Bisimulation Revisited
(or How point-freeness matters)

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Bisimulation as a Reynolds-arrow

Bisimulation as a relation closed for the coalgebra dynamics

For c and d are F-coalgebras, [Jac06] Def 3.1.2 (pg 67) defines bisimulation as a relation R st

$$(x, y) \in R \Rightarrow (c(x), d(y)) \in Rel(F)(R) \quad (1)$$

is PF-transformed to

$$R \subseteq c^{\circ} \cdot (\mathsf{F}\, R) \cdot d \quad (2)$$

Shunting on c° above (c is a function, not a relation), yields

$$c \cdot R \subseteq (\mathsf{F}\, R) \cdot d \quad (3)$$

Bisimulation as a Reynolds-arrow

Bisimulation as a relation closed for the coalgebra dynamics

This brings to mind the “Reynolds arrow combinator”-pattern:

$$f(R \leftarrow S)g \quad \equiv \quad f \cdot S \subseteq R \cdot g \tag{4}$$

leading to

$$R \text{ is a bisimulation} \quad \equiv \quad c(\mathsf{F}\,R \leftarrow R)d \tag{5}$$

Reasoning about Bisimulations: the Laws

$$id \leftarrow id \quad = \quad id \tag{6}$$

$$(R \leftarrow S)^\circ \quad = \quad R^\circ \leftarrow S^\circ \tag{7}$$

$$R \leftarrow S \;\subseteq\; V \leftarrow U \quad \Leftarrow \quad R \subseteq V \;\wedge\; U \subseteq S \tag{8}$$

Bisimulation as a Reynolds-arrow

Reasoning about Bisimulations: the Laws

from where one get monotony on the consequent side and thus,

$$S \leftarrow R \quad \subseteq \quad (S \cup V) \leftarrow R \tag{9}$$

$$\top \leftarrow S \quad = \quad \top \tag{10}$$

anti-monotony on the antecedent one

$$R \leftarrow \bot \quad = \quad \top \tag{11}$$

and two distributive laws:

$$S \leftarrow (R_1 \cup R_2) \quad = \quad (S \leftarrow R_1) \cap (S \leftarrow R_2) \tag{12}$$

$$(S_1 \cap S_2) \leftarrow R \quad = \quad (S_1 \leftarrow R) \cap (S_2 \leftarrow R) \tag{13}$$

Reasoning about Bisimulations

$\bot$ is a bisimulation for *any* pair of coalgebras c and d

$$
\begin{array}{ll}
 & \langle \forall\, c, d \ :: \ c(\mathsf{F} \bot \leftarrow \bot) b \rangle \\
\equiv & \qquad \{ \ \text{PF-transform} \ \} \\
 & \langle \forall\, c, d \ :: \ c(\mathsf{F} \bot \leftarrow \bot) b \equiv \text{TRUE} \rangle \\
\equiv & \qquad \{ \ \text{PF-transform} \ \} \\
 & \mathsf{F} \bot \leftarrow \bot = \top \\
\equiv & \qquad \{ \ (11) \ \} \\
 & \text{TRUE}
\end{array}
$$

Reasoning about Bisimulations

The converse of a bisimulation is a bisimulation

$$
\begin{array}{ll}
 & c(\mathsf{F}\,R \leftarrow R)d \\
\equiv & \quad \{ \text{ converse } \} \\
 & d(\mathsf{F}\,R \leftarrow R)^\circ c \\
\equiv & \quad \{ \text{ (7) } \} \\
 & d((\mathsf{F}\,R)^\circ \leftarrow R^\circ)c \\
\equiv & \quad \{ \text{ relator F } \} \\
 & d(\mathsf{F}(R^\circ) \leftarrow R^\circ)c \\
\equiv & \quad \{ \text{ (5) } \} \\
 & R^\circ \text{ is a bisimulation}
\end{array}
$$

Reasoning about Bisimulations

Bisimulations are closed under composition

Is a direct consequence of another generic law on the Reynolds-arrow combinator:

$$(R \leftarrow V) \cdot (S \leftarrow U) \quad \subseteq \quad (R \cdot S) \leftarrow (V \cdot U) \tag{14}$$

which expresses *fusion* (but not *fission*) and of which we shall need a special case (cf Jose's morning talk):

$$(r \cdot s^{\circ}) \leftarrow (f \cdot g^{\circ}) \quad = \quad (r \leftarrow f) \cdot (s \leftarrow g)^{\circ} \tag{15}$$

if pair r, s is a tabulation.

Reasoning about Bisimulations

Bisimulations are closed under union

$$
\begin{array}{ll}
 & (\mathsf{F}\,R_1 \leftarrow R_1) \cap (\mathsf{F}\,R_2 \leftarrow R_2) \\
\subseteq & \qquad \{ \text{ (9) (twice) ; monotonicity of meet } \} \\
 & ((\mathsf{F}\,R_1 \cup \mathsf{F}\,R_2) \leftarrow R_1) \cap ((\mathsf{F}\,R_1 \cup \mathsf{F}\,R_2) \leftarrow R_2) \\
= & \qquad \{ \text{ (12) } \} \\
 & ((\mathsf{F}\,R_1 \cup \mathsf{F}\,R_2) \leftarrow (R_1 \cup R_2) \\
= & \qquad \{ \text{ relators } \} \\
 & (\mathsf{F}(R_1 \cup R_2) \leftarrow (R_1 \cup R_2)
\end{array}
$$

Reasoning about Bisimulations

Behavioural equivalence is a bisimulation

$$uRv \equiv [\![c]\!]u = [\![d]\!]v \qquad R \text{ is a bisimulation}$$

$$
\begin{array}{ll}
& c(\mathsf{F}\,([\![c]\!]^\circ \cdot [\![d]\!]) \leftarrow [\![c]\!]^\circ \cdot [\![d]\!])d \\
\equiv & \quad \{ \text{ definition } \} \\
& [\![c]\!]^\circ \cdot [\![d]\!] \subseteq c^\circ \cdot \mathsf{F}\,([\![c]\!]^\circ \cdot [\![d]\!]) \cdot d \\
\equiv & \quad \{ \text{ relators } \} \\
& [\![c]\!]^\circ \cdot [\![d]\!] \subseteq c^\circ \cdot \mathsf{F}\,[\![c]\!]^\circ \cdot \mathsf{F}\,[\![d]\!] \cdot d \\
\equiv & \quad \{ \text{ converse } \} \\
& [\![c]\!]^\circ \cdot [\![d]\!] \subseteq (\mathsf{F}\,[\![c]\!] \cdot c)^\circ \cdot \mathsf{F}\,[\![d]\!] \cdot d
\end{array}
$$

Reasoning about Bisimulations

Behavioural equivalence is a bisimulation

$$[\![(c)]\!]^\circ \cdot [\![(d)]\!] \subseteq (\mathsf{F}\,[\![(c)]\!] \cdot c)^\circ \cdot \mathsf{F}\,[\![(d)]\!] \cdot d$$

$\equiv$ { universal property of coinductive extension }

$$[\![(c)]\!]^\circ \cdot [\![(d)]\!] \subseteq (\omega \cdot [\![(c)]\!])^\circ \cdot \omega \cdot [\![(d)]\!]$$

$\equiv$ { converse }

$$[\![(c)]\!]^\circ \cdot [\![(d)]\!] \subseteq [\![(c)]\!]^\circ \cdot \omega^\circ \cdot \omega \cdot [\![(d)]\!]$$

$\equiv$ { Lambek (final coalgebra is an isomorphism) }

true

Reasoning about Invariants

Invariants are coreflexive coalgebras

$$c(\mathsf{F}\,\Phi \leftarrow \Phi)c$$

Get for free:

- *id* (everywhere true predicate) is largest invariant
- $\perp$ (everywhere false) is the least one
- Invariants are closed by disjunction (ie. union), ...

Reasoning about Invariants

The *next* (○) combinator

[Jacobs,06] definition:

$$x' \bigcirc\Phi\, x \;\equiv\; cx' \,\mathsf{F}\,\Phi\, cx$$

PF-converts to

$$\bigcirc\Phi \;=\; c^{\circ} \cdot \mathsf{F}\,\Phi c$$

Φ invariant $\equiv$ $\Phi \subseteq \bigcirc\Phi$

$$\begin{aligned} c(\mathsf{F}\,\Phi \leftarrow \Phi)c \;&\equiv\; c \cdot \Phi \subseteq \mathsf{F}\,\Phi \cdot c \\ &\equiv\; \Phi \subseteq c^{\circ}\mathsf{F}\,\Phi \cdot c \\ &\equiv\; \Phi \subseteq \bigcirc\Phi \end{aligned}$$

Reasoning about Invariants

The *henceforth* ($\Box$) combinator

[Jacobs,06] definition 4.2.8:

$$\Box\Phi(x) \;\equiv\; (\exists Q \in X : Q \; inv \;\;\wedge\;\; Q \in \Phi \;\;\wedge\;\; Q(x))$$

'hides' a *supremum*:

$$(\bigcup Q : Q \; inv \;\;\wedge\;\; Q \in \Phi : Q)$$

Reasoning about Invariants

The *henceforth* (□) combinator

$$(\bigcup Q : Q\ inv \ \wedge \ Q \in \Phi : Q)$$

$$\equiv \quad \{ \text{ invariant definition } \}$$

$$(\bigcup Q : Q \subseteq \bigcirc Q \ \wedge \ Q \in \Phi : Q)$$

$$\equiv \quad \{ \ \cap\text{-universal } \}$$

$$(\bigcup Q : Q \subseteq \Phi \cap \bigcirc Q : Q)$$

$$\equiv \quad \{ \ \cap \text{ is } \cdot \text{ for coreflexives } \}$$

$$(\bigcup Q : Q \subseteq \Phi \cdot \bigcirc Q : Q)$$

Reasoning about Invariants

The *henceforth* (□) combinator

which means $\Box\Phi = (\nu_X : \Phi \cdot \bigcirc X)$

Reasoning about Invariants

$\Box\Phi = \Phi \;\equiv\; \Phi\ inv$

(cf, [Jacobs,06] Lemma 4.2.6, pg 109)

$\Box\Phi \subseteq \Phi$ is obvious from the definition, but

$$
\begin{array}{ll}
 & \Phi\ inv \\
\equiv & \quad \{\ \text{just proved}\ \} \\
 & \Phi \subseteq \bigcirc\Phi \\
\equiv & \quad \{\ \Phi\cdot\ \text{monotonic; composition of coreflexives is involutive}\ \} \\
 & \Phi \subseteq \Phi \cdot \bigcirc\Phi \\
\Rightarrow & \quad \{\ \text{greatest fixed point induction: } x \leq fx \;\Rightarrow\; x \leq \nu f\ \} \\
 & \Phi \subseteq \Box\Phi
\end{array}
$$

Reasoning about Invariants

$\square\Phi = \Phi \;\equiv\; \Phi\ inv$

$$
\begin{array}{ll}
& \Phi \subseteq \square\Phi \\
\Rightarrow & \{\ \square\Phi \subseteq f(\square\Phi) \text{ for } fx = \Phi \cdot \bigcirc x \text{ and gfp induction: } \nu_f \leq f\nu_f\ \} \\
& \Phi \subseteq \Phi \cdot \bigcirc(\square\Phi) \\
\equiv & \{\ \text{shunting of coreflexives: }\} \\
& \Phi \subseteq \bigcirc(\square\Phi) \\
\Rightarrow & \{\ \text{monotony; } \square\Phi \subseteq \Phi\ \} \\
& \Phi \subseteq \bigcirc\Phi \\
\equiv & \{\ \text{definition}\ \} \\
& \Phi\ inv
\end{array}
$$

Reasoning about Invariants

$\Box\Phi \subseteq \Box\Box\Phi$

$$\Box\Phi \subseteq \Box\Box\Phi$$

$\equiv$ { definition }

$$\Box\Phi \subseteq (\nu_X :: \Box\Phi \cdot \bigcirc X)$$

$\Leftarrow$ { gfp induction }

$$\Box\Phi \subseteq \Box\Phi \cdot \bigcirc(\Box\Phi)$$

$\equiv$ { $\Box\Phi \cdot \Phi = \Box\Phi$ because $\cap$ is composition and $\Box\Phi \subseteq \Phi$ }

$$\Box\Phi \subseteq \Box\Phi \cdot \Phi \cdot \bigcirc(\Box\Phi)$$

$\equiv$ { shunting of coreflexives }

$$\Box\Phi \subseteq \Phi \cdot \bigcirc(\Box\Phi)$$

Jacobs ≡ Aczel & Mendler

- It pays to have both around: compare in both settings the proof that *coalgebra morphisms entail bisimulation*
- ...

Coalgebra morphisms entail bisimulation

In the relational setting

Immediate, since inclusion of functions is equality:

$$c(\mathsf{F}\,h \leftarrow h)d \quad \equiv \quad c \cdot h = (\mathsf{F}\,h) \cdot d \tag{16}$$

Coalgebra morphisms entail bisimulation

In the Aczel & Mendler setting

Let $h : d \longleftarrow c$ a coalgebra morphism and conjecture $\gamma : \mathsf{F}\, h \longleftarrow h$

$$\gamma \;=\; \mathsf{F}\,(\pi_2)^{\circ} \cdot d \cdot \pi_2 \tag{17}$$

Now prove the diagram commutes: i.e., both π_1 and π_2 are coalgebra morphisms, i.e.,

$$\mathsf{F}\,\pi_1 . \gamma = c \cdot \pi_1 \qquad \mathsf{F}\,\pi_2 . \gamma = d \cdot \pi_2 \tag{18}$$

Clearly, π_2 is a coalgebra *iso*morphism. Then, prove that π_1 is also a colagebra morphism, i.e.,

$$c \cdot \pi_1 \;=\; \mathsf{F}\,\pi_1 \cdot \gamma \tag{19}$$

Coalgebra morphisms entail bisimulation

In the Aczel & Mendler setting

$$
\begin{array}{ll}
& c \cdot \pi_1 = \mathsf{F}\,\pi_1 \cdot \gamma \\
\equiv & \quad \{ \text{ conjecture on } \gamma\text{; functors } \} \\
& c \cdot \pi_1 = \mathsf{F}\,(\pi_1 \cdot (\pi_2)^\circ \cdot d \cdot \pi_2 \\
\equiv & \quad \{ \ h = \pi_1 \cdot (\pi_2)^\circ \ \} \\
& c \cdot \pi_1 = \mathsf{F}\,h \cdot d \cdot \pi_2 \\
\equiv & \quad \{ \ h \text{ morphism } \} \\
& c \cdot \pi_1 = c \cdot h \cdot \pi_2 \\
\equiv & \quad \{ \ \pi_2 \text{ iso, } h = \pi_1 \cdot (\pi_2)^\circ \ \} \\
& c \cdot \pi_1 = c \cdot \pi_1
\end{array}
$$

Coalgebra morphisms entail bisimulation

In the Aczel & Mendler setting

Now the converse direction: if h is a function st the diagram commutes, h is a coalgebra morphism.

$$
\begin{array}{ll}
 & c \cdot h = \mathsf{F}\, h \cdot d \\
\equiv & \quad \{\ h = \pi_1 \cdot (\pi_2)^\circ, \text{functors}\ \} \\
 & c \cdot \pi_1 \cdot (\pi_2)^\circ = \mathsf{F}\, \pi_1 \cdot \mathsf{F}\, (\pi_2)^\circ \cdot d \\
\equiv & \quad \{\ \text{hyp: (18)}\ \} \\
 & \mathsf{F}\, \pi_1 . \gamma \cdot (\pi_2)^\circ = \mathsf{F}\, \pi_1 \cdot \mathsf{F}\, (\pi_2)^\circ \cdot d \\
\equiv & \quad \{\ \gamma \text{ definition and } \pi_2 \text{ is iso}\ \} \\
 & \mathsf{F}\, \pi_1 . \gamma = \mathsf{F}\, \pi_1 \cdot \gamma
\end{array}
$$

Jacobs ≡ Aczel & Mendler

- ...
- Equivalence was proved in this morning talk, resorting to a basic (new) result (law 15):

$$(r \cdot s^\circ) \leftarrow (f \cdot g^\circ) \quad = \quad (r \leftarrow f) \cdot (s \leftarrow g)^\circ$$

if pair r, s is a tabulation.

Jacobs ≡ Aczel & Mendler

$$(r \cdot s^\circ) \leftarrow (f \cdot g^\circ) \subseteq (r \leftarrow f) \cdot (s \leftarrow g)^\circ$$

which equivales

$$c \cdot f \cdot g^\circ \subseteq r \cdot s^\circ \cdot d \quad \Rightarrow \quad \langle \exists\, k \;::\; c(r \leftarrow f)k \;\wedge\; d(s \leftarrow g)k \rangle$$

$\equiv$ { shunting and (**??**) }

$$c \cdot f \subseteq r \cdot s^\circ \cdot d \cdot g \quad \Rightarrow \quad \langle \exists\, k \;::\; c \cdot f = r \cdot k \;\wedge\; d \cdot g = s \cdot k \rangle$$

This, in turn, is an instance of

$$x \subseteq r \cdot s^\circ \cdot y \quad \Rightarrow \quad \langle \exists\, k \;::\; x = r \cdot k \;\wedge\; y = s \cdot k \rangle$$

$\equiv$ { shunting and split-universal, followed by split-fusion }

$$x \cdot y^\circ \subseteq r \cdot s^\circ \quad \Rightarrow \quad \langle \exists\, k \;::\; \langle x, y \rangle = \langle r, s \rangle \cdot k \rangle \qquad (20)$$

for $x, y := c \cdot f, d \cdot g$

Jacobs ≡ Aczel & Mendler

The righthand side of (20) is an assertion of *split-fission*.

- image of $\langle x, y\rangle$ must be be at most image of $\langle r, s\rangle$ which is exactly the antecendent of (20):

$$\mathrm{img}\,\langle x, y\rangle \subseteq \mathrm{img}\,\langle r, s\rangle$$
$$\equiv \qquad \{ \text{ split image transform, see } (\textbf{??}) \text{ below } \}$$
$$x \cdot y^\circ \subseteq r \cdot s^\circ$$

- $\langle r, s\rangle$ must be injective within the range of $\langle x, y\rangle$. Here we go stronger than required in forcing $\langle r, s\rangle$ to be *everywhere*-injective:

$$\ker\,\langle r, s\rangle \subseteq id$$
$$\equiv \qquad \{ \text{ kernels of splits ; functions kernels of reflexive } \}$$
$$\ker r \cap \ker s = id$$

Function *fission*

Given f and g, find a functional solution k to equation

$$g \quad = \quad f \cdot k$$

Clearly, a relational upperbound for k always exists, $f \setminus g = f^{\circ} \cdot g$, cf.

$$\begin{array}{ll} & g = f \cdot k \\ \equiv & \qquad \{ \text{ equality of functions } \} \\ & f \cdot k \subseteq g \\ \equiv & \qquad \{ \text{ shunting } \} \\ & k \subseteq f^{\circ} \cdot g \end{array}$$

Function *fission*

Conditions for such a (maximal) solution $f^\circ \cdot g$ to be a function:

- it must be entire

$$
\begin{array}{ll}
 & id \subseteq (f^\circ \cdot g)^\circ \cdot f^\circ \cdot g \\
\equiv & \{ \text{ shunting and definition of image } \} \\
 & \mathrm{img}\, g \subseteq \mathrm{img}\, f
\end{array}
$$

- and simple:

$$
\begin{array}{ll}
 & f^\circ \cdot g \cdot (f^\circ \cdot g)^\circ \subseteq id \\
\equiv & \{ \text{ converses } \} \\
 & f^\circ \cdot g \cdot g^\circ \cdot f \subseteq id
\end{array}
$$

So, for f more surjective than g and f injective within the image (range) of g, equation $f \cdot k = g$ has $k = f^\circ \cdot g$ as maximal (in fact, unique) functional solution.

Function *fission*

Summing up, we proved

$$\langle \exists\, k \ :: \ g = f \cdot k \rangle \equiv k = f^{\circ} \cdot g \quad \Leftarrow \quad \mathrm{img}\, g \subseteq \mathrm{img}\, f \ \wedge \ f^{\circ} \cdot g \cdot g^{\circ} \cdot f \subseteq id$$

Images of splits

$$\mathrm{img}\,\langle R, S\rangle \subseteq \mathrm{img}\,\langle U, V\rangle$$

$\equiv$ { switch to conditions }

$$\langle R, S\rangle \cdot !^{\circ} \subseteq \langle U, V\rangle \cdot !^{\circ}$$

$\equiv$ { “split twist” rule (21) }

$$\langle R, !\rangle \cdot S^{\circ} \subseteq \langle U, !\rangle \cdot V^{\circ}$$

$\equiv$ { (22) thanks to !-natural }

$$\langle id, !\rangle \cdot R \cdot S^{\circ} \subseteq \langle id, !\rangle \cdot U \cdot V^{\circ}$$

$\equiv$ { $\langle id, f\rangle$ is injective for any f, thus left-cancellable }

$$R \cdot S^{\circ} \subseteq U \cdot V^{\circ}$$

Images of splits

The "split twist" rule

$$\langle R, S\rangle \cdot T \subseteq \langle U, V\rangle \cdot X \quad \equiv \quad \langle R, T^{\circ}\rangle \cdot S^{\circ} \subseteq \langle U, X^{\circ}\rangle \cdot V^{\circ} \qquad (21)$$

is proved in [Oliveira,06], as is

$$\langle R, S\rangle \cdot T = \langle R \cdot T, S \cdot T\rangle \quad \Leftarrow \quad R \cdot (\mathrm{img}\ T) \subseteq R \vee S \cdot (\mathrm{img}\ T) \subseteq S$$

as a consequence of fusion results given in [Backhouse,04].

Conclusions & Current Work

- Towards an ”agile” theory for bisimulation?
- The powerset case:

$$
\begin{array}{ll}
& (\Lambda S)(\mathcal{P}R \leftarrow R)(\Lambda U) \\
\equiv & \quad \{ \ \ldots \ \} \\
& \ldots \\
\equiv & \quad \{ \ \ldots \ \} \\
& S \cdot R \subseteq R \cdot U \ \wedge \ U \cdot R^{\circ} \subseteq R^{\circ} \cdot S
\end{array}
$$

  vs recent work on *weak* bisimulation for generic process algebra [Ribeiro thesis]
- Revisiting modal logic for coalgebras.
- Simulations vs. current work on coalgebraic refinement

$$c(\sqsubseteq \cdot FR \cdot \sqsubseteq \leftarrow R)d$$