Lecture 2: Functors

Summary.
(1) Functors: motivation and formal definition.

(2) Examples of functors involving different categories. Forgetful and free functors.

(3) Contravariance. Examples: the covariant and contravariant powerset functor; Hom functors.

(4) Full and faithful functors. Isomorphism of categories. Properties preserved by functors.

Luís Soares Barbosa,
UNIV. MINHO *(Informatics Dep.)* & INL *(Quantum Software Engineering Group)*

Opening.

Intuitively, functors provide ways of moving from one mathematical universe to another, that is from one category to another. As John Baez put it *[in Mathematics] every sufficiently good analogy is yearning to become a functor* [1]. Looking at categories as algebraic structures themselves, functors are the corresponding homomorphisms.

Formally, a functor $F : C \longrightarrow D$ between categories C and D consists of an object $F(X)$ fo D for each object X of C, and an arrow $F(f) : F(X) \longrightarrow F(Y)$ for each arrow $f : A \longrightarrow B$, such that

- $F(id_X) = id_{FX}$ for all X in C
- $F(f) \cdot F(g) = F(f \cdot g)$ for any pair of composable arrows f and g in C

The adjective *functorial* means that a construction on objects can be extended to a construction on arrows that preserves composition and identities.

Exercise 1

Let $\mathcal{P}$ stand for the (finite) powerset construction, such that $\mathcal{P}(A) = \{X \mid X \subseteq A\}$ and $\mathcal{P}(f)(X) = \{f(x) \mid x \in X\}$. Prove that $\mathcal{P}$ is an endofunctor in Set.

Exercise 2

Show that there is a functor $R : \mathsf{Set} \longrightarrow \mathsf{Rel}$ which is the identity on objects, and maps each function $f : A \longrightarrow B$ to its graph, i.e.

$$R(f) \mathrel{\widehat{=}} \{(x, f(x)) \in A \times B \mid x \in A\}$$

Exercise 3

What is a functor between preorders regarded as categories?

Exercise 4

What is the effect on arrows of a functor $D : C^{\rightarrow} \longrightarrow C$ mapping each object $f : A \longrightarrow B$ to A?

Exercise 5

Let C/X be the slice category over C induced by an object X. An arrow $k : X \longrightarrow Y$ induces a functor $F_k : C/X \longrightarrow C/Y$ such that

$$\begin{aligned} F_k(f : A \longrightarrow X) &\mathrel{\widehat{=}} k \cdot f : A \longrightarrow Y \\ F_k(h : f \longrightarrow g) &\mathrel{\widehat{=}} h : k \cdot f \longrightarrow k \cdot g \end{aligned}$$

The action on arrows can be illustrated as follows:

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] & \mapsto & A \arrow[rr, "h"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & X & & & & X \arrow[d, "k"'] & \\ & & & & & Y & \end{tikzcd}$$

Show that the axioms for a functor hold for F_k.

Exercise 6

Functor $D : C^{\rightarrow} \longrightarrow C$, discussed in a previous exercise, forgets part of the structure of the source category. A more 'radical' example of a forgetful functor is

$$U : C/X \longrightarrow Set \quad \text{such that} \quad U(f : A \longrightarrow X) = A \ \text{ and } \ U(h : f \longrightarrow g) = h$$

Consider, now, a functor

$$S : C/X \longrightarrow C^{\rightarrow} \quad \text{such that} \quad S(f : A \longrightarrow X) = A \ \text{ and } \ S(h : f \longrightarrow g) = (h, id_x)$$

Prove that U and S are indeed functors. Show that $D \cdot S = U$.

Exercise 7

Free functors are somehow dual to forgetful functors. For example, given a set X one can construct a vector space (over a given field K) with basis X. This construction is canonical in the sense that it is defined without making any arbitrary choices[1]. Actually, the free vector space is the set of all formal K-linear combinations of elements of X, i.e. expressions

$$\sum_{x \in X} \alpha_x \, x$$

where α_x is a scalar in K such that $\alpha_x \neq 0$ for only finitely many values of x. Verify that this defines indeed a vector space, and note how it was obtained from the set X without imposing any equations other than those required by the definition of a vector space. Take the correspondence from X to the respective free vector space as the action on objects of a functor $F : \mathsf{Set} \longrightarrow \mathsf{Vect}_K$. Define the action on arrows and show that the functoriality axioms hold.

Exercise 8

A *contravariant* functor $F : C \longrightarrow D$ is a functor $F : C^{op} \longrightarrow D$. Note that, making the data explicit, an arrow $f : A \longrightarrow B$ in C is mapped to an arrow $F(f) : F(B) \longrightarrow F(A)$ in D, and $F(f) \cdot F(g) = F(g \cdot f)$.

The contravariant power set functor $P : \mathsf{Set}^{op} \longrightarrow \mathsf{Set}$ sends each set A to its power set $\mathcal{P}A$ and each function $f : A \longrightarrow B$ to its inverse image function $f^{-1} : \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$ which maps each $X \subseteq B$ into $f^{-1}(X) \subseteq A$. Verify it is indeed a functor.

Exercise 9

Given two categories C and D, the *product* category $C \times D$ has as objects (resp. arrows) ordered pairs of objects (resp. arrows) whose first element comoes from C and the second from D. A functor whose domain is a *product* category (that one may think as a functor of two variables) is called a *bifunctor*.

Define a functor $SWAP : C \times D \longrightarrow D \times C$ that swaps the order in objects and arrows of its argument and verify it is a functor indeed.

Exercise 10

Let $Vec_{\mathbb{C}}$ be the category of complex vector spaces. The correspondence between a vector space V and its dual V^* , i.e. the vector space whose elements are the linear transformations between V and $\mathbb{C}$ is functorial. The relevant (contravariant) functor is

$$^* : \mathrm{Vec}_{\mathbb{C}}^{op} \longrightarrow \mathrm{Vec}_{\mathbb{C}}$$

such that

[1] Such is the sense the word *canonical* has in Category Theory: a construction given by a deity...

- $V^* = \mathrm{Hom}(V, \mathbb{C})$
- $f^* : W^* \longrightarrow V^*$, for each $f : V \longrightarrow W$, is such that $f^*(t) = t \cdot f$.

Verify that $^* : \mathrm{Vec}_{\mathbb{C}} \longrightarrow \mathbb{C}$ is indeed a functor.

Exercise 11

Let $t : V \longrightarrow W$ be a linear transformation between (finite) Hilbert spaces V and W. Define its *adjoint* $t^\dagger$ by the unique linear transformation

$$t^\dagger : W \longrightarrow V$$

such that, for all $v \in V, w \in W$,

$$\langle t(v)|w\rangle \;=\; \langle v|t^\dagger(w)\rangle$$

Show that this construction is functorial.

Exercise 12

Show that any functor *preserves* isomorphisms, but not necessarily *reflects* them. For the second part, look for a counterexample, i.e. a functor F and an arrow f such that $F(f)$, but not f, is an isomorphism. What can you say about monic and epic arrows, and their split versions?

Exercise 13

Functors can be thought as homomorphisms between categories, i.e. as arrows in Cat whose objects are small categories (recall that a category is small if its collection of arrows is a set), and also in CAT whose objects are locally small categories (all homsets are sets[2]). In this setting, a *isomorphism of categories* is just the usual notion of an isomorphims in Cat or CAT.

Show that the category Mat_S is isomorphic to Mat_S^{op} via a functor which is the identity on objects, and carries a matrix to its transpose.

Exercise 14

In computing, partial operators are often characterised in the context of the category $\mathsf{Set}_\perp$ of pointed sets. A pointed set X is just a set with a distinguished element $\perp_X$, which are preserved by arrows in $\mathsf{Set}_\perp$. I. e. a function $f : X \longrightarrow Y$ in $\mathsf{Set}_\perp$ satisfies $f(\perp_X) = \perp_Y$. Show that $\mathsf{Set}_\perp$ is isomorphic to $\mathbf{1}/\mathsf{Set}$.

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[2]Note that CAT is not locally small and therefore does not belong to itself, which would contradict Russell's paradox.

Exercise 15

Let G be a group, regarded as a category. Characterise G^{op} and prove G is isomorphic to G^{op}.

Exercise 16

Functors may be classified in terms of the correspondences they induce between homsets. In particular, a functor $F : C \longrightarrow D$ is *faithful* (respectively, *full*) if the map $\mathrm{Hom}_C(X, Y) \to \mathrm{Hom}_D(F(X), F(Y))$ is injective (respectively, *surjective*). An *embedding* is a faithful functor which is, additionally, injective on morphisms. Show that full and faithful functors *reflect* isomorphisms, i.e. if $F(f)$ is an isomorphism so is f

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Exercise 17

A subcategory S of a category C is *full* if $\mathrm{Hom}_S(X, Y) = \mathrm{Hom}_C(X, Y)$ for all objects X and Y of S. Show that the inclusion functor $I : S \longrightarrow C$ defined as the identity on objects and arrows of S is always faithful, but is full only when S is a full subcategory.

References

[1] J. Baez. Quantum quandaries: a category-theoretic perspective. In D. Rickles, S. French, and J. T. Saatsi, editors, *The structural foundations of quantum gravity*, pages 240–265. Oxford University Press, 2006.