Lecture 11: Quantum λ-calculus[1]

Summary.
(1) Syntax and operational semantics.

(2) Typing system.

(2) Examples: representation of quantum programs.

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Syntax.

$$M, N, P \ \ni \ x \mid c \mid M\,N \mid \lambda x.M \mid \langle M, N\rangle \mid \mathit{let}\,\langle x, y\rangle = M\,\mathit{in}\,N \mid \mathit{if}\,M\,\mathit{then}\,N\,\mathit{else}\,P$$

where $x \in X$, for X an infinite set of variables, and c ranges over the following constants,

$$c \ \ni \ * \mid 0 \mid 1 \mid \mathit{new} \mid \mathit{ms} \mid U$$

where new stands for a function for state preparation (accepts a classical bit b, returns qubit $|b\rangle$), ms for a function performing a measurement (in the canonical basis), and U for the application of an unitary transformation. Common abbreviations include

$$\begin{aligned} \mathit{let}\,x = M\,\mathit{in}\,P &\overset{\text{abv}}{=} (\lambda x.P)\,M \\ \lambda\langle x, y\rangle.P &\overset{\text{abv}}{=} \lambda z.(\mathit{let}\,\langle x, y\rangle = z\,\mathit{in}\,P) \end{aligned}$$

The notions of α-equivalence, free variable and substitution are defined as usual. Terms encode quantum algorithms, e.g.

<u>Example [fair coin].</u>

$$\mathit{coin} \ = \ \lambda * .\mathit{ms}(H(\mathit{new}\,0))$$

At first sight, it seemed reasonable to include a term to directly represent a qubit, e.g. $|\phi\rangle$, as in a function $\lambda x.|\phi\rangle$ which constantly outputs $|\phi\rangle$. The problem comes from entanglement: given two qubits entangled (and therefore not representable in the form $|\phi\rangle \otimes |\phi'\rangle$) there are no ways to represent in a term the varaibles corresponding to the first and second qubits in the entangled pair.

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[1] These lecture sums up the seminar given by Benoît Valiron. Reference text: [2]

Operational semantics.

The operational semantics is given in terms of a reduction machine, which somehow represents a quantum processor acting over a quantum memory. The problem mentioned above requires some form of indirect representation of the quantum state of the underlying a program. This entails the notion of a quantum closure:

$$[Q, L, M]$$

where Q is a normalized vector in $\otimes^n \mathcal{C}^2$, M is a λ-term, and L is an ordered list $|x_1 \cdots x_n\rangle$ of term variables meaning that variable x_i is bound in term M to the qubit i.

<u>Example.</u>

$$[\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), |p, q\rangle, \lambda x.xpq]$$

where p and q represent, respectively, the two qubits inthe entangled state $|p, q\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

Given the probabilistic nature of measurement, the reduction machine is probabilistic:

$$(S, V, R, pr)$$

where S is a set of states, $V \subseteq S$ is the subset of value states (in which reduction terminates), $R \subseteq S - V \times S$ is a set of reductions, and $pr : R \longrightarrow [0, 1]$ is a probability function, such that the number states related by R with each state is finite and

$$\sum_{y \in \{y \mid (x,y) \in R\}} pr(x, y) \leq 1$$

Notation $x \rightarrow_\rho y$ stands for $pr(x, y) = \rho$, which extends, as expected, to n-step reductions: $x \rightarrow^n_\rho y \stackrel{\text{abv}}{=} (pr^n(x, y) = \rho)$, where

$$pr^n(x, y) \;=\; \sum_{z \in \{z \mid (x,z) \in R\}} pr(x, z)\, pr^{n-1}(z, y)$$

The basic relation is *reachability with non-zero probability* ($x \rightarrow^n_{>0} y$ for some $n \geq 0$).

- total V-probability: $pr_V(x) \;=\; \sum_{n=0}^{\infty} \sum_{v \in V} pr^n(x, v)$
- divergence-probability: $pr_\infty(x) \;=\; \lim_{n \to \infty} \sum_{x \in S} pr^n(x, y)$
- error-probability: $pr_{err}(x) \;=\; 1 - pr_V(x) - pr_\infty(x)$

In some situations it is useful to relax reachability to include null probability ($x \rightsquigarrow y$) because a null probability of getting to a certain state is not an absolute warranty of its impossibility, due to decoherence and imprecision of physical operations. Thus, a state $x \in S$ is *consistent* if there is no error state e such that $x \rightsquigarrow e$, where e is an *error state* if $e \notin V$ and $\sum_{y \in S} pr(e, y) < 1$.

Exercise 1

Show that $pr_{err}(x) = 0$ if x is consistent. Does the converse hold?

Operational semantics of the quantum λ-calculus

The reduction machine for the quantum λ-calculus is probabilistic and adopts a *call-by-value* reduction strategy[2]. Its purpose is to evaluate a quantum closure until a value state is reached. A value state is a quantum closure whose term is a value, defined by

$$V, V' \ \ni\ x \mid \lambda x.M \mid \langle V, V'\rangle \mid * \mid 0 \mid 1 \mid new \mid ms \mid U$$

Classical control:

$$[Q, L, (\lambda x.M)P] \ \longrightarrow_1\ [Q, L, M[x := P]]$$

$$[Q, L, let\,\langle x, y\rangle = \langle V, V'\rangle\, in\, N] \ \longrightarrow_1\ [Q, L, N[x := V, y := V']]$$

$$[Q, L, if\, 0\, then\, N\, else\, P] \ \longrightarrow_1\ [Q, L, P]$$

$$[Q, L, if\, 1\, then\, N\, else\, P] \ \longrightarrow_1\ [Q, L, N]$$

Quantum data:

$$[Q, |x_1, \cdots, x_n\rangle, new\, 0] \ \longrightarrow_1\ [Q \otimes |0\rangle, |x_1, \cdots, x_n, x_{n+1}\rangle, x_{n+1}]$$

$$[Q, |x_1, \cdots, x_n\rangle, new\, 1] \ \longrightarrow_1\ [Q \otimes |1\rangle, |x_1, \cdots, x_n, x_{n+1}\rangle, x_{n+1}]$$

$$[Q, L, U\langle x_1, \cdots, x_n\rangle] \ \longrightarrow_1\ [Q', L, \langle x_1, \cdots, x_n\rangle]$$

$$[\alpha|Q_0\rangle + \beta|Q_1\rangle, L, ms\, x_i] \ \longrightarrow_{|\alpha|^2}\ [|Q_0\rangle, L, 0]$$

$$[\alpha|Q_0\rangle + \beta|Q_1\rangle, L, ms\, x_i] \ \longrightarrow_{|\beta|^2}\ [|Q_1\rangle, L, 1]$$

In the rule dealing with $U\langle x_1, \cdots, x_n\rangle$, Q' is the state produced by applying U to qubits indexed by variables x_1 to x_n. In the rule for measurements, $|Q_0\rangle = \sum_j \alpha_j|\phi_j\rangle \otimes |0\rangle \otimes |\psi_j\rangle$ where $|\phi_j\rangle$ is a i-qubit state, so that the measured qubit is the one pointed to by x_i, and similarly for $|Q_1\rangle$.

[2]Note that adopting a *call-by-value* reduction strategy could result in measurements of the form $ms\, M$ being delayed along reductions, as there will be no way to force them to be executed.

Congruence rules:

$$\frac{[Q,L,N] \longrightarrow_{\rho} [Q',L',N']}{[Q,L,MN] \longrightarrow_{\rho} [Q',L,MN']} \qquad \frac{[Q,L,M] \longrightarrow_{\rho} [Q',L',M']}{[Q,L,MV] \longrightarrow_{\rho} [Q',L',M'V]}$$

$$\frac{[Q,L,N] \longrightarrow_{\rho} [Q',L',N']}{[Q,L,\langle M,N\rangle] \longrightarrow_{\rho} [Q',L',\langle M,N'\rangle]} \qquad \frac{[Q,L,M] \longrightarrow_{\rho} [Q',L',M']}{[Q,L,\langle M,V\rangle] \longrightarrow_{\rho} [Q',L',\langle M',V\rangle]}$$

$$\frac{[Q,L,M] \longrightarrow_{\rho} [Q',L',M']}{[Q,L,\mathit{if}\, M\, \mathit{then}\, N\, \mathit{else}\, P] \longrightarrow_{\rho} [Q',L,\mathit{if}\, M'\, \mathit{then}\, N\, \mathit{else}\, P]}$$

$$\frac{[Q,L,M] \longrightarrow_{\rho} [Q',L,M']}{[Q,L,\mathit{let}\,\langle x,y\rangle = M\, \mathit{in}\, N] \longrightarrow_{\rho} [Q',L,\mathit{let}\,\langle x,y\rangle = M'\, \mathit{in}\, N]}$$

Types.

The reduction machine can produce *error-states* — e.g. $[Q,L,H(\lambda x.x)]$ or $[Q,|x,y,z\rangle,U\langle x,x\rangle]$ — which correspond to run-time errors. The purpose of a type system is precisely to get rid of such states.

$$A,B \ \ni\ \mathtt{bit} \mid \mathtt{qubit} \mid !A \mid A \otimes B \mid A \multimap B \mid \top$$

where $A \otimes B$ types pairs of elements of type A and B, $A \multimap B$ is the type of functions from A to B, $\top$ is the type of constant $*$, and $!A$ is the type of *duplicable* elements of type A. Any value of type $!A$ can be used in a context in which a value of type A is expected (i.e. used only once, even if it is a duplicable value), leading to the following *subtyping* relation $\precsim$, defined under the overall condition $n = 0 \Rightarrow m = 0$:

$$\frac{}{!^n\mathtt{bit} \precsim !^m\mathtt{bit}}\,(\mathtt{bit}) \qquad \frac{}{!^n\mathtt{qubit} \precsim !^m\mathtt{qubit}}\,(\mathtt{qubit}) \qquad \frac{}{!^n\top; \precsim !^m\top}\,(\top)$$

$$\frac{A_1 \precsim B_1 \quad A_2 \precsim B_2}{!^n(A_1 \otimes A_2) \precsim !^m(B_1 \otimes B_2)}\,(\otimes) \qquad \frac{A \precsim A' \quad B \precsim B'}{!^n(A' \multimap B) \precsim !^m(A \otimes B')}\,(\multimap)$$

Exercise 2

Let QT denote the set of types for quantum λ-calculus. Show that $(\mathsf{QT}, \precsim)$ is a preorder and that the quotient of QT by $\precsim$-symmetric closure forms a poset under $\precsim$.

Terms in the calculus are typed through *typing judgements* — $\Delta \ \triangleright \ M : A$, where Δ is a set of typed variables $\{x_1 : A_1, \cdots, x_n : A_n\}$[3]. Each constant c has an associated type A_c as follows:

$$A_0, A_1 = \mathsf{bit} \qquad A_{new} = \mathsf{bit} \multimap \mathsf{qubit} \qquad A_U = \mathsf{qubit}^{\otimes n} \multimap \mathsf{qubit}^{\otimes n} \qquad A_{ms} = \mathsf{qubit} \multimap !\mathsf{bit}$$

Exercise 3

Rule (ax_2) establishes type $!A_c$ as the *most generic* type for c. Use this fact to show that no qubit created through new can have the type $!\mathsf{qubit}$.

Typing rules

$$\frac{A \precsim B}{\Delta, x : A \ \triangleright \ x : B}\,(ax_1) \qquad \frac{!A_c \precsim B}{\Delta \ \triangleright \ c : B}\,(ax_2) \qquad \frac{}{\Delta \ \triangleright \ * : !^n\top}\,(\top)$$

$$\frac{x : A, \Delta \ \triangleright \ M : B}{\Delta \ \triangleright \ \lambda x.M : A \multimap B}\,(\lambda_1) \qquad \frac{\Gamma, !\Delta, x : A \ \triangleright \ M : B}{\Gamma, !\Delta \ \triangleright \ \lambda x.M : !^{n+1}(A \multimap B)}\,(\lambda_2),\ \ \text{if}\, \mathcal{FV}(M) \cap |\Gamma| = \emptyset$$

$$\frac{\Gamma_1, !\Delta \ \triangleright \ M : A \multimap B \qquad \Gamma_2, !\Delta \ \triangleright \ N : A}{\Gamma_1, \Gamma_2, !\Delta \ \triangleright \ MN : B}\,(app)$$

$$\frac{\Gamma_1, !\Delta \ \triangleright \ M : \mathsf{bit} \multimap B \qquad \Gamma_2, !\Delta \ \triangleright \ N : A \qquad \Gamma_2, !\Delta \ \triangleright \ P : A}{\Gamma_1, \Gamma_2, !\Delta \ \triangleright \ \mathsf{if}\, M\, \mathsf{then}\, N\, \mathsf{else}\, P : A}\,(cond)$$

$$\frac{!\Delta, \Gamma_1 \ \triangleright \ M_1 : !^n A_1 \qquad !\Delta, \Gamma_2 \ \triangleright \ M_2 : !^n A_2}{!\Delta, \Gamma_1, \Gamma_2 \ \triangleright \ \langle M_1, M_2 \rangle : !^n(A_1 \otimes A_2)}\,(\otimes_{in})$$

$$\frac{!\Delta, \Gamma_1 \ \triangleright \ M : !^n(A_1 \otimes A_2) \qquad !\Delta, \Gamma_2, x : !^n A_1, y : !^n A_2 \ \triangleright \ N : A}{!\Delta, \Gamma_1, \Gamma_2 \ \triangleright \ \mathsf{let}\, \langle x, y \rangle = M\, \mathsf{in}\, N : A}\,(\otimes_{out})$$

[3]Whenever several contexts Δ_1, Δ_2, ... Δ_n, appear in a typing judgement they are assumed to be disjoint.

Well-typed quantum closure: $\Gamma \models [Q, L, M] : A$

A quantum closure $[Q, L, M]$ is well-typed of type A in a context Γ if $|L| \cap |\Gamma| = \emptyset$, $\mathcal{FV}(M) - |\Gamma| \subseteq |L|$, and

$$\Gamma, x_1 : \mathtt{qubit}, \cdots, x_n : \mathtt{qubit} \;\rhd\; M : A$$

is a valid typing judgement, where $\mathcal{FV}(M) - |\Gamma| = \{x_1, \cdots, x_n\}$.

A quantum closure is a *program* if $|\Gamma| = \emptyset$.

The properties of this typing system are similar to those of the one used in Lecture 7 for the simply-typed λ-calculus. In particular (see [2] for proof hints),

- Given a program $[Q, L, M]$ of type A and a derivation

  $$[Q, L, M] \;\rightsquigarrow^*\; [Q', L', M']$$

  $[Q', L', M']$ is still a program of type A. This property is known as *subject reduction* means that well-typedness is preserved by the reduction rules (i.e. by program execution), even in presence of decoherence and imprecision of the physical operations (cf, the use of $\rightsquigarrow$ in the statement).

- A well-typed program does not reach an error state. I.e. any probabilistic computation path of such a program is either infinite, or reaches a value state in a finite number of steps.This property is known as *type safety.*

- There exists a *type-inference algorithm* for the quantum λ-calculus.

Exercise 4

The type-inference algorithm mentioned above is described in detail in [1]. Provide a full implementation in Haskell of this algorithm.

Examples.

Example [fair coin]

$$\triangleright\ \mathsf{coin} : \top \multimap \mathsf{bit}$$

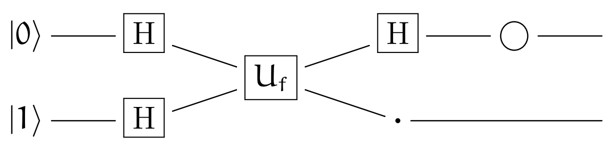
where $\mathsf{coin} \;=\; \lambda * .\mathsf{ms}(\mathsf{H}(\mathsf{new}\,0))$, as above.

Example [Deutsch algorithm]

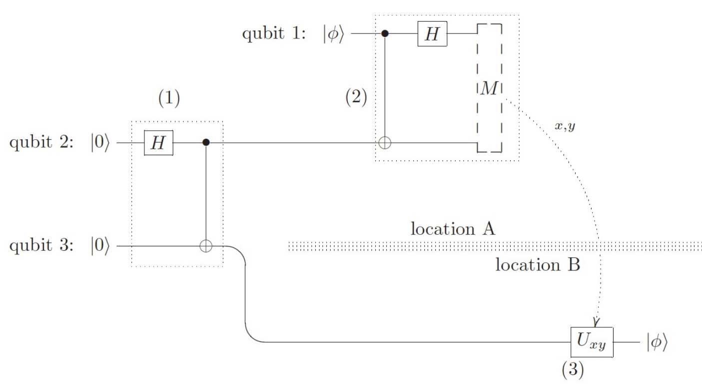
$$\triangleright\ \mathsf{Deutsch} : !((\mathsf{qubit} \otimes \mathsf{qubit} \multimap \mathsf{qubit} \otimes \mathsf{qubit}) \multimap \mathsf{bit})$$

where

$$\begin{aligned}\mathsf{Deutsch}\,\mathsf{U_f} \;=&\\ &\mathsf{let\,comb\,f\,g} \;=\; \lambda\langle x, y\rangle.\langle fx, gy\rangle\\ &\mathsf{in\,let}\langle x, y\rangle \;=\; (\mathsf{comb\,H}\,(\lambda x.x))\,(\mathsf{U_f}\langle \mathsf{H}(\mathsf{new}\,0), \mathsf{H}(\mathsf{new}\,1)\rangle)\\ &\mathsf{in\ ms}\ x\end{aligned}$$



Example [the teleportation protocol]



- Component (1): generates an EPR pair of entangled qubits:

$$\triangleright\ \mathsf{C_1} : !(\top \multimap \mathsf{qubit} \otimes \mathsf{qubit})$$

where $\mathsf{C_1} = \lambda x.\mathsf{CNOT}\,\langle \mathsf{H}(\mathsf{new}\,0), \mathsf{new}\,0\rangle$

- Component (2): performs a Bell measurement and outputs two classical bits::

$$\triangleright\ \mathsf{C}_2 : !(\mathsf{qubit} \multimap (\mathsf{qubit} \multimap \mathsf{bit} \otimes \mathsf{bit}))$$

  where $\mathsf{C}_2 = \lambda q_1.\lambda q_2.(\mathsf{let}\,\langle x, y\rangle = \mathsf{CNOT}\,\langle q_1, q_2\rangle\,\mathsf{in}\,\langle \mathsf{ms}(\mathsf{H}x), \mathsf{ms}y\rangle$

- Component (3): performs a correction::

$$\triangleright\ \mathsf{U} : !(\mathsf{qubit} \multimap (\mathsf{bit} \otimes \mathsf{bit} \multimap \mathsf{qubit}))$$

  where

$$\begin{aligned} \mathsf{U} = \lambda q.\lambda\langle x, y\rangle.\mathsf{if}\,x\,\mathsf{then}\,(\mathsf{if}\,y\,\mathsf{then}\,\mathsf{U}_{11}q\,\mathsf{else}\,,\mathsf{U}_{10}q) \\ \mathsf{else}\,(\mathsf{if}\,y\,\mathsf{then}\,\mathsf{U}_{01}q\,\mathsf{else}\,,\mathsf{U}_{00}q) \end{aligned}$$

  where

$$\mathsf{U}_{00} \mathrel{\widehat{=}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathsf{U}_{01} \mathrel{\widehat{=}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathsf{U}_{10} \mathrel{\widehat{=}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \mathsf{U}_{11} \mathrel{\widehat{=}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Thus yielding

$$\triangleright\ \mathsf{Teleportation} : (\mathsf{qubit} \multimap \mathsf{bit} \otimes \mathsf{bit}) \otimes (\mathsf{bit} \otimes \mathsf{bit} \multimap \mathsf{qubit})$$

where

$$\begin{aligned} \mathsf{Teleportation} \ &= \mathsf{let}\,\langle x, y\rangle = \mathsf{C}_1 * \,\mathsf{in} \\ &\quad \mathsf{let}\,f = \mathsf{C}_2\,x\,\mathsf{in} \\ &\quad \mathsf{let}\,g = \mathsf{U}\,y\,\mathsf{in}\,\langle f, g\rangle \end{aligned}$$

Thus, the teleportation protocol creates two functions f and g, non duplicable because they depend on the state of the pair of entangled qubits x and y, and such that $(g \cdot f)(z) = z$ for an arbitrary qubit z, and $(f \cdot g)(x, y) = (x, y)$ for bits x and y. This pair of mutually inverse functions can only be used once because each of them contains an embedded qubit. Actually, they witness a *single-use isomorphim* between the (otherwise non isomorphic) types qubit and $\mathsf{bit} \otimes \mathsf{bit}$.

Example [execution of the teleportation protocol]

In the sequel, consider the following abbreviations:

$$\begin{aligned} \mathsf{M}_{p,p'} &\mathrel{\widehat{=}} \ \mathsf{let}\,f = \mathsf{C}_2\,p\,\mathsf{in}\,\mathsf{let}\,g = \mathsf{U}p'\,\mathsf{in}\,g(fp_0) \\ \mathsf{B}_{p_1} &\mathrel{\widehat{=}} \ \lambda q_1.\mathsf{let}\,\langle p, p'\rangle = \mathsf{CNOT}\langle q_1, p_1\rangle \ \in \ \langle \mathsf{ms}(\mathsf{H}p), \mathsf{ms}p'\rangle \\ \mathsf{U}_{p_2} &\mathrel{\widehat{=}} \ \lambda\langle x, y\rangle.\mathsf{if}\,x\,\mathsf{then}\,(\mathsf{if}\,y\,\mathsf{then}\,\mathsf{U}_{11}p_2\,\mathsf{else}\,,\mathsf{U}_{10}p_2) \\ &\qquad\qquad \mathsf{else}\,(\mathsf{if}\,y\,\mathsf{then}\,\mathsf{U}_{01}p_2\,\mathsf{else}\,,\mathsf{U}_{00}p_2) \end{aligned}$$

$$[\alpha|0\rangle+\beta|1\rangle,\ \mathrm{let}\langle p,p'\rangle=C_1 * \ \mathrm{in\,let}\ f=C_2 p\ \mathrm{in\,let}\ g=Up'\ \mathrm{in}\ g(fp_0)]$$

$$\longrightarrow_1\quad [\alpha|0\rangle+\beta|1\rangle,\ \mathrm{let}\langle p,p'\rangle=\mathrm{CNOT}\,\langle H(new\,0),new\,0\rangle\ \mathrm{in}\ M_{p,p'}]$$

$$\longrightarrow_1\quad [(\alpha|0\rangle+\beta|1\rangle)\otimes|0\rangle,\ \mathrm{let}\,\langle p,p'\rangle=\mathrm{CNOT}\,\langle Hp1,new\,0\rangle\ \mathrm{in}\ M_{p,p'}]$$

$$\longrightarrow_1\quad [(\alpha|0\rangle+\beta|1\rangle)\otimes\tfrac{1}{\sqrt{2}}(|0\rangle+|1\rangle),\ \mathrm{let}\,\langle p,p'\rangle=\mathrm{CNOT}\,\langle p1,new\,0\rangle\ \mathrm{in}\ M_{p,p'}]$$

$$\longrightarrow_1\quad [(\alpha|0\rangle+\beta|1\rangle)\otimes\tfrac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes|0\rangle,\ \mathrm{let}\,\langle p,p'\rangle=\mathrm{CNOT}\,\langle p1,p2\rangle\ \mathrm{in}\ M_{p,p'}]$$

$$\longrightarrow_1\quad [(\alpha|0\rangle+\beta|1\rangle)\otimes\tfrac{1}{\sqrt{2}}(|00\rangle+|11\rangle),\ \mathrm{let}\,\langle p,p'\rangle=\langle p1,p2\rangle\ \mathrm{in}\ M_{p,p'}]$$

$$\longrightarrow_1\quad [(\alpha|0\rangle+\beta|1\rangle)\otimes\tfrac{1}{\sqrt{2}}(|00\rangle+|11\rangle),\ \mathrm{let}\ f=C_2\,p_1\ \mathrm{in\,let}\ g=U\,p2\ \mathrm{in}\ g(fp_0)]$$

$$\longrightarrow_1^*\quad [(\alpha|0\rangle+\beta|1\rangle)\otimes\tfrac{1}{\sqrt{2}}(|00\rangle+|11\rangle),\ U_{p_2}(Bp_1,p_0)]$$

$$\longrightarrow_1\quad [(\alpha|0\rangle+\beta|1\rangle)\otimes\tfrac{1}{\sqrt{2}}(|00\rangle+|11\rangle),\ U_{p_2}(\mathrm{let}\,\langle p,p'\rangle=\mathrm{CNOT}\langle p_0,p_1\rangle\ \mathrm{in}\ \langle ms(Hp),ms\,p'\rangle)]$$

$$\longrightarrow_1\quad [\tfrac{1}{\sqrt{2}}(\alpha|000\rangle+\alpha|011\rangle+\beta|110\rangle+\beta|101\rangle),\ U_{p_2}(\mathrm{let}\,\langle p,p'\rangle=\langle p_0,p_1\rangle\ \mathrm{in}\ \langle ms(Hp),ms\,p'\rangle)]$$

$$\longrightarrow_1\quad [\tfrac{1}{\sqrt{2}}(\alpha|000\rangle+\alpha|011\rangle+\beta|110\rangle+\beta|101\rangle),\ U_{p_2}\langle ms(Hp_0),ms\,p_1\rangle]$$

$$\longrightarrow_1\quad [\tfrac{1}{2}(\alpha|000\rangle+\alpha|011\rangle+\alpha|100\rangle+\alpha|111\rangle+\beta|010\rangle+\beta|001\rangle+\beta|110\rangle+\beta|101\rangle),\ U_{p_2}\langle msp_0,ms\,p_1\rangle]$$

$$\begin{cases}\longrightarrow_{\frac{1}{2}} & [\frac{1}{\sqrt{2}}(\alpha|000\rangle+\alpha|011\rangle+\beta|010\rangle+\beta|001\rangle,U_{p_2}\langle 0,msp_1\rangle]\\ \longrightarrow_{\frac{1}{2}} & [\frac{1}{\sqrt{2}}(\alpha|100\rangle+\alpha|111\rangle+\beta|110\rangle+\beta|101\rangle,U_{p_2}\langle 1,msp_1\rangle]\end{cases}$$

$$\begin{cases}\longrightarrow_{\frac{1}{2}} & [(\alpha|000\rangle+\beta|001\rangle\,U_{p_2}\langle 0,0\rangle] & \longrightarrow_1^* & [(\alpha|000\rangle+\beta|001\rangle\,U_{00}p_2]\\ \longrightarrow_{\frac{1}{2}} & [(\alpha|011\rangle+\beta|010\rangle\,U_{p_2}\langle 0,1\rangle] & \longrightarrow_1^* & [(\alpha|011\rangle+\beta|010\rangle\,U_{01}p_2]\\ \longrightarrow_{\frac{1}{2}} & [(\alpha|100\rangle+\beta|101\rangle\,U_{p_2}\langle 1,0\rangle] & \longrightarrow_1^* & [(\alpha|100\rangle+\beta|101\rangle\,U_{10}p_2]\\ \longrightarrow_{\frac{1}{2}} & [(\alpha|111\rangle+\beta|110\rangle\,U_{p_2}\langle 1,1\rangle] & \longrightarrow_1^* & [(\alpha|111\rangle+\beta|110\rangle\,U_{11}p_2]\end{cases}$$

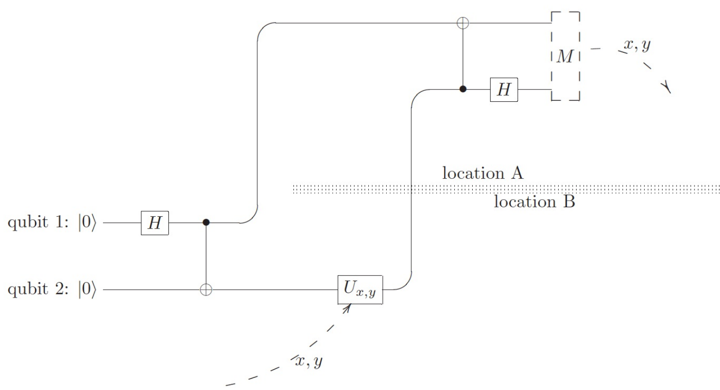
$$\begin{cases}\longrightarrow_1 & [(\alpha|000\rangle+\beta|001\rangle\,p_2] & = & [|00\rangle\otimes(\alpha|0\rangle+\beta|1\rangle\,p_2]\\ \longrightarrow_1 & [(\alpha|010\rangle+\beta|011\rangle\,p_2] & = & [|01\rangle\otimes(\alpha|0\rangle+\beta|1\rangle\,p_2]\\ \longrightarrow_1 & [(\alpha|100\rangle+\beta|101\rangle\,p_2] & = & [|10\rangle\otimes(\alpha|0\rangle+\beta|1\rangle\,p_2]\\ \longrightarrow_1 & [(\alpha|110\rangle+\beta|111\rangle\,p_2] & = & [|11\rangle\otimes(\alpha|0\rangle+\beta|1\rangle\,p_2]\end{cases}$$

Exercise 5

Justify each step of the reduction above.

Exercise 6

Consider now the dense coding protocol depicted below:



Reduce the following quantum closure

$$[|\ \rangle, \mathsf{let}\langle \mathsf{p}, \mathsf{p}' \rangle = \mathsf{C}_1 * \ \mathsf{in}\,\mathsf{let}\,\mathsf{f} = \mathsf{C}_2\mathsf{p}\,\mathsf{in}\,\mathsf{let}\,\mathsf{g} = \mathsf{U}\mathsf{p}'\,\mathsf{in}\,\mathsf{f}(\mathsf{g}\langle 0, 1\rangle)]$$

Exercise 7

Reference [2] extends the calculus with

- a term for recursive function definition;
- the possibility to accommodate infinite data types in the language.

Read the paper and discuss typing and reduction for these new terms. Give examples.

References

[1] Peter Selinger and Benoît Valiron. A lambda calculus for quantum computation with classical control. *Mathematical Structures in Computer Science*, 16(3):527–552, 2006.

[2] Peter Selinger and Benoît Valiron. Quantum lambda calculus. In Simon Gay and IanEditors Mackie, editors, *Semantic Techniques in Quantum Computation*, pages 135–172. Cambridge University Press, 2009.