Coinduction by calculation

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Outline

Motivation

- Initial algebras and final coalgebras provide abstract descriptions of a variety of phenomena in programming

	Definition	Proof
initial algebras	recursion	induction
final coalgebras	co-recursion	co-induction

- Initiality and finality, as universal properties, entail proof principles

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- The role of such universals has been fundamental to a whole discipline of model transformation (the Bird-Meertens style).
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What will we show?

We will show how…

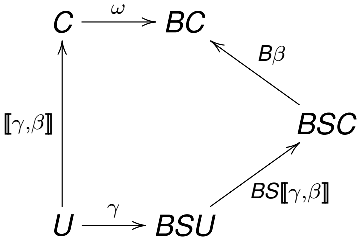
- … to derive *traditional* laws for λ-coinduction
- … the general kit specializes to well known corecursive schemes
- … an example of application
- … bisimulation up-to arises in the calculi

Generalized λ-coinduction

λ-coinduction	=	Functor	+	Comonad
		$\Downarrow$		$\Downarrow$
		type signature		recursive pattern call

Rephrasing as an universal property

For any arrow $\gamma : U \to BSU$, the morphism $k = [\![\gamma, \beta]\!] : U \to C$ is the unique arrow that makes the following diagram commute:



i.e. satisfying the universal property:

$$k = [\![\gamma, \beta]\!] \text{ iff } \omega . k = B(\beta . S\, k) . \gamma \tag{1}$$

Calculational kit

λ**-Reflexion** For $k = id$, we get λ-reflexion.

$$
\begin{aligned}
& id = [\![\gamma, \beta]\!] \\
\equiv & \quad \{ \text{UNIVERSAL PROPERTY} \} \\
& \omega \cdot \mathrm{id} = B(\beta \cdot S\,\mathrm{id}) \cdot \gamma \\
\equiv & \quad \{ \mathrm{id};\ \text{FUNCTOR} \} \\
& \omega = B\beta \cdot \gamma
\end{aligned}
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λ**-Cancellation** $\omega \cdot [\![\gamma, \beta]\!] = B(\beta \cdot S[\![\gamma, \beta]\!]) \cdot \gamma$

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Calculational kit

λ**-Fusion** For $k = [\![\gamma, \beta]\!] \cdot h$, we get λ-Fusion.

$$
\begin{array}{ll}
& [\![\gamma, \beta]\!] \cdot h = [\![\alpha, \beta]\!] \\
\equiv & \quad \{ \text{universal property} \} \\
& \omega \cdot ([\![\gamma, \beta]\!] \cdot h) = B(\beta \cdot S([\![\gamma, \beta]\!] \cdot h)) \cdot \alpha \\
\equiv & \quad \{ \text{composition is associative; Functor} \} \\
& (\omega \cdot [\![\gamma, \beta]\!]) \cdot h = B(\beta \cdot S([\![\gamma, \beta]\!])) \cdot BSh \cdot \alpha \\
\equiv & \quad \{ \text{universal property} \} \\
& B(\beta \cdot S[\![\gamma, \beta]\!]) \cdot h = B(\beta \cdot S([\![\gamma, \beta]\!])) \cdot BSh \cdot \alpha \\
\Leftarrow & \quad \{ \text{function} \} \\
& h = BSh \cdot \alpha
\end{array}
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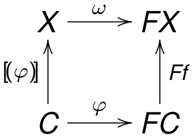
Instances of λ-coinduction

- For $SX = X$ and $\beta = id$, (1) degenerates in the anamorphism universal property;



and we derive the following kit:

ana-Reflexion	$[\!(\omega)\!] = id$
ana-Cancellation	$\omega.[\!(\gamma)\!] = F[\!(\gamma)\!].\gamma$
ana-Fusion	$[\!(\gamma)\!].h = [\!(\alpha)\!] \equiv h = Fh.\alpha$

Instances of λ-coinduction

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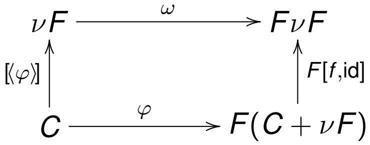

and we derive the following kit:

apo-Reflexion	$[\!(F\ i_1.\omega)\!] = id$
apo-Cancellation	$\omega.[\!(\gamma)\!] = F[[\!(\gamma)\!], \mathrm{id}].\gamma$
apo-Fusion	$[\!(\gamma)\!].h = [\!(\alpha)\!] \Leftarrow \gamma.h = F(h + \mathrm{id}).\alpha$

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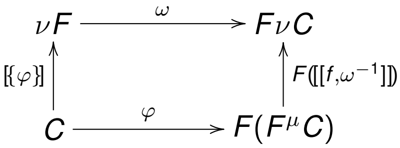
- For $S = F^{\mu}$ and $\beta = (\![[id, \omega^{-1}]]\!)$, (1) degenerates in the futomorphism universal property.



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Exercise

Exercise — Prove that shuffle product is commutative.

But... Because we love streams we read:

Exercise — Prove that shuffle product on streams is commutative.

What we know about streams – $\mathbb{R}^\omega$

- Greatest fix point of $FX = \mathbb{R} \times X$
- Functions from $\mathbb{N}$ to $\mathbb{R}$ (Formal power series over 1^*) and...
- Formal power series have a general product definition (in terms of derivatives)

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Stream product

$(\sigma \otimes \tau)(0) = \sigma(0) \times \tau(0)$
$(\sigma \otimes \tau)' = \sigma' \otimes \tau \oplus \sigma \otimes \tau'$

Looking at the pattern we identify that

$$\otimes = [\![\langle \mathsf{tl} \times \mathsf{id}, \mathsf{id} \times \mathsf{tl} \rangle, \oplus]\!]$$

$\otimes \cdot s = \otimes$ COMMUTATIVITY

s natural transformation $A \times B \to B \times A$

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And now... Calculate

$$\otimes \cdot s = \otimes$$

$\equiv$ { UNIV-λ }

$$\omega \cdot \otimes \cdot s = (\mathrm{id} \times \oplus \cdot (\otimes \cdot s \times \otimes \cdot s)) \cdot \gamma$$

$\equiv$ { CANC-λ }

$$(\mathrm{id} \times \oplus \cdot (\otimes \times \otimes)) \cdot \gamma \cdot s = (\mathrm{id} \times \oplus \cdot (\otimes \cdot s \times \otimes \cdot s)) \cdot \gamma$$

$\equiv$ { SUM COMMUTATIVITY; FUNCTOR-$\times$ }

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$\equiv$ { $\times$ IS COMMUTATIVE; s-NAT; FUSION-$\times$ }

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And now... Calculate

$$\otimes \cdot s = \otimes$$

$\equiv$ $\{$ Univ-λ $\}$

$$\omega \cdot \otimes \cdot s = (\mathrm{id} \times \oplus \cdot (\otimes \cdot s \times \otimes \cdot s)) \cdot \gamma$$

$\equiv$ $\{$ Canc-λ $\}$

$$(\mathrm{id} \times \oplus \cdot (\otimes \times \otimes)) \cdot \gamma \cdot s = (\mathrm{id} \times \oplus \cdot (\otimes \cdot s \times \otimes \cdot s)) \cdot \gamma$$

$\equiv$ $\{$ Sum commutativity; Functor-$\times$ $\}$

$$(\mathrm{id} \times \oplus \cdot s \cdot (\otimes \times \otimes)) \cdot \gamma \cdot s = (\mathrm{id} \times \oplus \cdot (\otimes \times \otimes)) \cdot (\mathrm{id} \times (s \times s)) \cdot \gamma$$

$\equiv$ $\{$ s-nat; Functor-$\times$ $\}$

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$\Leftarrow$ $\{$ Function; Absor-$\times$ $\}$

$$\langle \gamma_1, s \cdot \langle \mathrm{tl} \times \mathrm{id}, \mathrm{id} \times \mathrm{tl} \rangle \rangle \cdot s = \langle \gamma_1, (s \times s) \cdot \langle \mathrm{tl} \times \mathrm{id}, \mathrm{id} \times \mathrm{tl} \rangle \rangle$$

$\equiv$ $\{$ $s \cdot \langle f, g \rangle = \langle g, f \rangle$; Absor-$\times$; s-nat $\}$

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$\equiv$ $\{$ $\times$ is commutative; s-nat; Fusion-$\times$ $\}$

$$\langle \gamma_1, \langle \mathrm{id} \times \mathrm{tl}, \mathrm{tl} \times \mathrm{id} \rangle \rangle \cdot s = \langle \gamma_1, \langle (\mathrm{id} \times \mathrm{tl}), (\mathrm{tl} \times \mathrm{id}) \rangle \rangle \cdot s$$

Remark

- We could have applied the fusion law

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  which is one of the most used laws in other constructions.

- Strategy: the sequence of steps above provides us a proof that can be re-used

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$$
\begin{array}{ll}
& \otimes \cdot s = \otimes \\
\equiv & \{ \text{UNIV-}\lambda \} \\
\\
\equiv & \{ \text{CANC-}\lambda \} \\
\\
\equiv & \{ \text{SUM COMMUTATITVITY; FUNCTOR-}\times \} \\
\\
\equiv & \{ s\text{-NAT; FUNCTOR-}\times \} \\
\\
\Leftarrow & \{ \text{FUNCTION; ABSOR-}\times \} \\
\\
\equiv & \{ s \cdot \langle f, g \rangle = \langle g, f \rangle\text{; ABSOR-}\times\text{; } s\text{-NAT} \} \\
\\
\equiv & \{ \times \text{ IS COMMUTATIVE; } s\text{-NAT; FUSION-}\times \}
\end{array}
$$

Exercise(cont.)

So, if we realise we shouldn't have used streams — $\mathbb{R}^\omega$ — but binary trees, we can use the same strategy to do the proof

What we know about trees

- Greatest fix point of $FX = A \times X \times X$
- Formal power series of the form $\{0,1\}^* \to A$
- Coalgebraic structure

$$T_A \xrightarrow{\langle i, \langle l, r \rangle \rangle} A \times T_A \times T_A$$

- Product definition

$$\otimes = [\![\langle (\times) \cdot (i \times i), \langle \langle l \times \mathrm{id}, \mathrm{id} \times l \rangle, \langle r \times \mathrm{id}, \mathrm{id} \times r \rangle \rangle \rangle, \oplus^2]\!]$$

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$\equiv$ $\{ \cdots \}$

$\equiv$ $\{$ $\times$ IS COMMUTATIVE; s-NAT; FUSION-$\times$ $\}$

$$\langle \gamma_1, \langle \langle \mathrm{id} \times l, l \times \mathrm{id} \rangle, \langle \mathrm{id} \times r, r \times \mathrm{id} \rangle \rangle \rangle \cdot s = \langle \gamma_1, \langle \langle \mathrm{id} \times l, l \times \mathrm{id} \rangle, \langle \mathrm{id} \times r, r \times \mathrm{id} \rangle \rangle \rangle \cdot s$$

Bisimulation up-to

$$\otimes \cdot (\mathsf{id} \times \oplus) = \oplus \cdot (\otimes \times \otimes) \cdot \langle \mathsf{id} \times \pi_1, \mathsf{id} \times \pi_2 \rangle$$

After some calculation on the rhs:

$$\otimes \cdot (\mathsf{id} \times \oplus) = [\![\langle \mathsf{tl} \times \mathsf{id}, \mathsf{id} \times (\mathsf{tl} \times \mathsf{tl}) \rangle, \oplus]\!]$$

But...

$$\begin{aligned}
& \omega \cdot \oplus \cdot (\otimes \times \otimes) \cdot \langle \mathsf{id} \times \pi_1, \mathsf{id} \times \pi_2 \rangle \\
\equiv \quad & \{ \text{CANC-ANA... AND A FEW MORE STEPS} \} \\
& \oplus \cdot (\oplus \cdot (\otimes \times \otimes))^2 \cdot \langle \pi_1^2, \pi_2^2 \rangle \cdot \langle \mathsf{id} \times \pi_1, \mathsf{id} \times \pi_2 \rangle^2 \cdot \langle \mathsf{tl} \times \mathsf{id}, \mathsf{id} \times (\mathsf{tl} \times \mathsf{tl}) \\
\equiv \quad & \{ \text{ARITHMETIC} \} \\
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Arithmetic : $(A + B) + (C + D) = (A + C) + (B + D)$

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Conclusions

- We have derived a calculational kit for generalised coinduction and have shown an application
- We have shown that such a kit specialises to well-know corecursion schemes
- We have shown how this calculational proof style has the advantage of offering a strategy that can be repeated in different proofs
- and is suitable to automate
- Bisimulations up-to arise as arithmetic properties, which is very nice

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