Quantitative higher-order equational theories

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Quantalic Equality

Equality is ubiquitous in mathematics

Central for example in equational logic (algebraic theories)

$$t = s \Longrightarrow [\![t]\!] = [\![s]\!] \text{ in category } \mathrm{C}$$

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This talk: about a generalised notion of equality

Quantalic Equality

Equations are labelled by elements of a quantale $\mathcal{V}$

$\mathcal{V}$-generalisations of eq. laws emerge; others become apparent

Examples

$$\frac{t =_q s \qquad s =_r u}{t =_{q \otimes r} u} \ (\mathcal{V}\text{-trans}) \qquad\qquad \frac{\forall i \leq n.\, t =_{q_i} s}{t =_{\vee q_i} s} \ (\mathcal{V}\text{-join})$$

Covers *inter alia* classical, (ultra)metric, and fuzzy (in)equations

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7. Concluding notes

Programming theory

Quantitative deductive systems for program equivalence

Example (metric equations and real-time computation)

$t =_q s \Longrightarrow$ difference of execution times between $[\![t]\!]$ and $[\![s]\!]$ does not exceed q time units

Applications in *e.g.* probabilistic and quantum programming as well

$$t =_q s \Longrightarrow \text{relation between } [\![t]\!] \text{ and } [\![s]\!] \text{ in category } \mathsf{C}$$

Linear lambda-calculus

Natural to see programs t and s as terms of λ-calculus

But many paradigms we wish to harbour impose "resource constraints"

Example

Qubits cannot be cloned nor discarded in pure quantum theory

Se we use instead linear λ-calculus as language (copying and discarding is forbidden)

Some main results

A $\mathcal{V}$-deductive system for linear λ-calculus

Soundness and (approximate) completeness theorems

Syntax-semantics equivalence theorem

Some main results

$$\mathcal{V}\lambda\text{-theories} \simeq (\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sep}})\text{-autonomous categories}$$

λ-theories	Class of categories
classical	locally small autonomous categories
ordered	Pos-autonomous categories
metric	Met-autonomous categories
ultrametric	UMet-autonomous categories
...	...

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Types and contexts

Types formed according to grammar

$$\mathbb{A} ::= X \mid \mathbb{I} \mid \mathbb{A} \otimes \mathbb{A} \mid \mathbb{A} \multimap \mathbb{A} \qquad (X \in G)$$

Definition

Contexts are lists $x_1 : \mathbb{A}_1, \ldots, x_n : \mathbb{A}_n$ s.t. each x_i appears at most once

Contexts denoted by Greek letters $\Gamma, \Delta, E \ldots$

A fragment of the term formation rules

$$\frac{\Gamma_i \vdash t_i : \mathbb{A}_i \quad f : \mathbb{A}_1, \ldots, \mathbb{A}_n \to \mathbb{A} \in \Sigma}{\Gamma_1, \ldots, \Gamma_n \vdash f(t_1, \ldots, t_n) : \mathbb{A}} \qquad \frac{}{x : \mathbb{A} \vdash x : \mathbb{A}}$$

$$\frac{\Gamma \vdash t : \mathbb{A} \quad \Delta \vdash s : \mathbb{B}}{\Gamma, \Delta \vdash t \otimes s : \mathbb{A} \otimes \mathbb{B}} \qquad \frac{\Gamma, x : \mathbb{A} \vdash t : \mathbb{B}}{\Gamma \vdash \lambda x : \mathbb{A}.\, t : \mathbb{A} \multimap \mathbb{B}}$$

Examples

- $x : \mathbb{A}, y : \mathbb{B} \vdash f(x) \otimes g(y) : \mathbb{A} \otimes \mathbb{B}$
- $- \vdash \lambda x : \mathbb{A}.\, \texttt{wait}_1(x) : \mathbb{A} \multimap \mathbb{A}$ ($\texttt{wait}_1$ seen as a wait call)

Interpretation on autonomous categories

Linear λ-calculus standardly interpreted on autonomous categories

- types $\mathbb{A}$ interpreted as objects $[\![\mathbb{A}]\!] \in \mathsf{C}$
- contexts $x_1 : \mathbb{A}_1, \ldots, x_n : \mathbb{A}_n$ as tensors $[\![\mathbb{A}_1]\!] \otimes \cdots \otimes [\![\mathbb{A}_n]\!] \in \mathsf{C}$
- judgements $\Gamma \vdash t : \mathbb{A}$ as morphisms $[\![\Gamma \vdash t : \mathbb{A}]\!] : [\![\Gamma]\!] \to [\![\mathbb{A}]\!]$

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Computational set-up of $\mathcal{V}$

Definition

Take complete lattice L and $x, y \in L$. $x \ll y \Leftrightarrow$ for every subset $X \subseteq L$ whenever $x \leq \bigvee X$ there exists finite subset $A \subseteq X$ s.t. $y \leq \bigvee A$

L is continuous iff for every $x \in L$

$$x = \bigvee \{y \mid y \in L \text{ and } y \ll x\}$$

Definition

A subset $B \subseteq L$ is a basis if for all $x \in L$ the set below is directed and

$$x = \bigvee B \cap \{y \mid y \in L \text{ and } y \ll x\}$$

Computational set-up of $\mathcal{V}$

A basis B permits working with only a specified subset of $\mathcal{V}$-equations, chosen *e.g.* for computational reasons

Examples

In the lattice $([0,\infty],\wedge)$ the relation $\ll$ is $> \cup \{(\infty,\infty)\}$. It is continuous and $[0,\infty] \cap \mathbb{Q}$ is a basis

Boolean lattice $(\{0 \leq 1\}, \vee)$ is finite and thus continuous. Underlying set itself is a basis.

$\mathcal{V}$-equations-in-context

Definition

A $\mathcal{V}$-equation-in-context is an expression $\Gamma \vdash t =_q s : \mathbb{A}$ with $\Gamma \vdash t : \mathbb{A}$ and $\Gamma \vdash s : \mathbb{A}$

Definition

Classical equation-in-context $\Gamma \vdash t = s : \mathbb{A}$ encoded as

$$\Gamma \vdash t =_k s : \mathbb{A} \text{ and } \Gamma \vdash s =_k t : \mathbb{A} \qquad (k \text{ the unit of } \mathcal{V})$$

Example (metric equations and real-time computation)

- $x : \mathbb{A} \vdash \mathtt{wait}_2(\mathtt{wait}_1(x)) = \mathtt{wait}_3(x) : \mathbb{A}$
- $x : \mathbb{A} \vdash \mathtt{wait}_1(x) =_1 \mathtt{wait}_2(x) : \mathbb{A}$

A fragment of the $\mathcal{V}$-equational system

$$\frac{t =_q s \qquad s =_r u}{t =_{q \otimes r} u} \qquad \frac{t =_q s \qquad r \leq q}{t =_r s} \qquad \frac{\forall i \leq n.\ t =_{q_i} s}{t =_{\vee q_i} s}$$

$$\frac{}{t =_k t} \qquad \frac{\forall r \ll q.\, t =_r s}{t =_q s}$$

. . . .

$$\frac{\forall i \leq n.\ t_i =_{q_i} s_i}{f(t_1, \ldots, t_n) =_{\otimes q_i} f(s_1, \ldots, s_n)} \qquad \frac{t =_q s}{\lambda x : \mathbb{A}.\ t =_q \lambda x : \mathbb{A}.\ s}$$

. . . .

$$(\lambda x : \mathbb{A}.\, t)\, s = t[s/x] \qquad \lambda x : \mathbb{A}.\, (t\, x) = t$$

We close the basis under the above operations and this is again a basis

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The general idea

Classically $t = s \Longrightarrow [\![t]\!] = [\![s]\!] \in \mathsf{C}([\![\Gamma]\!], [\![\mathbb{A}]\!])$

For $\mathcal{V}$-equations $t =_q s$ we need extra structure on the hom-set $\mathsf{C}([\![\Gamma]\!], [\![\mathbb{A}]\!])$

This suggests an enrichment of autonomous categories

$\mathcal{V}$-categories (the basis of enrichment)

Definition

A $\mathcal{V}$-category is a pair (X, a) where X is a set and $a : X \times X \to \mathcal{V}$ a function s.t.

$$k \leq a(x, x) \text{ and } a(x, y) \otimes a(y, z) \leq a(x, z)$$

Definition

A $\mathcal{V}$-functor $f : (X, a) \to (Y, b)$ is a function $f : X \to Y$ s.t.

$$a(x, y) \leq b(f(x), f(y))$$

$\mathcal{V}$-categories (the basis of enrichment)

Small $\mathcal{V}$-categories and $\mathcal{V}$-functors form a category called $\mathcal{V}$-Cat

A $\mathcal{V}$-category is symmetric if $a(x, y) = a(y, x)$. Let $\mathcal{V}\text{-Cat}_{\mathsf{sym}}$ be the corresponding full subcategory

$\mathcal{V}$-categories carry an order $x \leq y \Leftrightarrow k \leq a(x, y)$ and called separated if it is anti-symmetric. Let $\mathcal{V}\text{-Cat}_{\mathsf{sep}}$ be the corresponding full subcategory

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Theorem

$\mathcal{V}$-Cat *and the full subcategories previously listed are autonomous*

Interpretation of $\mathcal{V}$-equations

Definition

A $\mathcal{V}$-Cat-autonomous category C is an autonomous $\mathcal{V}$-Cat-category C s.t. $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ is $\mathcal{V}$-Cat-enriched and $(- \otimes X) \dashv (X \multimap -)$ is a $\mathcal{V}$-Cat-adjunction

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Finally $t =_q s$ satisfied $\Leftrightarrow q \leq a(\llbracket t \rrbracket, \llbracket s \rrbracket)$ in $\mathsf{C}(\llbracket \Gamma \rrbracket, \llbracket \mathbb{A} \rrbracket)$

Interpretation of $\mathcal{V}$-equations

The following categories form instances of the previous definition

Examples

- Pos of partially ordered sets and monotone maps
- Set of sets and functions
- (U)Met of (ultra)metric spaces and non-expansive maps
- Ban of Banach spaces and short linear maps
- $[\mathsf{C}^{\mathsf{op}}, \mathsf{Met}]$ of Met-enriched presheaves with C small and Met-symmetric monoidal

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Theories and models

Definition

Take a signature Σ of basic types and operation symbols. A $\mathcal{V}\lambda$-theory (Σ, Ax) is a tuple s.t. Ax is a set of $\mathcal{V}$-equations over terms built from Σ

Elements of Ax are called axioms and $\mathcal{V}$-equations provable from Ax and the $\mathcal{V}$-equational system are called theorems

Definition

Take a theory $\mathscr{T}$ and $\mathcal{V}$-Cat-autonomous category C. Suppose for each basic type G we have $[\![G]\!] \in \mathrm{C}$ and analogously for operations. This interpretation is a model of $\mathscr{T}$ if all of its axioms are satisfied

Syntactic category

From $\mathscr{T}$ generate a syntactic category $\mathrm{Syn}(\mathscr{T})$

- objects are types $\mathbb{A}$
- morphisms $\mathbb{A} \to \mathbb{B}$ are equivalence classes of $x : \mathbb{A} \vdash t : \mathbb{B}$

$$t \sim s \Leftrightarrow t =_k s \wedge s =_k t$$

- function $a : \mathrm{Syn}(\mathscr{T}) \times \mathrm{Syn}(\mathscr{T}) \to \mathcal{V}$ defined as

$$a([t],[s]) = \bigvee \{q \mid t =_q s \text{ a theorem of } \mathscr{T}\}$$

Theorem

$\mathrm{Syn}(\mathscr{T})$ *is* $\mathcal{V}$-$\mathrm{Cat}_{\mathrm{sep}}$*-autonomous*

Soundness and completeness

Theorem

Take a theory $\mathscr{T}$ and a model M of $\mathscr{T}$. If $t =_q s$ ($q \in B$) is a theorem of $\mathscr{T}$ it is satisfied by M

Theorem

Take a theory $\mathscr{T}$. If $t =_q s$ ($q \in B$) is satisfied by all models of $\mathscr{T}$ then $t =_q s$ is a theorem of $\mathscr{T}$

Proof.

Uses syntactic category and the rule involving infinitely many premisses

$$\frac{\forall r \ll q.\, t =_r s}{t =_q s}$$

□

Approximate completeness

The following theorem holds without the previous syllogism

Theorem

Take a theory $\mathscr{T}$. If $t =_q s$ $(q \in B)$ is satisfied by all models of $\mathscr{T}$ then for all approximations $r \ll q$ $(r \in B)$ the equation $t =_r s$ is a theorem. If q is compact (i.e. $q \ll q$) then $t =_q s$ is a theorem

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Copying, discarding, and reuse under precise control

Simply forbidding copying or discarding is often too restrictive

Example

Full quantum theory allows to freely discard qubits. Bits can be cloned even if qubits cannot

Frequently there is a limit to the nº of times a resource can be used

Example

Sampling from a distribution

Wish to extend our results to this fine-grained control of resources

Graded lambda-calculus

Types formed according to grammar

$$\mathbb{A} ::= X \mid \mathbb{I} \mid \mathbb{A} \otimes \mathbb{A} \mid \mathbb{A} \multimap \mathbb{A} \mid !_n \, \mathbb{A} \qquad (X \in G, n \in \mathbb{N})$$

The new term formation rules (in simplified form)

$$\frac{\Gamma \vdash t : !_1 \, \mathbb{A}}{\Gamma \vdash \mathtt{dr}(t) : \mathbb{A}} \qquad \frac{x : !_r \, \mathbb{A} \vdash t : \mathbb{B}}{x : !_{k \cdot r} \, \mathbb{A} \vdash !_k t : !_k \, \mathbb{B}} \qquad \frac{\Gamma \vdash t : !_0 \, \mathbb{A} \quad \Delta \vdash s : \mathbb{B}}{\Gamma, \Delta \vdash \mathtt{ds}(t).\, s : \mathbb{B}}$$

$$\frac{\Gamma \vdash t : !_{n+m} \, \mathbb{A} \quad \Delta, x : !_n \, \mathbb{A}, y : !_m \, \mathbb{A} \vdash s : \mathbb{B}}{\Gamma, \Delta \vdash \mathtt{cp}_{n,m}(t) \, \mathtt{to} \, x, y.\, s : \mathbb{B}}$$

How to extend previous interpretation of $\mathcal{V}$-equations to the graded setting?

Graded exponential comonads

Recall that a comonad is an oplax monoidal functor $1 \to ([\mathsf{C}, \mathsf{C}], \mathrm{Id}, \cdot)$

Definition

An $\mathbb{N}$-graded comonad is an oplax monoidal functor

$$D_{(-)} : (\mathbb{N}, 1, \cdot) \to ([\mathsf{C}, \mathsf{C}], \mathrm{Id}, \cdot)$$

Definition

An $\mathbb{N}$-graded comonad is called exponential if it equips every C-object X with the structure of a graded commutative comonoid

$$D_0 X \to I \qquad\qquad D_{n+m} X \to D_n X \otimes D_m X$$

satisfying certain laws

Interpretation of graded lambda-calculus

Take an autonomous category C with a $\mathbb{N}$-graded exponential comonad

Some of the interpretation rules

$$[\![!_k \, \mathbb{A}]\!] = D_k [\![\mathbb{A}]\!] \qquad \frac{[\![\Gamma \vdash t : !_1 \, \mathbb{A}]\!] = m}{[\![\Gamma \vdash \mathtt{dr}(t) : \mathbb{A}]\!] = \epsilon_{[\![\mathbb{A}]\!]} \cdot m}$$

$$\frac{[\![x : !_k \mathbb{A} \vdash t : \mathbb{B}]\!] = m}{[\![x : !_{k \cdot r} \, \mathbb{A} \vdash !_k \, t : !_k \, \mathbb{B}]\!] = D_k(m) \cdot \delta_{k,r,[\![\mathbb{A}]\!]}}$$

A fragment of the $\mathcal{V}$-equational system

$$\frac{t =_q s \qquad s =_r u}{t =_{q \otimes r} u} \qquad \frac{t =_q s \qquad r \leq q}{t =_r s} \qquad \frac{\forall i \leq n.\ t =_{q_i} s}{t =_{\vee q_i} s}$$

$$\frac{}{t =_k t} \qquad \frac{\forall r \ll q.\, t =_r s}{t =_q s}$$

$$\frac{t =_q s}{\lambda x : \mathbb{A}.\ t =_q \lambda x : \mathbb{A}.\ s} \qquad \frac{t =_q s}{!_k\, t =_{k \cdot q}\, !_k\, s}$$

$$(\lambda x : \mathbb{A}.\, t)\, s = t[s/x] \qquad \mathtt{dr}(!_1\, t) = t$$

Interpretation of $\mathcal{V}$-equations

As before we take a $\mathcal{V}$-Cat-autonomous category C

Take also an $\mathbb{N}$-graded exponential comonad with the Lipschitz condition

$$k \cdot a(m_1, m_2) \leq a(D_k m_1, D_k m_2)$$

Example (comonad of dilations in Met)

$D_k(X, d) = (X, k \cdot d)$ and other operations defined trivially.
$D_k X \to Y$ is a k-Lipschitz continuous map

Soundness and (approximate) completeness

Soundness and completeness theorems hold similarly to before

Currently working on approximate completeness

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Some other results

Syntax-semantics equivalence theorem

Models of linear $\mathcal{V}\lambda$-theories $\mathscr{T}$ as $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-autonomous functors

$$\mathrm{Syn}(\mathscr{T}) \to \mathsf{C}$$

Canonical construction of Lipschitz $\mathbb{N}$-graded exponential comonads

Current work

Exploration of $\mathcal{V}$-equational systems for different quantales

Addition of recursion constructs

Working out connections to $\mathcal{V}$-universal algebra and toposes