paraconsistent Kripke structures.

Moreover, in this paper both crisp and graded notions of simulation and bisimulation of the corresponding (paraconsistent) Kripke structures were also characterised and a number of modal preservation results proved. It remains to consider other notions of bisimulation from coalgebraic trace equivalence [UH18] to logic-induced bisimulations [BGKM23] and their corresponding preservation results.

A lot remains to be done, namely in what concerns the development of the logic's proof-theoretic perspective, which is not addressed here. Such is crucial from a Software Engineering point of view, as a major step towards (semi-)automatic support to reasoning about such complex, often weird, but actually quite common phenomena. The development of a proper specification theory *à la* Sanella and Tarlecki [DS12] is also worth to explore, following some preliminary work by the authors documented in [CMB23b, CMB23a].

The characterisation of different application scenarios, from AI models to quantum computation and robotics, constitutes another main challenge for the future, once the basic mathematical structure has been unveiled.

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