Lecture 14: The Curry-Howard-Lambek correspondence for classical computation

Summary.
(1) The Curry-Howard-Lambek correspondence: from logic to categories and back.

(2) The Curry-Howard-Lambek correspondence: from programs to categories and back.

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Overview.

The general triangle

$$\begin{tikzcd} \text{Computation} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \text{Categories} \arrow[dl, leftrightarrow] \\ & \text{Logic} & \end{tikzcd}$$

is instantiated to

$$\begin{tikzcd} \lambda\text{-calculus} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \text{Cartesian-closed categories} \arrow[dl, leftrightarrow] \\ & \text{Intuitionistic logic} & \end{tikzcd}$$

A previous lecture already discussed the link between (intuitionistic) logic and (simply-typed) λ-calculus under the *motto*

$$\boxed{\text{Formulas-as-Types and Proofs-as-Programs}}$$

It was emphasized that exploring the computational content of proofs is, indeed, fully aligned with the constructive (BHK) interpretation of intuitionistic logic under which, for example, a proof of $A \wedge B$ is a pair of proofs of both A and B, and a proof of $A \longrightarrow B$ is a procedure to transform any proof of A into a proof of B. We turn now to the links that both logic and computation keep with the mathematical structures which provide their semantical models, i.e with *categories*.

Given a Cartesian-closed category (CCC) $\mathcal{C}$, the Lambek's part of the diagram identifies

$$\boxed{\text{Formulas-as-Objects and Proofs-as-Arrows}}$$

Recall the basic structure of a CCC:

- *Products*: $A \times B$, with projections π_1, π_2 and a split arrow $\langle f, g \rangle : C \longrightarrow A \times B$ defined by a universal property from $f : C \longrightarrow A$ and $g : C \longrightarrow g$. The product construction is functorial: $f \times g = \langle f \cdot \pi_1, g \cdot \pi_2 \rangle$.
- *Exponentials*: B^A, given through the natural isomorphism between

  $$f : A \times B \longrightarrow C \quad \Leftrightarrow \quad \overline{f} : A \longrightarrow C^B$$

  expressed through another universal property

  $$k = \overline{f} \;\; \Leftrightarrow \;\; f = ev \cdot (k \times id)$$

  $$\begin{tikzcd} A \times B \arrow[r, "\overline{f} \times id"] \arrow[dr, "f"'] & C^B \times B \arrow[d, "ev"] \\ & Y \end{tikzcd}$$

  $$A \xrightarrow{\quad \overline{f} \quad} C^B$$

  Construction $-^C$ extends to a functor, the covariant *exponential* functor, by defining

  $$h^C : X^C \longrightarrow Y^C \;\; = \;\; \overline{(h \cdot ev)}$$

  for $h : X \longrightarrow Y$.

  Note that the exponential object X^C represents as an object in the category, the arrows from C to X. Consequently, the action of $-^C$ on each arrow $f : X \longrightarrow Y$ should *internalise* composition with h. In Set it is easy to verify that this is indeed the case. For $g : C \longrightarrow X$ and $c \in C$, a simple calculation yields,

  $$\begin{aligned} & h^C(g)(c) \\ = & \quad \{ \; h^C = \overline{(h \cdot ev)} \} \\ & \overline{(h \cdot ev)}(g)(c) \\ = & \quad \{ \text{ uncurrying } \} \\ & h \cdot ev \; (g, c) \\ = & \quad \{ \text{ function composition } \} \end{aligned}$$

$$
\begin{aligned}
& h(ev(g,c)) \\
= \quad & \quad \{\ ev \text{ definition}\} \\
& h(g(c)) \\
= \quad & \quad \{\ \text{function composition}\ \} \\
& (h \cdot g)\ (c)
\end{aligned}
$$

which means that $h^C \;=\; h \cdot \_$.

The link to logic.

Formulas in intuitionistic logic correspond to objects in $\mathcal{C}$; proofs correspond to morphisms in $\mathcal{C}$. The correspondence is as follows:

Intuitionistic logic	CCC
$\dfrac{}{\Gamma, x : A \;\vdash\; A}$	$\dfrac{}{\pi_2 : \Gamma \times A \longrightarrow A}$
$\dfrac{\Gamma \;\vdash\; A \qquad \Gamma \;\vdash\; B}{\Gamma \;\vdash\; A \wedge B}$	$\dfrac{f : \Gamma \longrightarrow A \qquad g : \Gamma \longrightarrow B}{\langle f, g\rangle : \Gamma \longrightarrow A \times B}$
$\dfrac{\Gamma \;\vdash\; A \wedge B}{\Gamma \;\vdash\; A}$	$\dfrac{f : \Gamma \longrightarrow A \times B}{\pi_1 \cdot f : \Gamma \longrightarrow A}$
$\dfrac{\Gamma \;\vdash\; A \wedge B}{\Gamma \;\vdash\; B}$	$\dfrac{f : \Gamma \longrightarrow A \times B}{\pi_2 \cdot f : \Gamma \longrightarrow B}$
$\dfrac{\Gamma, A \;\vdash\; B}{\Gamma \;\vdash\; A \longrightarrow B}$	$\dfrac{f : \Gamma \times A \longrightarrow B}{\overline{f} : \Gamma \longrightarrow B^A}$
$\dfrac{\Gamma \;\vdash\; A \longrightarrow B \quad \Gamma \;\vdash\; A}{\Gamma \;\vdash\; B}$	$\dfrac{f : \Gamma \longrightarrow B^A \quad g : \Gamma \longrightarrow A}{ev_{A,B} \cdot \langle f, g\rangle : \Gamma \longrightarrow B}$

Exercise 1

Extend the CHL correspondence to capture the propositional intuitionistic logic is enriched with disjunction, i.e. connectives $\vee$ and $\perp$.

The link to computation.

$\boxed{\text{Types-as-Objects and Programs-as-Arrows}}$

Types in the simply-typed λ-calculus correspond objects in a CCC $\mathcal{C}$. Programs, i.e. terms in the simply-typed λ-calculus, on the other hand, correspond to morphisms in $\mathcal{C}$. Moreover, the β, η-reduction is suitably derived from the axioms of a CCC. The correspondence is captured by a semantic function which translates each term

$$x_1 : A_1, \cdots, x_n : A_n \vdash u : B$$

into an arrow in $\mathcal{C}$:

$$[\![u]\!] : [\![A_1]\!] \times \cdots \times [\![A_n]\!] \longrightarrow [\![B]\!]$$

The correspondence is defined recursively on types by

$$\begin{aligned} [\![A \times B]\!] &\mathrel{\widehat{=}} [\![A]\!] \times [\![B]\!] \\ [\![A \longrightarrow B]\!] &\mathrel{\widehat{=}} [\![B]\!]^{[\![A]\!]} \end{aligned}$$

assuming a set of distinguished objects in $\mathcal{C}$ as semantic domains for the basic types.

Similarly, for terms,

$$\frac{}{[\![\Gamma, x : A \vdash x : A]\!] \mathrel{\widehat{=}} \pi_2 : [\![\Gamma]\!] \times [\![A]\!] \longrightarrow [\![A]\!]}$$

$$\frac{[\![\Gamma \vdash u : A \times B]\!] = f : [\![\Gamma]\!] \longrightarrow [\![A]\!] \times [\![B]\!]}{[\![\Gamma \vdash \pi_1\, u : A]\!] \mathrel{\widehat{=}} \pi_1 \cdot f : [\![\Gamma]\!] \longrightarrow [\![A]\!]}$$

$$\frac{[\![\Gamma \vdash u : A]\!] = f : [\![\Gamma]\!] \longrightarrow [\![A]\!] \qquad [\![\Gamma \vdash v : B]\!] = g : [\![\Gamma]\!] \longrightarrow [\![B]\!]}{[\![\Gamma \vdash \langle u, v\rangle : A \times B]\!] \mathrel{\widehat{=}} \langle f, g\rangle : [\![\Gamma]\!] \longrightarrow [\![A]\!] \times [\![B]\!]}$$

$$\frac{[\![\Gamma, x : A \vdash u : B]\!] = f : [\![\Gamma]\!] \times [\![A]\!] \longrightarrow [\![B]\!]}{[\![\Gamma \vdash \lambda x \,.\, u : A \longrightarrow B]\!] \mathrel{\widehat{=}} \overline{f} : [\![\Gamma]\!] \longrightarrow [\![B]\!]^{[\![A]\!]}}$$

$$\frac{[\![\Gamma \vdash u : A \longrightarrow B]\!] = f \qquad [\![\Gamma \vdash v : A]\!] = g}{[\![\Gamma \vdash u\, v : B]\!] \mathrel{\widehat{=}} ev \cdot \langle f, g\rangle : [\![A]\!] \longrightarrow [\![B]\!]}$$

Soundness of $[\![-]\!]$.

Soundness of the translation of simply-typed λ-calculus to a CCC means that β,η-equivalence, which equates terms that are derived one from the other through the rules of β,η-reduction, correspond to semantic equality, i.e.

$$\boxed{u =_{\beta,\eta} v \quad\Rightarrow\quad [\![u]\!] = [\![v]\!]}$$

Let $\Gamma = x_1 : A_1 \ \cdots\ A_n$. Given terms $\Gamma \vdash u : A$ and, for all $1 \leq i \leq n$, $\Gamma \vdash u_i\ A$,

$$[\![u[x_1 := u_1, \cdots, x_n := u_n]]\!] \;=\; [\![u]\!] \cdot \langle [\![u_1]\!] \ \cdots\ , [\![u_n]\!]\rangle$$

This statement, known as the *substitution lemma*, is proved by induction on the structure of terms. The base case is that of variables: x_i. Actually,

$$[\![x_i[\mathbf{x} := \mathbf{u}]]\!] \;=\; [\![u_i]\!] \;=\; \pi_i \cdot \langle [\![u_1]\!], \cdots, [\![u_k]\!]\rangle \;=\; [\![x_i]\!] \cdot \langle [\![u_1]\!], \cdots, [\![u_k]\!]\rangle$$

For the inductive process, consider, for example, $\lambda x \,.\, u$. Thus,

$$
\begin{array}{cl}
 & [\![\lambda x \,.\, u[\mathbf{x} := \mathbf{v}]]\!] \\
= & \{\ \text{substitution}\ \} \\
 & [\![\lambda x \,.\, u[\mathbf{x}, x := \mathbf{v}, x]]\!] \\
= & \{\ [\![-]\!]\ \text{definition}\ \} \\
 & \overline{[\![u[\mathbf{x}, x := \mathbf{v}, x]]\!]} \\
= & \{\ \text{induction hypothesis}\ \} \\
 & \overline{[\![u]\!] \cdot (\langle \mathbf{v} \rangle \times id)} \\
= & \{\ \text{fusion law for exponentials: } \overline{f} \cdot g \;=\; \overline{f \cdot (g \times id)}\ \} \\
 & \overline{[\![u]\!]} \cdot \langle \mathbf{v} \rangle \\
= & \{\ [\![-]\!]\ \text{definition}\ \} \\
 & [\![\lambda x \,.\, u]\!] \cdot \langle \mathbf{v} \rangle
\end{array}
$$

$\boxed{\textbf{Exercise } 2}$

Complete the proof of the *substitution lemma* above for the remaining cases.

To establish soundness of the semantic interpretation $[\![\ ]\!]$, all we need to show is that the interpretation of both sides of a β, η-reduction corresponds to a valid equation in a CCC. The substitution lemma is an important tool in this proof.

Let us start with β-conversion, considering the interpretation of

$$(\lambda x \,.\, u)v \;=_\beta\; u[x := v]$$

$$\begin{array}{ll} & [\![(\lambda x \,.\, u)v]\!] \\ = & \{\ [\![-]\!] \text{ definition }\} \\ & ev \cdot \langle \overline{[\![u]\!]}, [\![v]\!] \rangle \\ = & \{\ \times\text{-absorption law}\} \\ & ev \cdot (\overline{[\![u]\!]} \times id) \cdot \langle id, [\![v]\!] \rangle \\ = & \{\text{currying definition}\} \\ & [\![u]\!] \cdot \langle id, [\![v]\!] \rangle \\ = & \{\text{substitution lemma}\} \\ & [\![u[\mathbf{x}, v := \mathbf{x}, x]]\!] \end{array}$$

$\boxed{\textbf{Exercise } 3}$

Verify the second $=_\beta$-conversion

$$\pi_1 \langle u, v \rangle \;=\; u \qquad \text{and} \qquad \pi_2 \langle u, v \rangle \;=\; v$$

$\boxed{\textbf{Exercise } 4}$

Verify the two $=_\eta$-conversions

$$u \;=\; \lambda x \,.\, u\, x$$

and

$$u \;=\; \langle \pi_1\, u, \pi_2\, u \rangle$$

Completeness of $[\![-]\!]$.

To show completeness one has to come up with a concrete CCC, Λ, in which equalities between arrows correspond to β,η-conversions between terms, i.e.

$$\boxed{u =_{\beta,\eta} v \quad \Leftarrow \quad [\![u]\!] = [\![v]\!]}$$

where $[\![-]\!]$ is an interpretation of λ-terms in Λ.

The category Λ has an object $\hat{A}$ for each type A in the λ-calculus, plus a final object $\mathbf{1}$. An arrow from $\hat{A}$ to $\hat{B}$ is an equivalence class of the following relation defined on variable-term pairs:

$$(x,u) \approx (y,v) \quad \text{iff} \quad x : A \vdash u : B \;\text{ and }\; y : A \vdash v : B \;\text{ and }\; u =_{\beta,\eta} v[y := x]$$

which extends to pairs $(*,u)$, where $*$ represents the single inhabitant of $\mathbf{1}$, as follows:

$$(*,u) \approx (*,v) \quad \text{iff} \quad \vdash u : B \;\text{ and }\; \vdash v : B \;\text{ and }\; u =_{\beta,\eta} v$$

As usual, the equivalence class $[(x,u)]$, for the element (x,u), is the set $\{(y,v) \mid (x,u) \approx (y,v)\}$. Thus, the homsets of Λ are as follows:

$$\begin{aligned} \Lambda\,[\hat{A},\hat{B}] &= \{[(x,u)] \mid x : A \vdash u : B\} \\ \Lambda\,[\mathbf{1},\hat{B}] &= \{[(*,u)] \mid \;\vdash u : B\} \\ \Lambda\,[\hat{A},\mathbf{1}] &= \{!_{\hat{A}}\} \end{aligned}$$

Exercise 5

In Λ define,

- Identities: $id_{\hat{A}} \mathrel{\widehat{=}} [(x,x)]$ and $id_{\mathbf{1}} \mathrel{\widehat{=}} \;!_{\mathbf{1}}$
- Composition:

$$\begin{aligned} [(x,u)] \cdot [(y,v)] &\mathrel{\widehat{=}} [(y, u[x := v])] \\ [(x,u)] \cdot [(*,v)] &\mathrel{\widehat{=}} [(*, u[x := v])] \\ [(*,u)] \cdot !_Z &\mathrel{\widehat{=}} \begin{cases} [(y,u)] & \Leftarrow Z = \hat{A} \\ [(*,u)] & \Leftarrow Z = \mathbf{1} \end{cases} \\ !_W \cdot h &\mathrel{\widehat{=}} \;!_Z \quad \text{for } h : Z \longrightarrow W \end{aligned}$$

Prove that Λ is a category.

The category Λ has finite products and exponentials, and provides what is called a *term* (i.e. built on top of the syntax) model for the simply-typed λ-calculus (see, e.g. [1] for proofs).

References

[1] S. Abramsky and N. Tzevelekos. Introduction to categories and categorical logic. In B. Coecke, editor, *New Structures for Physics*, pages 3–94. Springer Lecture Notes on Physics (813), 2011.