Paraconsistency for the working software engineer (extended abstract) *

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Abstract. Modelling complex information systems often entails the need for dealing with scenarios of inconsistency in which several requirements either reinforce or contradict each other. This lecture summarises recent joint work with Juliana Cunha, Alexandre Madeira and Ana Cruz on a variant of transition systems endowed with positive and negative accessibility relations, and a metric space over the lattice of truth values. Such structures are called *paraconsistent* transition systems, the qualifier stressing a connection to paraconsistent logic, a logic taking inconsistent information as potentially informative. A coalgebraic perspective on this family of structures is also discussed.

To the memory of Newton da Costa, in the year of his death at the age of 94, on 16 April 2024.

1 A scenario

Age-related macular degeneration (AMD) is a disease of the macula, the central part of the eye responsible for vision, consensually recognised as the leading cause of vision loss in western countries in people aged over 55 years. It is a multifactorial disease, with a complex pathophysiology, for which onset and progression different risk factors, environmental and genetic, contribute. The disease being multifactorial, its progression is still not well understood. For example, the rate of progression is different between patients, often with similar profiles, and even between the two eyes of the same patient.

Large epidemiological studies have helped understanding AMD risk factors and pathophysiology. Typically, the data obtained from such studies refers not only to the genetics of the cohort, but also to deep phenotyping analysis performed in both healthy controls and participants with AMD. The latter resorts to multimodal imaging techniques, from near-infrared capture to spectral-domain

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optical coherence tomography. All patients had both eyes classified in a central reading centre and graded into 5 stages of disease severity status.

This data is very rich, but, at the same time, highly heterogeneous. Heterogeneous not only with respect to origins, formats and time span, but also in what concerns its assessment by different expert teams. The latter is the crucial observation here. Actually, the evidence level assigned to each data factor (e.g. in an image) as an enabler for a specific future development of the disease may vary from an expert to another, often leading to contradicting conclusions. Data consolidation becomes even more complex if a temporal dimension is introduced allowing for comparative assessments of patients along a time axis of several years. Indeed, potentially contradicting medical judgements on what dictates the evolution of AMD cannot be swept under the carpet.

An effective inference framework to explore AMD data requires the ability to reason about contradictory, or even inconsistent data in a sound and effective way. Each piece of data, as well as all sorts of (empirical) relationships among them, can be assigned a pair of weights, a positive and a negative one. Informally, the positive weight captures the degree of effectiveness (*confidence*) and the negative weight captures the degree of impossibility (*absence*) of a particular judgement on data or a relationship. The framework will thus be able to capture both vagueness, whenever both weights sum less than 1, as usual e.g. in fuzzy systems, and potential inconsistency, when their sum exceeds 1. This labelling is crucial to data classification, but requires suitable logics to take them into consideration in the inference process. Such is the purpose of the research line discussed in this talk.

2 Paraconsistency in Computer Science

Actually, the world of data, pervasive to all natural and artificial ecosystems, is a mined field. Not only the values and structure of data changes from one computation to another, but also the logic under which this information needs to be understood changes as well (e.g. from classical to probabilistic, from fuzzy to linear). On the other hand, informational states may exhibit potentially inconsistent (or partially consistent) data, reflecting the diversity of judgements (e.g. from different domain experts). Moreover, it may be linked by both positive transitions (witnessing e.g. the existence of a computational step, or its cost, or its probability, etc.) and negative transitions (recording whatever prevents such a step to occur, and also possibly expressed as a specific weight). Finally, the weights of such transitions are, in most cases, non-complementary, opening an inference arena encompassing both classical, vague and even (controlled forms of) inconsistent reasoning.

This motivates the qualifier *paraconsistent* used in our title. The word is borrowed from the work on paraconsistent logic [19,13,12], a branch of logic which accommodates inconsistency in a controlled way, treating inconsistent information as potentially informative. Although extremely expressive, such logics are still under studied and lack suitable computational support.

But what do we mean by *accommodating inconsistency* in logical reasoning? Indeed, absence of contradiction is considered core within the method of Science, out of which only nonsense prevails. However, as briefly shown above, reality entails the need to be able to deal with contradictory scenarios. Paraconsistency, in such a setting, is the study of logical systems in which the presence of a contradiction does not imply triviality. This means, logics in which the simultaneous presence of an assertion and of its negation does not always trivialize the conclusions. Recall that trivialization occurs when every statement in the theory can be proved, i.e. becomes a theorem. The principle according to which any statement can be proven from a contradiction is know as the principle of explosion, a hallmark of all forms of classical logic. Paraconsistency, on the other hand, takes consistency as a primitive, independent notion, therefore separating the concept of contradiction from that of deductive triviality, and, as a result, inconsistency from contradiction, and, dually, consistency from absence of contradiction.

Exploring formal ways to deal with logical theories which may be inconsistent but not trivial has a long tradition in Logic and Philosophy. The idea that denying the law of noncontradiction would lead to still meaningful, although non-Aristotelian logics, appeared in the 1910's in the independent work of Jan Łukasiewicz [37] and Nicolai Vasiliev [40], later formalised, by the middle of the century, by Stanislaw Jaśkowski whose main interest was in the study of empirical theories including contradictory assumptions [33] (see also [21] for a broader discussion). In 1958 Newton da Costa seminal paper [18] went behind the propositional level of previous works. The influence of the so-called Brazilian school was remarkable. Quickly the topic attracted larger attention, and the original scope broadened out[1]. In an interview for a video documentary [43] in 2019, da Costa summed up his programme as follows: *I decided to do it the other way round: mathematics with contradictions. Existence in mathematics means anything but the absence of contradiction. Contradictions begin to appear at the edges of mathematics. There are always problems.* To a large extent, current work from this initial trend is known as LFI (the *logic of formal inconsistency* [12] which is endowed with a syntactic annotation, in the form a unary operator, to express the fact that a sentence is consistent.

In 1990, *Mathematical Reviews* added a new entry, 03B53, called *Paraconsistent Logic*, later expanded to *Logics admitting inconsistency (paraconsistent logics, discussive logics, etc.)*. But the real, unexpected impact was on applications. Actually, a bit surprisingly, applications quickly emerged ranging from Philosophy of Science [39,5], to Mathematics [28,44,29], from Economics [45], to the Foundations of Quantum Mechanics [14,20,27,31].

In Computer Science this family of logics found application in reasoning models [7,17], namely in the context of AI, but also in other domains from databases [3] and semantics of concurrency [9,34] to quantum computation [15,1]. A metodologiacal perspective on paraconsistency in systems modelling, and fur-

[1] As da Costa shared in an interview [43], the qualifier paraconsistent, expressing a logic *at the side of consistency*, was suggested to him by Peruvian logician Francisco Miró-Quesada in 1976

ther applications are addressed in a 2016 book significantly entitled *Towards Paraconsistent Engineering* [2].

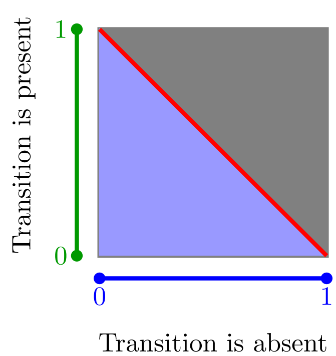
3 Paraconsistent transition systems and their logics

The work reported in this talk emerged in the context of software formal modelling. The original motivation comes from the domain of many-valued logics to deal with informational contexts in which the classical bivalent distinction is not enough, in particular when facing the need to capture vagueness or uncertainty. Residuated lattices, adding a commutative monoidal structure to a complete lattice such that the monoid composition has a right adjoint, the residue, provide the semantic universe for such logics. A suitable choice of the lattice carrier, which stands for the set of truth values, does the job — a typical example being the real $[0, 1]$ interval. Reference [11] explores, in a systematic way, the modal extensions of many-valued logics whose Kripke frames are defined over (variants of) residuated lattices.

This is not yet, however, the whole picture. In a number of modelling scenarios, as the one outlined in section 1, there is a need to deal simultaneously with what could be called *positive* and *negative* accessibility relations, one weighting the possibility of a transition to be present, the other weighting the possibility of being absent. In the concrete contexts we are interested in, such weights are not complementary, and thus both relations must be explicitly incorporated in the Kripke frame.

Assume, for example that weights for both transitions come from a residuated lattice over the real $[0, 1]$ interval. Then, jointly, the two accessibility relations in the frame express

– *inconsistency*, when the positive and negative weights are contradictory, i.e. they sum to some value greater than 1 (cf, the upper triangle filled in grey in the figure below).
– *vagueness*, when the sum is less than 1 (cf, the lower, periwinkle triangle);
– *strict consistency*, when the sum is exactly 1, which means that the measures of the factors enforcing or preventing a transition are complementary, corresponding to the red line in the figure.



Choosing a residuated lattice A to weight transitions, we have defined in a serie of papers starting with reference [23], the notion of a *paraconsistent transition system* over A by explicitly considering a *positive* and a *negative* accessibility relation, taking weights from that common universe. Equivalently, a transition between two states can be regarded as weighted by a pair of values. In a similar way, the valuation of a proposition in a state is a pair of (not necessarily complementary) values capturing, respectively, the degree upon which it may be considered to hold or to fail. A *distance* between two such weights needs to be computed, namely to assess the level of vagueness or inconsistency in a given transition. As a smooth generalization of the example above, this is taken into account by a suitable metric over the carrier of A.

Not only the algebraic structure underlying such systems, but also the generated modal logic, as a generalization of Belnap's four-valued logic [8], were developed in [22]. Later in [25] this was extended to the multi-modal case, thus giving rise to a *structured specification logic* [42] equipped with specific versions of the standard structured specification operators *à la CASL* [38]. This offers to the working software engineer the (formal) tools to specify such systems in a compositional way. Technically, the price to be paid to support this move consists of framing the logic as an institution [30]. Most recent results, including suitable notions of simulation and bisimulation, appear in references [24,26], the later providing a detailed overview to the whole approach.

Applications to typical computing scenarios were discussed in [17], on reactive graphs, and [4], on noise analysis in quantum circuits. Current research efforts address the scenario sketched in section 1, and the implementation of a devoted reasoning engine able to represent data and data relationships from weighted, but opposing, points of view.

4 A coalgebraic perspective

The final part of this talk will propose a different perspective over the semantic structures and logics underlying the concept of *paraconsistent transition system*. Over time, coalgebra theory [41,32] emerged as the right mathematics to express and reason about any sort of state-based, transition system. As an illustrative example, among many others, the reader is referred to the extensive work by F. Bartels, A. Sokolova and E. de Vink [6] on a coalgebraic rendering of probabilistic transition systems.

To take a more elementary example, recall that to define an inductive data structure one essentially specifies its 'assembly process'. For example, one builds a sequence in a data domain D, either by taking an empty list or by adjoining a fresh element to an existing sequence. Thus, declaring a sequence data type yields a function $\zeta : \mathbf{1} + D \times U \longrightarrow U$, where U stands for the data type being defined. The structured domain of function ζ captures a signature of *constructors* ($nil : \mathbf{1} \longrightarrow U$, $cons : D \times U \longrightarrow U$), composed additively. The whole procedure resembles the way in which an algebraic structure is defined.

Reversing the 'assembly process' swaps structure from the domain to the codomain of the arrow, which now captures the result of a 'decomposition' or 'observation' process. In the example this is performed by the familiar *head* and *tail selectors* joined together into

$$\alpha : U \longrightarrow \mathbf{1} + D \times U$$

which either returns a token $*$, when observing an empty sequence, or its decomposition in the top element and the remaining tail.

This reversal of perspective also leads to a different understanding of what U may stand for. The product $D \times U$ captures the fact that both the head and the tail of a sequence are selected (or *observed*) simultaneously. In fact, once one is no longer focused on how to construct U, but simply on what can be observed of it, finiteness is no longer required: both finite or infinite sequences can be observed through the process above. Therefore, U can be more accurately thought of as a *state space* of a machine generating a finite or infinite sequence of values of type D. Elements of U, in this example, can no longer be distinguished by construction, but should rather be identified when generating the same sequence. That is to say, when it becomes impossible to distinguish them through the observations allowed by the 'shape' structuring the codomain of α.

Function α above is an example of a *coalgebra*. Its ingredients are: a carrier U (intuitively the state space of a transition system), the *shape* of allowed observations, technically a functor $\mathcal{F}(X) = \mathbf{1} + D \times X$, and the observation *dynamics* given by function α, *i.e.* the machine itself. Formally, a $\mathcal{F}$-coalgebra is a pair $\langle U, \alpha \rangle$ consisting of an object U and a map $\alpha : U \longrightarrow \mathcal{F}\, U$. The latter maps states to structured collections of successor states. By varying $\mathcal{F}$, *i.e.* the shape of the underlying transitions, one may capture a large class of semantic structures used to model computational phenomena as (more or less complex) transition systems. Going even further, $\mathcal{F}$ is not restricted to be an endofunctor in *Set*, the category of sets and functions.

A morphism between two $\mathcal{F}$-coalgebras, $\langle U, \alpha \rangle$ and $\langle V, \beta \rangle$, is a map h between carriers U and V which preserves the dynamics, *i.e.* such that $\beta \cdot h = \mathcal{F}\, h \cdot \alpha$. As one would expect, $\mathcal{F}$-coalgebras and their morphisms form a category $C_{\mathcal{F}}$ where both composition and identities are inherited from the host category C.

This sets Coalgebra as a suitable mathematical framework for the study of dynamical systems in both a *compositional* and *uniform* way. The qualifier *uniform* requires some extra explanation: coalgebraic concepts (*i.e.* models, constructions, logics, and proof principles) are parametric on, or *typed* by, the functor that characterises the underlying transition structure.

This applies to the semantic structures, their morphisms and notions of (observational) equivalence, but also to the modal logics that are systematically derived, again, from the underlying functor. The literature on what is now called coalgebraic logic is vast [36,16,35]. Modalities, and therefore modal reasoning, also acquires a shape.

We will play this exercise for paraconsistent transition systems, with their double positive/negative weights. A first characterization, in the category of sets and set-theoretic functions, will be refined to the (finite, deterministic) case in which the set of states form a vector of (pairs of) weights, generalising the construction proposed in reference [10]. The coalgebraic methodology, as remarked above, provides a semantics for our systems induced by the unique morphism to the final coalgebra, a canonical notion of observational equivalence, and specific modal logic. This will provide an insightful comparison with our previous results and may pave the way for a more systematic study of paraconsistent transition systems.

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