Lecture 3: Universal Properties

Summary.
(1) Universal properties: concept, examples and ubiquity.

(2) Initial and final objects in a category.

(3) Universal characterisation of Cartesian product in Set. The categorial product construction.

(4) Universal properties 'come in pairs': the coproduct construction. Properties of products and coproducts.

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Opening.

If there is a 'main topic' in category theory, this is certainly the study of *universal* properties. Roughly speaking, an entity ϵ is universal among a family of 'similar' entities if it is the case that every other entity in the family can be *reduced* or *traced back* to ϵ. For example, an object T is said to be *final* in a category C if, from every other object X in C, there exists a unique arrow $!_X$ to T. Therefore, there is a canonical, *unique* way to relate every object in C to T — *finality* is thus an universal property.

A nice thing about universal properties is the fact they always 'come in pairs': the *dual* of an universal is still an universal. Dualizing finality, we arrive at *initiality*: an object is *initial* in C if there is one and only one arrow in C from it to any other object in the category.

Universal properties, like finality or initiality, can be recognised, usually under a different terminology, in many branches of Mathematics. Moreover, they happen to play a major role in the structure of 'mathematical spaces'. Therefore, category theory provides a setting for studying abstractly such 'spaces' and their relationships.

Let us consider an illustrative example (adapted from [2]). The study of bilinear (i.e. linear in both arguments) maps out of two vector spaces U and V can be reduced to the study of linear maps because there is a *universal* bilinear map $\epsilon : U \times V \longrightarrow T$ through which all the others factor, i.e. for all $f : U \times V \longrightarrow X$, there exists one and only one linear map $\overline{f} : T \longrightarrow X$ such that $f = \overline{f} \cdot \epsilon$. Look for a moment how uniqueness is proved. Suppose both ϵ and $\epsilon' : U \times V \longrightarrow T'$ satisfy the universal property above. Thus, we obtain linear maps $\overline{\epsilon}$ and $\overline{\epsilon'}$, such that

$$\epsilon' = \overline{\epsilon'} \cdot \epsilon \quad \text{and} \quad \epsilon = \overline{\epsilon} \cdot \epsilon'$$

because, respectively, ϵ and ϵ' are universal by assumption. Clearly, $\overline{\epsilon} \cdot \overline{\epsilon'} \cdot \epsilon = \overline{\epsilon} \cdot \epsilon' = \epsilon$ as depicted in the following diagram:

$$
\begin{tikzcd}
 & U \times W \arrow[dl, "\epsilon"'] \arrow[d, "\epsilon'"] \arrow[dr, "\epsilon"] & \\
T \arrow[r, "\overline{\epsilon'}"] & T' \arrow[r, "\overline{\epsilon}"] & T
\end{tikzcd}
$$

However, $id_T \cdot \epsilon = \epsilon$, which entails $\overline{\epsilon} \cdot \overline{\epsilon'} = id_T$ by the uniqueness of ϵ. A similar argument, relying on the universality of ϵ', yields $\overline{\epsilon'} \cdot \overline{\epsilon} = id_{T'}$. Thus, $\overline{\epsilon}$ is an isomorphism witnessing $T \cong T'$.

Vector space T is the *tensor* product of U and V, often written as $U \otimes V$; and what the universal property tells is that it is essentially unique. The way it is constructed is, to a large extent, irrelevant: the universal property is enough.

Exercise 1

Characterise the initial and final objects in a preorder regarded as a category. Give an example of a preorder in which such objects do not exist.

Exercise 2

Show that any singleton set is both initial and final in $\mathsf{Set}_\perp$ (and, therefore, called a *zero* object). Can you think of another familiar category with a zero object?

Exercise 3

Let Rng be the category of rings and consider $Z = \langle \mathbb{Z}, +, 0, -, \cdot, 1 \rangle$ the ring of integer numbers. Show that there is a unique ring homomorphism h from Z to any other ring $\langle S, +', 0', -', \cdot', 1' \rangle$ given by

$$
h(n) \;\widehat{=}\; \begin{cases} 0' & \Leftarrow\; n = 0 \\ -'h(-n) & \Leftarrow\; n < 0 \\ \underbrace{1' +' 1' +' \cdots +' 1'}_{n} & \Leftarrow\; n > 0 \end{cases}
$$

Exercise 4

Based on the previous exercise, conclude that Z is the initial object in Rng, showing that any other ring satisfying the universal property is isomorphic to Z. Appreciate that for the proof it does not matter ... what a ring is (just as, in the example discussed in the introduction to this Lecture, the meaning of bilinear map or vector space is indeed irrelevant to establish the uniqueness of the tensor product).

$\boxed{\textbf{Exercise } 5}$

Show that any map from a final object in a category to an initial one is an isomorphism.

$\boxed{\textbf{Exercise } 6}$

Coalgebras are a generic way represent transition systems. Formally, a *coalgebra* for a functor $F : C \longrightarrow C$, thought of as the type of the allowed transitions, is an object U, called its carrier, or state space, and an arrow $c : U \longrightarrow T(U)$ of C. A morphism between coalgebras c and c' is an arrow $h : U \longrightarrow V$ in C making the following diagram comute:

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "c"'] & V \arrow[d, "c'"] \\ F(U) \arrow[r, "F(h)"'] & F(V) \end{tikzcd}$$

1. Instantiate the definition for $C = Set$ and $F(X) = \mathcal{P}(L \times X)$, where $\mathcal{P}$ is the finite powerfunctor and L an arbitrary set (of labels, say). What sort of transition systems correspond to this type of coalgebras?
2. Show that coalgebras and their morphisms form a category.
3. Prove that, if coalgebra $(W, \omega : W \longrightarrow F(W))$ is final in the category of F-coalgebras, ω is an isomorphism.

$\boxed{\textbf{Exercise } 7}$

Dualise the definition of a coalgebra given above to arrive to the dual concept of a F-*algebra*, $(A, a : F(A) \longrightarrow A)$. Show that an initial algebra in the corresponding category is also an isomorphism — notice the proof strucuture is exactly the same used in the last question of the previous exercise.

$\boxed{\textbf{Exercise } 8}$

Characterise product and coproduct in a poset regarded as a category. Do the same for the category Pos whose objects are posets and arrows are monotone functions.

$\boxed{\textbf{Exercise } 9}$

Resorting to the corresponding universal property, show that the product (respectively, coproduct) construction in a category is functorial. Show, in particular that, given two arrows $f : A \longrightarrow B$ and $g : C \longrightarrow D$, $f \times g : A \times C \longrightarrow B \times D = \langle f \cdot \pi_1, g \cdot \pi_2 \rangle$. What about $f + g$?

Exercise 10

Derive, from the universal property of products, the equality $\langle f, g\rangle \cdot h = \langle f \cdot h, g \cdot h\rangle$, for f, g and h suitably typed, and $\langle \mathsf{id}_A, \mathsf{id}_B\rangle = \mathsf{id}_{A\times B}$. These results are known in classical program calculi [1], as the product *fusion* and *reflection* laws, respectively.

Exercise 11

A coproduct in Rel is given by disjoint union, with the universal arrow in the diagram below defined as

$$[R, S] \;\widehat{=}\; R \cdot \iota_1^{\circ} \cup S \cdot \iota_1^{\circ}$$

$$\begin{tikzcd} A \arrow[r, "\iota_1"] \arrow[dr, "R"'] & A + B \arrow[d, "{[R,S]}"] & B \arrow[l, "\iota_2"'] \arrow[dl, "S"] \\ & C & \end{tikzcd}$$

Define product in Rel by dualising this construction. Recall that Rel is a self-dual category.

Exercise 12

The product of two vector spaces U, V over a field K, in Vect_K usually represented as $U \oplus V$, is given by $U \times V = \{(u, v) \mid u \in U, v \in V\}$ made into a vector space by defining addition and scalar multiplication as follows:

$$(x, y) + (x', y') \;=\; (x + x', y + y') \;\text{ and }\; k\,(x, y) \;=\; (k\,x, k\,y)$$

Projections and the universal arrow are as in Set but required to be linear. Show such is the case indeed.

Exercise 13

The disjoint sum $U \oplus V$ of two vector spaces U, V over a field K, in Vect_K, is simultaneously their product (as discussed in the previous exercise) and coproduct. Define the embeddings $\iota_1 : U \longrightarrow U \oplus V$ and $\iota_2 : V \longrightarrow U \oplus V$ as

$$\iota_1(x) \;=\; (x, 0_V) \;\text{ and }\; \iota_2(y) \;=\; (0_U, y)$$

where $0_U, 0_V$ are the additive identities in U and V, respectively. For $f : U \longrightarrow Z$ and $g : V \longrightarrow Z$, define the universal arrow $[f, g] : U \oplus V \longrightarrow Z$ by

$$[f, g](x, y) \;=\; f(x) + g(y)$$

and prove that the relevant arrows are linear and this construction defines indeed a coproduct.

References

[1] R. Bird and O. Moor. *The Algebra of Programming*. Series in Computer Science. Prentice-Hall International, 1997.

[2] T. Leinster. *Basic Category Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2014.