An Internal Language for Categories enriched over Generalised Metric Spaces

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Motivation

Equivalence between two programs is standardly interpreted as equality between their denotations: $v = w \Longrightarrow [\![v]\!] = [\![w]\!]$

Often one needs a more 'quantitative' notion of program equivalence and consequently of equality as well ...

- v and w are at most at distance ϵ from each other
- v and w are very similar
- ...

An example - Wait calls

Take a language with a ground type X and a signature Σ of operations $\{\mathtt{wait_n} : X \to X \mid n \in \mathbb{N}\}$ where ...

$\mathtt{wait_n(x)}$ adds a latency of n sec. to computation $\mathtt{x}$.

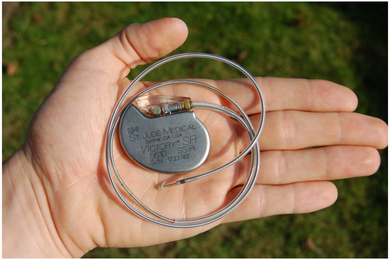
The following metric equations then naturally arise

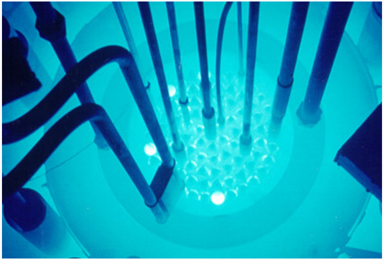
$$\frac{}{\mathtt{wait_0(x)} =_0 \mathtt{x}} \qquad \frac{}{\mathtt{wait_n(wait_m(x))} =_0 \mathtt{wait_{n+m}(x)}}$$

$$\frac{\epsilon = |m - n|}{\mathtt{wait_n(x)} =_\epsilon \mathtt{wait_m(x)}}$$

Context - Hybrid Systems

Computational devices that interact with their physical environment





Contributions

We explore the idea of equivalence taking values in a quantale $\mathcal{V}$ which covers e.g. (in)equations, fuzzy (in)equations, and (ultra)metric equations

We introduce a $\mathcal{V}$-equational system for linear λ-calculus and show that it is sound and complete (in fact, an internal language) for a certain class of enriched autonomous categories

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Quantales and the notion of a $\mathcal{V}$-equation

Definition

A quantale is a complete lattice $\mathcal{V}$ equipped with an associative operation $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ such that,

$$x \otimes (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \otimes y_i) \qquad \text{and} \qquad (\bigvee_{i \in I} y_i) \otimes x = \bigvee_{i \in I} (y_i \otimes x)$$

Definition

Take a quantale $\mathcal{V}$. A $\mathcal{V}$-equation $v =_q w$ is an equation between terms v and w labelled by an element $q \in \mathcal{V}$

Quantales and the notion of a $\mathcal{V}$-equation

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Definition

Take a quantale $\mathcal{V}$. A $\mathcal{V}$-equation $v =_q w$ is an equation between terms v and w labelled by an element $q \in \mathcal{V}$

The quantale structure takes a key role in establishing a notion of $\mathcal{V}$-congruence and a corresponding completeness result ...

Reflexivity, transitivity, symmetry . . .

$$\frac{}{v =_\top v}\ (\mathrm{refl}) \qquad \frac{v =_q w \qquad w =_r u}{v =_{q \otimes r} u}\ (\mathrm{trans}) \qquad \frac{v =_q w}{w =_q v}\ (\mathrm{sym})$$

Example

Boolean quantale $((\{0 \leq 1\}, \vee), \otimes := \wedge)$ yields (in)equations,

$$\frac{}{v =_1 v} \qquad \frac{v =_q w \qquad w =_r u}{v =_{q \wedge r} u} \qquad \frac{v =_q w}{w =_q v}$$

Example

Metric quantale $(([0, \infty], \wedge), \otimes := +)$ yields metric equations,

$$\frac{}{v =_0 v} \qquad \frac{v =_q w \qquad w =_r u}{v =_{q + r} u} \qquad \frac{v =_q w}{w =_q v}$$

. . . join and weakening

$$\frac{\forall i \leq n.\ v =_{q_i} w}{v =_{\vee q_i} w}\ (\text{join}) \qquad \frac{v =_q w \quad r \leq q}{v =_r w}\ (\text{weak})$$

Example

For the Boolean quantale $((\{0 \leq 1\}, \vee), \otimes := \wedge)$

$$\frac{\forall i \leq n.\ v =_{q_i} w}{v =_{\max q_i} w} \qquad \frac{v =_q w \quad r \leq q}{v =_r w}$$

Example

For the metric quantale $(([0, \infty], \wedge), \otimes := +)$

$$\frac{\forall i \leq n.\ v =_{q_i} w}{v =_{\min q_i} w} \qquad \frac{v =_q w \quad r \geq q}{v =_r w}$$

Our goal

- Integrate a $\mathcal{V}$-equational deductive system in linear λ-calculus
- show that it is sound and complete
- and establish an internal language theorem

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Types and contexts in linear λ-calculus

$$\mathbb{A} ::= X \in G \mid \mathbb{I} \mid \mathbb{A} \otimes \mathbb{A} \mid \mathbb{A} \multimap \mathbb{A}$$

Definition

A context Γ is a non-repet. list of variables $x_1 : \mathbb{A}_1, \ldots, x_n : \mathbb{A}_n$

Definition

A shuffle $E \in \mathrm{Sf}(\Gamma_1; \ldots; \Gamma_n)$ is a permutation of $\Gamma_1, \ldots, \Gamma_n$ such that $\forall i \leq n$ the relative order of the variables in Γ_i is preserved

Example

Take $\Gamma_1 = x : \mathbb{A}, y : \mathbb{B}$ and $\Gamma_2 = z : \mathbb{C}$. Then $z : \mathbb{C}, x : \mathbb{A}, y : \mathbb{B}$ is a shuffle but $y : \mathbb{B}, x : \mathbb{A}, z : \mathbb{C}$ is not

Judgement derivation rules

$$\frac{\Gamma_i \rhd v_i : \mathbb{A}_i \quad f : \mathbb{A}_1, \dots, \mathbb{A}_n \to \mathbb{A} \in \Sigma \quad E \in \mathrm{Sf}(\Gamma_1; \dots; \Gamma_n)}{E \rhd f(v_1, \dots, v_n) : \mathbb{A}} \ (\mathrm{ax}) \qquad \frac{}{x : \mathbb{A} \rhd x : \mathbb{A}} \ (\mathrm{hyp})$$

$$\frac{}{- \rhd * : \mathbb{I}} \ (\mathbb{I}_{\mathrm{i}}) \qquad \frac{\Gamma \rhd v : \mathbb{I} \quad \Delta \rhd w : \mathbb{A} \quad E \in \mathrm{Sf}(\Gamma; \Delta)}{E \rhd v \ \mathtt{to}\ *.\ w : \mathbb{A}} \ (\mathbb{I}_{\mathrm{e}})$$

$$\frac{\Gamma \rhd v : \mathbb{A} \quad \Delta \rhd w : \mathbb{B} \quad E \in \mathrm{Sf}(\Gamma; \Delta)}{E \rhd v \otimes w : \mathbb{A} \otimes \mathbb{B}} \ (\otimes_{\mathrm{i}})$$

$$\frac{\Gamma \rhd v : \mathbb{A} \otimes \mathbb{B} \quad \Delta, x : \mathbb{A}, y : \mathbb{B} \rhd w : \mathbb{C} \quad E \in \mathrm{Sf}(\Gamma; \Delta)}{E \rhd \mathtt{pm}\ v\ \mathtt{to}\ x \otimes y.\ w : \mathbb{C}} \ (\otimes_{\mathrm{e}})$$

$$\frac{\Gamma, x : \mathbb{A} \rhd v : \mathbb{B}}{\Gamma \rhd \lambda x : \mathbb{A}.\, v : \mathbb{A} \multimap \mathbb{B}} \ (\multimap_{\mathrm{i}}) \qquad \frac{\Gamma \rhd v : \mathbb{A} \multimap \mathbb{B} \quad \Delta \rhd w : \mathbb{A} \quad E \in \mathrm{Sf}(\Gamma; \Delta)}{E \rhd v\, w : \mathbb{B}} \ (\multimap_{\mathrm{e}})$$

Uniqueness of derivations, exchange, and substitution

Theorem

If $\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \rhd v : \mathbb{C}$ *then* $\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \rhd v : \mathbb{C}$.

Moreover all judgements $\Gamma \rhd v : \mathbb{A}$ *have a unique derivation*

Proof.

Crucially relies on the notion of a shuffle $\square$

Lemma

If $\Gamma, x : \mathbb{A} \rhd v : \mathbb{B}$ *and* $\Delta \rhd w : \mathbb{A}$ *we can derive* $\Gamma, \Delta \rhd v[w/x] : \mathbb{B}$

Proof.

Follows by structural induction on $\Gamma, x : \mathbb{A} \rhd v : \mathbb{B}$ $\square$

A fragment of the equational system

$$\begin{array}{rcl}
\mathtt{pm}\ v \otimes w\ \mathtt{to}\ x \otimes y.\ u & = & u[v/x, w/y] \\
\mathtt{pm}\ v\ \mathtt{to}\ x \otimes y.\ u[x \otimes y/z] & = & u[v/z] \\
* \ \mathtt{to}\ * .\ v & = & v \\
v\ \mathtt{to}\ * .\ w[*/z] & = & w[v/z]
\end{array}$$

(a) Monoidal structure

$$\begin{array}{rcl}
(\lambda x : \mathbb{A}.\ v)\ w & = & v[w/x] \\
\lambda x : \mathbb{A}.\ (v\ x) & = & v
\end{array}$$

(b) Higher-order structure

Semantics of linear λ-calculus pt. I

Linear λ-calculus is interpreted on autonomous categories ...

- types $\mathbb{A}$ interpreted as objects $[\![\mathbb{A}]\!] \in \mathsf{C}$
- contexts $x_1 : \mathbb{A}_1, \ldots, x_n : \mathbb{A}_n$ interpreted as tensors $[\![\mathbb{A}_1]\!] \otimes \cdots \otimes [\![\mathbb{A}_n]\!] \in \mathsf{C}$
- judgements $\Gamma \rhd v : \mathbb{A}$ interpreted as C-morphisms $[\![\Gamma \rhd v : \mathbb{A}]\!] : [\![\Gamma]\!] \to [\![\mathbb{A}]\!]$

Semantics of linear λ-calculus pt. II

Theorem (Soundness)

For any provable equation $\Gamma \rhd v = w : \mathbb{A}$ *we have* $[\![v]\!] = [\![w]\!] \in \mathsf{C}([\![\Gamma]\!], [\![\mathbb{A}]\!])$

Proof.

Follows from the previous substitution lemma and the axiomatics of autonomous categories □

Semantics of linear λ-calculus pt. II

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Proof.

Follows from the previous substitution lemma and the axiomatics of autonomous categories $\square$

Theorem (Completeness)

If $[\![v]\!] = [\![w]\!]$ *for every possible interpretation* $[\![-]\!]$ *then* $v = w$

Proof.

Build a syntactic category whose objects are the available types and morphisms $\mathbb{A} \to \mathbb{B}$ are equivalence classes of judgements $x : \mathbb{A} \rhd v : \mathbb{B}$ w.r.t. provable equality $\square$

Internal language

From an autonomous category C we build a λ-theory $\mathrm{Lang}(\mathsf{C})$

- the ground types are the objects of C
- operation symbols $f : X \to Y$ are the C-morphisms $f : X \to Y$
- we include as axioms 'all the equations in C'

Conversely we build $\mathrm{Syn}(\mathrm{Lang}(\mathsf{C}))$ the syntactic category of $\mathrm{Lang}(\mathsf{C})$, as described earlier

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Theorem (Internal language)

There exists an equivalence of categories $\mathrm{Syn}(\mathrm{Lang}(\mathsf{C})) \simeq \mathsf{C}$

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Congruence in linear λ-calculus

$$\frac{}{v = v} \qquad \frac{v = w \qquad w = u}{v = u} \qquad \frac{v = w}{w = v}$$

$$\frac{\forall i \leq n.\ v_i = w_i}{f(v_1, \ldots, v_n) = f(w_1, \ldots, w_n)} \qquad \frac{v = w \quad v' = w'}{v \otimes v' = w \otimes w'}$$

$$\frac{v = w \qquad v' = w'}{\texttt{pm}\ v\ \texttt{to}\ x \otimes y.\ v' = \texttt{pm}\ w\ \texttt{to}\ x \otimes y.\ w'} \qquad \frac{v = w \quad v' = w'}{v\ v' = w\ w'}$$

$$\frac{v = w \qquad v' = w'}{v\ \texttt{to}\ *.\ v' = w\ \texttt{to}\ *.\ w'} \qquad \frac{v = w}{\lambda x : \mathbb{A}.\ v = \lambda x : \mathbb{A}.\ w}$$

$$\frac{\Gamma \triangleright v = w : \mathbb{A} \qquad \Delta \in perm(\Gamma)}{\Delta \triangleright v = w : \mathbb{A}} \qquad \frac{v = w \qquad v' = w'}{v[v'/x] = w[w'/x]}$$

$\mathcal{V}$-congruence in linear λ-calculus

$$\frac{}{v =_\top v} \qquad \frac{v =_q w \qquad w =_r u}{v =_{q \otimes r} u} \qquad \frac{v =_q w \qquad r \leq q}{v =_r w} \qquad \frac{\forall i \leq n.\ v =_{q_i} w}{v =_{\vee q_i} w}$$

$$\frac{\forall i \leq n.\ v_i =_{q_i} w_i}{f(v_1, \dots, v_n) =_{\otimes q_i} f(w_1, \dots, w_n)} \qquad \frac{v =_q w \quad v' =_r w'}{v \otimes v' =_{q \otimes r} w \otimes w'}$$

$$\frac{v =_q w \qquad v' =_r w'}{\mathtt{pm}\ v\ \mathtt{to}\ x \otimes y.\ v' =_{q \otimes r} \mathtt{pm}\ w\ \mathtt{to}\ x \otimes y.\ w'} \qquad \frac{v =_q w \quad v' =_r w'}{v\, v' =_{q \otimes r} w\, w'}$$

$$\frac{v =_q w \qquad v' =_r w'}{v\ \mathtt{to}\ *.\ v' =_{q \otimes r} w\ \mathtt{to}\ *.\ w'} \qquad \frac{v =_q w}{\lambda x : \mathbb{A}.\ v =_q \lambda x : \mathbb{A}.\ w}$$

$$\frac{\Gamma \triangleright v =_q w : \mathbb{A} \qquad \Delta \in \mathit{perm}(\Gamma)}{\Delta \triangleright v =_q w : \mathbb{A}} \qquad \frac{v =_q w \qquad v' =_r w'}{v[v'/x] =_{q \otimes r} w[w'/x]}$$

Semantics of $\mathcal{V}$-equations

An equation $v = w$ is interpreted as $[\![v]\!] = [\![w]\!] \in \mathsf{C}([\![\Gamma]\!], [\![\mathbb{A}]\!])$
which presupposes that $\mathsf{C}([\![\Gamma]\!], [\![\mathbb{A}]\!])$ is a set

A $\mathcal{V}$-equation $v =_q w$ is interpreted as $a([\![v]\!], [\![w]\!]) \geq q \in \mathcal{V}$
with $a : \mathsf{C}([\![\Gamma]\!], [\![\mathbb{A}]\!]) \times \mathsf{C}([\![\Gamma]\!], [\![\mathbb{A}]\!]) \to \mathcal{V}$ a function

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A $\mathcal{V}$-equation $v =_q w$ is interpreted as $a([\![v]\!], [\![w]\!]) \geq q \in \mathcal{V}$ with $a : \mathsf{C}([\![\Gamma]\!], [\![\mathbb{A}]\!]) \times \mathsf{C}([\![\Gamma]\!], [\![\mathbb{A}]\!]) \to \mathcal{V}$ a function

This suggests a certain enrichment on autonomous categories, which we detail next

$\mathcal{V}$-categories pt. I

From now on assume that $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ has a unit k which coincides with the top element $\top \in \mathcal{V}$

Definition

A (small) $\mathcal{V}$-category is a pair (X, a) where X is a class (set) and $a : X \times X \to \mathcal{V}$ is a function such that,

$$k \leq a(x, x) \qquad \text{and} \qquad a(x, y) \otimes a(y, z) \leq a(x, z)$$

Definition

A $\mathcal{V}$-functor $f : (X, a) \to (Y, b)$ between $\mathcal{V}$-categories (X, a) and (Y, b) is a function $f : X \to Y$ such that $a(x, y) \leq b(f(x), f(y))$

$\mathcal{V}$-categories pt. II

Small $\mathcal{V}$-categories and $\mathcal{V}$-functors form a category which we denote by $\mathcal{V}$-Cat

A $\mathcal{V}$-category is symmetric if $a(x, y) = a(y, x)$. We denote by $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym}}$ the full subcategory of symmetric $\mathcal{V}$-categories

Every $\mathcal{V}$-category carries an order $x \leq y$ iff $k \leq a(x, y)$, and the former is separated if $\leq$ is anti-symmetric. We denote by $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$ the full subcategory of separated $\mathcal{V}$-categories

A zoo of categories of $\mathcal{V}$-categories

- For $\mathcal{V}$ the Boolean quantale, $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sep}}$ is the category Pos of partially ordered sets and monotone maps ...
- and $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym,sep}}$ is the category Set of sets and functions
- For $\mathcal{V}$ the metric quantale, $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym,sep}}$ is the category Met of metric spaces and non-expansive maps
- For $\mathcal{V}$ the ultrametric quantale, $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym,sep}}$ is the category of ultrametric spaces and non-expansive maps
- ...

A basis of enrichment

Theorem

The category $\mathcal{V}$-Cat *is autonomous and the full subcategories* $\mathcal{V}$-Cat$_{\mathsf{sym}}$, $\mathcal{V}$-Cat$_{\mathsf{sep}}$, *and* $\mathcal{V}$-Cat$_{\mathsf{sym,sep}}$ *inherit the autonomous structure of* $\mathcal{V}$-Cat

This allows us to consider the following notion of a category enriched over $\mathcal{V}$-categories

Definition

A $\mathcal{V}$-Cat-enriched autonomous category C is an autonomous $\mathcal{V}$-Cat-category C such that $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ is a $\mathcal{V}$-Cat-functor and $(- \otimes X) \dashv (X \multimap -)$ is a $\mathcal{V}$-Cat-adjunction

Semantics of linear $\mathcal{V}\lambda$-calculus pt. I

Linear $\mathcal{V}\lambda$-calculus is interpreted on $\mathcal{V}$-Cat-enriched autonomous categories, in the same way that linear λ-calculus is interpreted on autonomous categories

Theorem (Soundness)

All $\mathcal{V}$-congruence rules previously listed are sound for $\mathcal{V}$-Cat-enriched autonomous categories

Proof.

Crucially relies on the $\mathcal{V}$-Cat-enriched structure of C $\square$

Semantics of linear $\mathcal{V}\lambda$-calculus pt. II

Theorem (Completeness)

If $a([\![v]\!],[\![w]\!]) \geq q$ for every possible interpretation $[\![-]\!]$ then $v =_q w$

Proof.

We build a syntactic category akin to before and make it enriched: for $\Gamma \triangleright v : \mathbb{A}$ and $\Gamma \triangleright w : \mathbb{A}$ we define $v \sim w$ iff $v =_\top w$ and $w =_\top v$ are provable equalities. Then take

$$C(\mathbb{A},\mathbb{B}) := \{ \ [v] \mid x : \mathbb{A} \triangleright v : \mathbb{B} \ \}$$

and define $a([v],[w]) = \bigvee\{q \mid v =_q w \text{ is a provable equality}\}$

This yields a (separated) $\mathcal{V}$-category on $C(\mathbb{A},\mathbb{B})$ □

Internal language

From a $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-enriched autonomous category C we build a $\mathcal{V}\lambda$-theory $\mathrm{Lang}(\mathsf{C})$

- the ground types are the objects of C
- operation symbols $f : X \to Y$ are the C-morphisms $f : X \to Y$
- we include as axioms ‘all the $\mathcal{V}$-equations in C’

Conversely we build $\mathrm{Syn}(\mathrm{Lang}(\mathsf{C}))$ the syntactic category of $\mathrm{Lang}(\mathsf{C})$, as described in the previous slide

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Theorem (Internal language)

There is a $\mathcal{V}$-Cat-equivalence of categories $\mathrm{Syn}(\mathrm{Lang}(\mathrm{C})) \simeq \mathrm{C}$

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Application 1. Wait calls and metric equations

Recall the language with a ground type X a signature of operations $\{\mathtt{wait_n} : X \to X \mid n \in \mathbb{N}\}$ and the following metric equations

$$\frac{}{\mathtt{wait_0(x)} =_0 \mathtt{x}} \qquad \frac{}{\mathtt{wait_n(wait_m(x))} =_0 \mathtt{wait_{n+m}(x)}}$$

$$\frac{\epsilon = |m - n|}{\mathtt{wait_n(x)} =_\epsilon \mathtt{wait_m(x)}}$$

We build a model of this theory on Met which is a $\mathcal{V}$-Cat-enriched autonomous category:

fix a metric space A, interpret the ground type X as $\mathbb{N} \otimes A$ and the operation symbol $\mathtt{wait_n}$ as the non-expansive map $[\![\mathtt{wait_n}]\!] : \mathbb{N} \otimes A \to \mathbb{N} \otimes A$, $(i, a) \mapsto (i + n, a)$

Application 2. Wait calls and inequations

$$\frac{}{\mathtt{wait_0(x) = x}} \qquad \frac{}{\mathtt{wait_n(wait_m(x)) = wait_{n+m}(x)}}$$

$$\frac{n \leq m}{\mathtt{wait_n(x) \leq wait_m(x)}}$$

We build a model of this theory on Pos which is a $\mathcal{V}$-Cat-enriched autonomous category:

fix a poset A, interpret the ground type X as $\mathbb{N} \times A$ and the operation symbol $\mathtt{wait_n}$ as the monotone map $[\![\mathtt{wait_n}]\!] : \mathbb{N} \times A \to \mathbb{N} \times A$, $(i, a) \mapsto (i + n, a)$

Application 3. Probabilistic programming

Consider a language with ground types `real` and `unit`, an operation $\texttt{bernoulli} : \texttt{real}, \texttt{real}, \texttt{unit} \to \texttt{real}$ and the axiom

$$\frac{p, q \in [0,1] \cap \mathbb{Q}}{\texttt{bernoulli}(x_1, x_2, p) =_{|p-q|} \texttt{bernoulli}(x_1, x_2, q)}$$

We build a model over Banach spaces and linear contractions, which form a $\mathcal{V}$-Cat-enriched autonomous category:

`real` and `unit` are interpreted as the spaces $\mathcal{M}\mathbb{R}$ and $\mathcal{M}[0,1]$ of Borel measures equipped with the total variation norm. For finite spaces the latter is the taxicab norm $\|\mu\| = \sum_{i=1}^{n} |\mu(x_i)|$

$[\![\texttt{bernoulli}]\!]$ is the pushforward of the Markov kernel $\mathbb{R}^3 \to \mathcal{M}\mathbb{R}$, $(u, v, p) \mapsto p\delta_u + (1-p)\delta_v$

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Summing up ...

Introduced the notion of a $\mathcal{V}$-equation which covers (in)equations and metric equations, among others

Introduced a sound and complete $\mathcal{V}$-equational system for linear λ-calculus

Illustrations with real-time and probabilistic programming

All details at: `https://arxiv.org/pdf/2105.08473.pdf`

Current work

Application of this work to quantum and hybrid programming

Development of a $\mathcal{V}$-equational system for linear λ-calculus extended with graded modalities