A brief overview of simply-typed λ-calculus

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The four chapters of this course

1. CCS and its semantics: focus on distributed systems comprised of processing units that communicate with each other

2. Adaption of previous notions to real-time systems: semantics via timed labelled transition systems

3. Going cyber-physical: simple imperative while-language and (as usual) program analysis via its semantics

4. Programming with algebraic effects: a uniform approach to the previous chapters

Overview

Modern programming typically involves different effects

- memory cell manipulation
- communication
- exception raising operations
- probabilistic operations
- real-time behaviour
- cyber-physical behaviour

In the following lectures we will study the mathematical foundations of

Programming with effects

in a uniform way

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Deductive Reasoning

The process of reasoning via assumptions and logical rules to obtain new knowledge: for example ...

if every crow is black and x is a crow then x is black

Deductive reasoning has been studied in the last millenia, Aristotle being one of the fathers ...

long before the age of artificial computers

So what does it have to do with programming?

A Logical Rule-Based System for Deductive Reasoning pt. I

Let $\mathbb{A}, \mathbb{B}, \mathbb{C} \ldots$ denote propositions (i.e. a property or a statement) and 1 denote a special proposition that always holds. Next, if $\mathbb{A}$ and $\mathbb{B}$ are propositions then:

- $\mathbb{A} \times \mathbb{B}$ is a proposition – it denotes the conjunction of $\mathbb{A}$ and $\mathbb{B}$
- $\mathbb{A} \to \mathbb{B}$ is a proposition – it says that $\mathbb{A}$ implies $\mathbb{B}$

A Logical Rule-Based System for Deductive Reasoning pt. II

Let Γ denote a list of propositions. $\Gamma \vdash \mathbb{A}$ means "if the propositions in Γ hold then we deduce that $\mathbb{A}$ also holds"

$$\frac{\mathbb{A} \in \Gamma}{\Gamma \vdash \mathbb{A}} \text{(ass)} \qquad \frac{}{\Gamma \vdash 1} \text{(trv)} \qquad \frac{\Gamma \vdash \mathbb{A} \times \mathbb{B}}{\Gamma \vdash \mathbb{A}} (\pi_1) \qquad \frac{\Gamma \vdash \mathbb{A} \times \mathbb{B}}{\Gamma \vdash \mathbb{B}} (\pi_2)$$

$$\frac{\Gamma \vdash \mathbb{A} \qquad \Gamma \vdash \mathbb{B}}{\Gamma \vdash \mathbb{A} \times \mathbb{B}} \text{(prd)} \qquad \frac{\Gamma, \mathbb{A} \vdash \mathbb{B}}{\Gamma \vdash \mathbb{A} \to \mathbb{B}} \text{(cry)} \qquad \frac{\Gamma \vdash \mathbb{A} \to \mathbb{B} \quad \Gamma \vdash \mathbb{A}}{\Gamma \vdash \mathbb{B}} \text{(app)}$$

Exercise

Show that $\mathbb{A} \times \mathbb{B} \vdash \mathbb{B} \times \mathbb{A}$

Building New Rules from the Original Ones

The following rules are derivable from the previous system

$$\frac{\Gamma \vdash \mathbb{A}}{\Gamma, \mathbb{B} \vdash \mathbb{A}} \qquad \frac{\Gamma, \mathbb{A}, \mathbb{B}, \Delta \vdash \mathbb{C}}{\Gamma, \mathbb{B}, \mathbb{A}, \Delta \vdash \mathbb{C}}$$

Exercise

Prove that $\mathbb{A} \to \mathbb{B}, \mathbb{B} \to \mathbb{C} \vdash \mathbb{A} \to \mathbb{C}$ and also that $\mathbb{A} \to \mathbb{B}, \mathbb{A} \to \mathbb{C} \vdash \mathbb{A} \to \mathbb{B} \times \mathbb{C}$. Are these deductions familiar?

Going back to programming ...

The Essentials of Programming

In order to study effectful programming, we should think of what are the basic features of (higher-order) programming ...

- variables
- function application
- function abstraction
- pairing ...

and base our study on the simplest programming language containing these features ...

Simply-typed λ-calculus

It is the basis of `Haskell`, `ML`, `Eff`, `F#`, `Agda`, `Elm` and many other programming languages

Simply-Typed λ-Calculus

Types $\mathbb{A} \ni 1 \mid \mathbb{A} \times \mathbb{A} \mid \mathbb{A} \to \mathbb{A}$

Γ is now a non-repetitive list of typed variables $x_1 : \mathbb{A}_1 \ldots x_n : \mathbb{A}_n$

Programs are built according to the previous deduction rules

$$\frac{x : \mathbb{A} \in \Gamma}{\Gamma \vdash x : \mathbb{A}}\ (\mathrm{ass}) \qquad \frac{}{\Gamma \vdash * : 1}\ (\mathrm{triv}) \qquad \frac{\Gamma \vdash V : \mathbb{A} \times \mathbb{B}}{\Gamma \vdash \pi_1 V : \mathbb{A}}\ (\pi_1)$$

$$\frac{\Gamma \vdash V : \mathbb{A} \qquad \Gamma \vdash U : \mathbb{B}}{\Gamma \vdash \langle V, U \rangle : \mathbb{A} \times \mathbb{B}}\ (\mathrm{prd}) \qquad \frac{\Gamma, x : \mathbb{A} \vdash V : \mathbb{B}}{\Gamma \vdash \lambda x : \mathbb{A}.\, V : \mathbb{A} \to \mathbb{B}}\ (\mathrm{cry})$$

$$\frac{\Gamma \vdash V : \mathbb{A} \to \mathbb{B} \quad \Gamma \vdash U : \mathbb{A}}{\Gamma \vdash V\, U : \mathbb{B}}\ (\mathrm{app})$$

Examples of λ-terms

$x : \mathbb{A} \vdash x : \mathbb{A}$ (identity)

$x : \mathbb{A} \vdash \langle x, x \rangle : \mathbb{A} \times \mathbb{A}$ (duplication)

$x : \mathbb{A} \times \mathbb{B} \vdash \langle \pi_2\ x, \pi_1\ x \rangle : \mathbb{B} \times \mathbb{A}$ (swap)

$f : \mathbb{A} \to \mathbb{B}, g : \mathbb{B} \to \mathbb{C} \vdash \lambda x : \mathbb{A}.\ g(f\ x) : \mathbb{A} \to \mathbb{C}$ (composition)

Exercise

Build a λ-term $f : \mathbb{A} \to \mathbb{B}, g : \mathbb{A} \to \mathbb{C} \vdash ? : \mathbb{A} \to \mathbb{B} \times \mathbb{C}$ that pairs the outputs given by f and g

Semantics for Simply-Typed λ-Calculus

We wish to assign a mathematical meaning to λ-terms

$$[\![-]\!]\colon \lambda\text{-Terms} \longrightarrow \ldots$$

so that we can reason about them in a rigorous way, and take advantage of known mathematical theories

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This is the goal of the next slides: we will study how to interpret λ-terms as functions. But first . . .

Basic Facts about Functions

For every set X, there is a 'trivial' function

$$! : X \longrightarrow \{\star\} = 1, \qquad !\,(x) = \star$$

We can always pair two functions $f : X \to A$, $g : X \to B$ into

$$\langle f, g \rangle : X \to A \times B, \qquad \langle f, g \rangle(x) = (f\,x, g\,x)$$

Consider two sets X, Y. There exist 'projection' functions

$$\pi_1 : X \times Y \to X, \qquad \pi_1(x, y) = x$$
$$\pi_2 : X \times Y \to Y, \qquad \pi_2(x, y) = y$$

Basic Facts about Functions

We can always 'curry' a function $f : X \times Y \to Z$ into

$$\lambda f : X \to Z^Y, \qquad \lambda f(x) = (y \mapsto f(x, y))$$

Consider sets X, Y, Z. There exists an 'application' function

$$\mathrm{app} : Z^Y \times Y \to Z, \qquad \mathrm{app}(f, y) = f\ y$$

Functional Semantics for the Simply-Typed λ-Calculus

Types $\mathbb{A}$ are interpreted as sets $[\![\mathbb{A}]\!]$

$$[\![1]\!] = \{\star\}$$
$$[\![\mathbb{A} \times \mathbb{B}]\!] = [\![\mathbb{A}]\!] \times [\![\mathbb{B}]\!]$$
$$[\![\mathbb{A} \to \mathbb{B}]\!] = [\![\mathbb{B}]\!]^{[\![\mathbb{A}]\!]}$$

A typing context Γ is interpreted as

$$[\![\Gamma]\!] = [\![x_1 : \mathbb{A}_1 \times \cdots \times x_n : \mathbb{A}_n]\!] = [\![\mathbb{A}_1]\!] \times \cdots \times [\![\mathbb{A}_n]\!]$$

A λ-term $\Gamma \vdash V : \mathbb{A}$ is interpreted as a function

$$[\![\Gamma \vdash V : \mathbb{A}]\!] : [\![\Gamma]\!] \longrightarrow [\![\mathbb{A}]\!]$$

Functional Semantics for the Simply-Typed λ-Calculus

A λ-term $\Gamma \vdash V : \mathbb{A}$ is interpreted as a function

$$[\![\Gamma \vdash V : \mathbb{A}]\!] : [\![\Gamma]\!] \longrightarrow [\![\mathbb{A}]\!]$$

in the following way

$$\frac{x_i : \mathbb{A} \in \Gamma}{[\![\Gamma \vdash x_i : \mathbb{A}]\!] = \pi_i} \qquad \frac{}{[\![\Gamma \vdash * : 1]\!] = !} \qquad \frac{[\![\Gamma \vdash V : \mathbb{A} \times \mathbb{B}]\!] = f}{[\![\Gamma \vdash \pi_1 V : \mathbb{A}]\!] = \pi_1 \cdot f}$$

$$\frac{[\![\Gamma \vdash V : \mathbb{A}]\!] = f \qquad [\![\Gamma \vdash U : \mathbb{B}]\!] = g}{[\![\Gamma \vdash \langle V, U \rangle : \mathbb{A} \times \mathbb{B}]\!] = \langle f, g \rangle} \qquad \frac{[\![\Gamma, x : \mathbb{A} \vdash V : \mathbb{B}]\!] = f}{[\![\Gamma \vdash \lambda x : \mathbb{A}.\, V : \mathbb{A} \to \mathbb{B}]\!] = \lambda f}$$

$$\frac{[\![\Gamma \vdash V : \mathbb{A} \to \mathbb{B}]\!] = f \quad [\![\Gamma \vdash U : \mathbb{A}]\!] = g}{[\![\Gamma \vdash V\, U : \mathbb{B}]\!] = \mathrm{app} \cdot \langle f, g \rangle}$$

Exercises

Show that the following equations hold.

$$
\begin{aligned}
[\![x : \mathbb{A}, y : \mathbb{B} \vdash \pi_1 \langle x, y \rangle : \mathbb{A}]\!] &= [\![x : \mathbb{A}, y : \mathbb{B} \vdash x : \mathbb{A}]\!] \\
[\![\Gamma \vdash V : \mathbb{A}]\!] &= [\![\Gamma \vdash \langle \pi_1 V, \pi_2 V \rangle : \mathbb{A}]\!] \\
[\![- \vdash (\lambda x.\, x + 1)\ 2 : \mathbb{N}]\!] &= [\![- \vdash 3 : \mathbb{N}]\!]
\end{aligned}
$$