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A hands-on introduction to categories

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Abstract

This document contains the summaries of an introductory module to Category Theory (20h). The module is part of the Quantum Logic curricular unit included in the syllabus of 5th year of MIEFis (MSc on Engineering Physics), offered at the University of Minho. Each summary includes a number of exercises intended as a study roadmap for the course. The course adopts a problem-solving approach, trying to build up the correct intuitions, and emphasising a number of core concepts, and a selection of results and theirs proofs, rather than attempting to provide a broader, all-inclusive account of the theory.

Lecture 1: Categories

Summary.

(1) The concept of a category: motivation and formal definition.

(2) Examples. The category of sets and functions as a prototypical category for classical computation. The category of sets and binary relations as a step towards quantum physics and quantum computation. Algebraic structures as categories and categories of algebraic structures.

(3) Special arrows in a category: (purely categorical) definitions of isomorphism, monomorphism and epimorphism. Sub-objects.

(4) New categories from old: product of categories, arrow and slice categories. Subcategories.

Opening.

Roughly speaking, categories deal with *arrows* and their composition in the same sense that sets deal with *elements*, their aggregation and membership. An *arrow* is an abstraction of the familiar notion of a function in set theory or of a homomorphism in algebra. Depicted as $f : X \longrightarrow Y$, it may be thought of as a transformation, or, simply, a connection, between two *objects* X and Y, called its *source* (or domain) and *target* (or codomain), respectively. The sources and targets of all the arrows in a category, form the class of its *objects*. If the same object is both the target of an arrow f and the source of another arrow g, f and g are said to be composable. Arrow composition is thus a partial operation and what the axioms for a category say is that arrows and arrow composition form a sort of generalised monoid.

Category theory provides a very general language to describe, organise and understand many different mathematical structures. It does so by neglecting what objects are, and focusing entirely on how they relate to each other. This includes the usual algebraic structures (e.g. preorders, groups or rings), but also data and algorithms in Computer Science, assertions and implications in Logic, systems and processes in Physics. Actually, as Colin McLarty put it [10]:

> *The spread of applications led to a general theory, and what had been a tool for handling structures became more and more a means of defining them. (...) In the 1960s, Lawvere began to give purely categorical descriptions of new and old structures, and developed several styles of categorical foundations for mathematics. This lead to new applications, notably in logic and computer science.*

Categories find striking application not only in Mathematics, but also in Computer Science, providing a semantic framework for programming concepts like parametrization, abstraction and compositionality, in Logic, with a syntax-independent version of the underlying structures, and even in Physics, as witnessed by numerous application to quantum information and quantum-based computational processes.

There are a number of good introductions to Category Theory. Reference [3] would be my choice for a comprehensive, accessible textbook. Abramsky and Tzevelekos introductory chapter [1] in the *New Structures for Physics* sort of handbook [7], is also an excellent starting point. Both can be complemented by more extensive introductions such as [2] or [11], both excellent, even if more oriented to a ‘traditional’ Mathematics graduate student. Shorter, sharp introductions are offered in [10], which includes a smooth introduction to toposes, and T. Leinster’s recent book [9]. To read on the beach, as pleasant motivations for further study, references [8] and [6] should be mentioned.

Exercise 1

Show that partially ordered sets and monotone functions form a category Pos. Generalise your argument to show that other algebraic structures, and the corresponding homomorphisms, also give rise to categories. Illustrate with the case of monoids and groups.

Exercise 2

Verify that vector spaces over a field K, say the complex numbers, and linear maps, form a category Vect_K.

Exercise 3

A preorder $\langle P, \leq \rangle$ may itself be regarded as a category[1] whose objects are the elements of P and there exists a unique arrow $p \to q$ iff $p \leq q$. Explain why transitivity (respectively, reflexivity) of relation $\leq$ implies the existence of the required composite (respectively, identity) arrow. Note that, on the other hand, a category with at most an arrow between any two objects determines a preorder on its objects with the relation $A \leq B$ iff there is an arrow from A to B.

Exercise 4

A category is *skeletal* if isomorphic objects are equal. Explain why a poset, but not a preorder, regarded as a category, is skeletal.

Exercise 5

Show that sets and binary relations $R : A \longrightarrow B$, defined as $R \subseteq A \times B$, can be organised into a category Rel, with relational composition

$$S \cdot R \mathrel{\widehat{=}} \{\langle a, c\rangle \in A \times C \mid \exists_{b \in B} \,.\, \langle a, b\rangle \in R \wedge \langle b, c\rangle \in S\}$$

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[1] Compare with the first exercise in this lecture. As noted in [1], *the ability to capture mathematics both ’in the large’ and ’in the small’ is a first indication of the flexibility and power of categories.*

for each pair of composable relations $R : A \longrightarrow B, S : B \longrightarrow C$, and, identities

$$\mathsf{id}_A \;\widehat{=}\; \{\langle a, a\rangle \mid a \in A\}$$

for each set A.

Exercise 6

Given a semiring $(S, +, \times)$, a category Mat_S of matrices over S, has non-negative integers as objects and $r \times c$ matrices as arrows $c \to r$ from (the number of columns) c to (the number of rows) r. Note that the way a matrix is typed provides its interface, syntactically governing the possibility of composition, and, obviously, does not bear any relationship to the structure of the matrix elements. Composition

$$\begin{tikzcd} c \arrow[r, "N"] \arrow[rr, bend right, "M \cdot N"'] & l \arrow[r, "M"] & r \end{tikzcd}$$

is matrix multiplication

$$i\,(M \cdot N)\,j \;\widehat{=}\; \sum_k (i\,M\,k) \times (k\,N\,j)$$

and diagonal matrices

$$i\,(id)\,j \;\widehat{=}\; (i = j)$$

serve as the identity arrows. Verify that Mat_S is a category indeed. Take a few minutes to explore the similarities between Mat_S and Rel. In particular, note that a binary relation can be regarded as a Boolean-valued matrix, and composition in Rel rephrased as

$$c\,(S \cdot R)\,a \;\widehat{=}\; \bigvee_k (c\,S\,b) \wedge (b\,R\,a)$$

i. e. as composition in $\mathsf{Mat}_{(2,\wedge,\vee)}$.

Exercise 7

The *dual category* C^{op} of a category C has the same objects as C and all the arrows of C reversed. Characterise C^{op} in detail, defining composition and identities in terms of the corresponding operations in C.

Exercise 8

Prove that Rel^{op} is self-dual, i.e. isomorphic to Rel. Explain why Rel is not obtained by dualizing Set.

Exercise 9

Given category C and one of its objects X, the *slice* category C/X has as objects all the arrows to X. An arrow between $f : A \longrightarrow X$ and $g : B \longrightarrow X$, is an arrow $h : A \longrightarrow B$ in C making the following diagram to commute:

$$\begin{tikzcd}
A \arrow[rr, "h"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\
& X &
\end{tikzcd}$$

Define identities and composition in C/X and show the axioms for a category hold. Show that if C_P is a poset $(P, \leq)$ regarded as a category, the slice C_P/x, for an element $x \in P$, is a principal ideal $x \downarrow = \{y \in P \mid y \leq x\}$.

Exercise 10

The so-called *co-slice* category X/C, for a category C and an object X of C, may be defined as

$$X/C \ \widehat{=}\ (C^{op}/X)^{op}$$

Unfold the definition in detail to arrive to a direct characterisation of X/C.

Exercise 11

If one takes the arrows of a category C and uses them as objects of a new category, the result is called an *arrow category* and denoted by $C^{\rightarrow}$. An arrow in $C^{\rightarrow}$ from f to g, is a pair of arrows (h_1, h_2) in C such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "h_1"] \arrow[d, "f"'] & C \arrow[d, "g"] \\
B \arrow[r, "h_2"] & D
\end{tikzcd}$$

Define identities and composition in $C^{\rightarrow}$ and show the axioms for a category hold.

Exercise 12

Recall that an arrow $f : A \longrightarrow B$ is an *isomorphism* if it has an inverse, i.e. an arrow $h : B \longrightarrow A$ such that

$$h \cdot f = id_A \quad \text{and} \quad f \cdot h = id_B$$

Show that inverses are unique. In Set isomorphisms correspond exactly to bijective functions, but this fact does not necessarily "scale" to categories whose objects carry extra structure. Prove, by building a suitable counter-example, that in Pos a monotone bijection may not be an isomorphism.

Exercise 13

Show that any group can be regarded as a category with only one object and whose arrows are isomorphisms. Conversely, show that a category of this type always determines a group.

Exercise 14

An arrow $h : A \longrightarrow B$ is *monic* (a monomorphism) if for all $f, g : C \longrightarrow A$ as in

$$\begin{tikzcd} C \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \arrow[r, "h"] & B \end{tikzcd}$$

the following holds

$$h \cdot f = h \cdot g \;\Rightarrow\; f = g$$

Dually, an arrow is *epic* (or a epimorphism) in a category C if it is monic in C^{op}. Unfold this statement to arrive to a direct definition of an epimorphism. Show that in Set monic (respectively, epic) arrows correspond to injective (respectively, surjective) functions.

Exercise 15

Show that the composition of monomorphisms is a monomorphism and that, if a composition $g \cdot f$ is monic, so is f. Formulate the dual results for epimorphims. Do you need to prove them?

Exercise 16

Show that, in any category, an isomorphism is always a monic and epic arrow. The converse, however, is not true in general. Show that the inclusion of integers into the set of rational numbers, although being monic and epic in the category of rings, fails to be an isomorphism.

Exercise 17

Consider arrows

$$\begin{tikzcd} A \arrow[r, bend right, "s"'] & B \arrow[l, bend right, "r"'] \end{tikzcd}$$

such that $r \cdot s = id_A$. Arrow s is called a *section*, or right inverse to r, whereas r is a *retraction*, or left inverse to s. Show that s is always monic and r epic. Note that, to witness these one-sided inverses, s (respectively, r) is said to be a *split* monomorphism (respectively, epimorphism). Show that in Set every epimorphism is a split one, and every monomorphism, but for the inclusion of the empty set in any other set, is a split monomorphism.

Exercise 18

A category S is a subcategory of another category C if it is defined by restricting to a sub-collection of objects and sub-collection of arrows such that the domain and codomain of any such arrow is in S, and S is closed for identities and composition. Show that the category of finite sets and bijections is a subcategory of Set.

Exercise 19

Show that in any category C monomorphisms (respectively, epimorphisms) define a subcategory of C.

Exercise 20

Show that isomorphisms in Rel are the graphs of bijections: a relation $S : X \longrightarrow Y$ is an isomorphism if there is some bijection $h : X \longrightarrow Y$ such that ySx iff $h(x) = y$.

Exercise 21

One way of thinking of an arrow $x : Z \longrightarrow X$ is as an 'element' of X, which is not given once and for all, but depends on Z. Such x is often called a *generalised element* of X, and Z its *stage of definition*. This suggests the alternative, set-inspired notation $x \in_Z X$ for arrow $x : Z \longrightarrow X$. The composite $f \cdot x$, for $f : X \longrightarrow Y$, can thus be written as $f(x)$. A special kind of elements of an object X consists of arrows into X whose source is the final object $\mathbf{1}$ (see Lecture 4) in the category (if it exists). These are called *points* (or *global elements*) of X.

In some categories every arrow $f : X \longrightarrow Y$ is fully determined by its effect on the points of X. Should this be the case, the category is said to be *well pointed*. Again Set is a good example: being well pointed is just a categorical version of the well known fact that 'a set is determined by its elements'. Moreover, in Set the correspondence between elements $x \in X$ and points $\underline{x} : \mathbf{1} \longrightarrow X$ is made explicit by denoting function application $f \cdot \underline{x}$ by $f(x)$.

Rephrase the definition of monic and epic arrow in an arbitrary category in the language of generalised elements.

Lecture 2: Functors

Summary.
(1) Functors: motivation and formal definition.

(2) Examples of functors involving different categories. Forgetful and free functors.

(3) Contravariance. Examples: the covariant and contravariant powerset functor; Hom functors.

(4) Full and faithful functors. Isomorphism of categories. Properties preserved by functors.

Opening.

Intuitively, functors provide ways of moving from one mathematical universe to another, that is from one category to another. As John Baez put it *[in Mathematics] every sufficiently good analogy is yearning to become a functor* [4]. Looking at categories as algebraic structures themselves, functors are the corresponding homomorphisms.

The adjective *functorial* means that a construction on objects can be extended to a construction on arrows that preserves composition and identities.

Exercise 1

Let $\mathcal{P}$ stand for the (finite) powerset construction, such that $\mathcal{P}(A) = \{X \mid X \subseteq A\}$ and $\mathcal{P}(f)(X) = \{f(x) \mid x \in X\}$. Prove that $\mathcal{P}$ is an endofunctor in Set.

Exercise 2

Show that there is a functor $R : Set \longrightarrow Rel$ which is the identity on objects, and maps each function $f : A \longrightarrow B$ to its graph, i.e.

$$R(f) \mathrel{\widehat{=}} \{(x, f(x)) \in A \times B \mid x \in A\}$$

Exercise 3

What is a functor between preorders regarded as categories?

Exercise 4

What is the effect on arrows of a functor $D : C^{\rightarrow} \longrightarrow C$ mapping each object $f : A \longrightarrow B$ to A?

$\boxed{\textbf{Exercise } 5}$

Let C/X be the slice category over C induced by an object X. An arrow $k : X \longrightarrow Y$ induces a functor $F_k : C/X \longrightarrow C/Y$ such that

$$\begin{aligned} F_k(f : A \longrightarrow X) &\ \widehat{=}\ k \cdot f : A \longrightarrow Y \\ F_k(h : f \longrightarrow g) &\ \widehat{=}\ h : k \cdot f \longrightarrow k \cdot g \end{aligned}$$

The action on arrows can be illustrated as follows:

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] & \mapsto & A \arrow[rr, "h"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & X & & & & X \arrow[d, "k"'] & \\ & & & & & Y & \end{tikzcd}$$

Show that the axioms for a functor hold for F_k.

$\boxed{\textbf{Exercise } 6}$

Functor $D : C^{\rightarrow} \longrightarrow C$, discussed in a previous exercise, forgets part of the structure of the source category. A more 'radical' example of a forgetful functor is

$$U : C/X \longrightarrow Set \quad \text{such that} \quad U(f : A \longrightarrow X) = A \ \text{ and } \ U(h : f \longrightarrow g) = h$$

Consider, now, a functor

$$S : C/X \longrightarrow C^{\rightarrow} \quad \text{such that} \quad S(f : A \longrightarrow X) = A \ \text{ and } \ S(h : f \longrightarrow g) = (h, id_x)$$

Prove that U and S are indeed functors. Show that $D \cdot S = U$.

$\boxed{\textbf{Exercise } 7}$

Free functors are somehow dual to forgetful functors. For example, given a set X one can construct a vector space (over a given field K) with basis X. This construction is canonical in the sense that it is defined without making any arbitrary choices[2]. Actually, the free vector space is the set of all formal K-linear combinations of elements of X, i.e. expressions

$$\sum_{x \in X} \alpha_x \, x$$

where α_x is a scalar in K such that $\alpha_x \neq 0$ for only finitely many values of x. Verify that this defines indeed a vector space, and note how it was obtained from the set X without imposing any equations other than those required by the definition of a vector space. Take the correspondence from X to the respective

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[2] Such is the sense the word *canonical* has in Category Theory: a construction given by a deity...

free vector space as the action on objects of a functor $F : \mathcal{S}et \longrightarrow \mathsf{Vect}_K$. Define the action on arrows and show that the functoriality axioms hold.

Exercise 8

A *contravariant* functor $F : C \longrightarrow D$ is a functor $F : C^{op} \longrightarrow D$. Note that, making the data explicit, an arrow $f : A \longrightarrow B$ in C is mapped to an arrow $F(f) : F(B) \longrightarrow F(A)$ in D.

The contravariant power set functor $P : \mathcal{S}et^{op} \longrightarrow \mathcal{S}et$ sends each set A to its power set $\mathcal{P}A$ and each function $f : A \longrightarrow B$ to its inverse image function $f^{-1} : \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$ which maps each $X \subseteq B$ into $f^{-1}(X) \subseteq A$. Verify it is indeed a functor.

Exercise 9

Show that any functor *preserves* isomorphisms, but not necessarily *reflects* them. For the second part, look for a counterexample, i.e. a functor F and an arrow f such that $F(f)$, but not f, is an isomorphism. What can you say about monic and epic arrows, and their split versions?

Exercise 10

Functors can be thought as homomorphisms between categories, i.e. as arrows in $\mathcal{C}at$ whose objects are small categories (recall that a category is small if its collection of arrows is a set), and also in CAT whose objects are locally small categories (all homsets are sets[3]). In this setting, a *isomorphism of categories* is just the usual notion of an isomorphims in $\mathcal{C}at$ or CAT.

Show that the category $\mathcal{M}at_S$ is isomorphic to $\mathcal{M}at_S^{op}$ via a functor which is the identity on objects, and carries a matrix to its transpose.

Exercise 11

In computing, partial operators are often characterised in the context of the category $\mathcal{S}et_\perp$ of pointed sets. A pointed set X is just a set with a distinguished element $\perp_X$, which are preserved by arrows in $\mathcal{S}et_\perp$. I. e. a function $f : X \longrightarrow Y$ in $\mathcal{S}et_\perp$ satisfies $f(\perp_X) = \perp_Y$. Show that $\mathcal{S}et_\perp$ is isomorphic to $\mathbf{1}/\mathcal{S}et$.

Exercise 12

Let G be a group, regarded as a category. Characterise G^{op} and prove G is isomorphic to G^{op}.

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[3] Note that CAT is not locally small and therefore does not belong to itself, which would contradict Russel's paradox.

Exercise 13

Functors may be classified in terms of the correspondences they induce between homsets. In particular, a functor $\mathsf{F} : \mathsf{C} \longrightarrow \mathsf{D}$ is *faithful* (respectively, *full*) if the map $\mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{D}}(F(X), F(Y))$ is injective (respectively, *surjective*). An *embedding* is a faithful functor which is, additionally, injective on morphisms. Show that full and faithful functors *reflect* and *create* isomorphisms, i.e. if $F(f)$ is an isomorphism so is f; and if every isomorphism in the image of F on objects is the image of an isomorphism in C.

Exercise 14

A subcategory S of a category C is *full* if $\mathrm{Hom}_{\mathsf{S}}(X, Y) = \mathrm{Hom}_{\mathsf{C}}(X, Y)$ for all objects X and Y of S. Show that the inclusion functor $\mathrm{I} : \mathsf{S} \longrightarrow \mathsf{C}$ defined as the identity on objects and arrows of S is always faithful, but is full only when S is a full subcategory.

Lecture 3: Universal Properties

Summary.
(1) Universal properties: concept, examples and ubiquity.

(2) Initial and final objects in a category.

(3) Universal characterisation of Cartesian product in Set. The categorial product construction.

(4) Universal properties 'come in pairs': the coproduct construction. Properties of products and coproducts.

Opening.

If there is a 'main topic' in category theory, this is certainly the study of *universal* properties. Roughly speaking, an entity ϵ is universal among a family of 'similar' entities if it is the case that every other entity in the family can be *reduced* or *traced back* to ϵ. For example, an object T is said to be *final* in a category C if, from every other object X in C, there exists a unique arrow $!_X$ to T. Therefore, there is a canonical, *unique* way to relate every object in C to T — *finality* is thus an universal property.

A nice thing about universal properties is the fact they always 'come in pairs': the *dual* of an universal is still an universal. Dualizing finality, we arrive at *initiality*: an object is *initial* in C if there is one and only one arrow in C from it to any other object in the category.

Universal properties, like finality or initiality, can be recognised, usually under a different terminology, in many branches of Mathematics. Moreover, they happen to play a major role in the structure of 'mathematical spaces'. Therefore, category theory provides a setting for studying abstractly such 'spaces' and their relationships.

Let us consider an illustrative example (adapted from [9]). The study of bilinear (i.e. linear in both arguments) maps out of two vector spaces U and V can be reduced to the study of linear maps because there is a *universal* bilinear map $\epsilon : U \times V \longrightarrow T$ through which all the others factor, i.e. for all $f : U \times V \longrightarrow X$, there exists one and only one linear map $\overline{f} : T \longrightarrow X$ such that $f = \overline{f} \cdot \epsilon$. Look for a moment how uniqueness is proved. Suppose both ϵ and $\epsilon' : U \times V \longrightarrow T'$ satisfy the universal property above. Thus, we obtain linear maps $\overline{\epsilon}$ and $\overline{\epsilon'}$, such that

$$\epsilon' = \overline{\epsilon'} \cdot \epsilon \quad \text{and} \quad \epsilon = \overline{\epsilon} \cdot \epsilon'$$

because, respectively, ϵ and ϵ' are universal by assumption. Clearly, $\overline{\epsilon} \cdot \overline{\epsilon'} \cdot \epsilon = \overline{\epsilon} \cdot \epsilon' = \epsilon$ as depicted in the following diagram:

$$\begin{tikzcd}
& U \times W \arrow[dl, "\epsilon"'] \arrow[d, "\epsilon'"] \arrow[dr, "\epsilon"] & \\
T \arrow[r, "\overline{\epsilon}"] & T' \arrow[r, "\overline{\epsilon'}"] & T
\end{tikzcd}$$

However, $\mathsf{id}_T \cdot \epsilon = \epsilon$, which entails $\overline{\epsilon} \cdot \overline{\epsilon'} = \mathsf{id}_T$ by the uniqueness of ϵ. A similar argument, relying on the universality of ϵ', yields $\overline{\epsilon'} \cdot \overline{\epsilon} = \mathsf{id}_{T'}$. Thus, $\overline{\epsilon}$ is an isomorphism witnessing $T \cong T'$.

Vector space T is the *tensor* product of U and V, often written as $U \otimes V$; and what the universal property tells is that it is essentially unique. The way it is constructed is, to a large extent, irrelevant: the universal property is enough.

Exercise 1

Characterise the initial and final objects in a preorder regarded as a category. Give an example of a preorder in which such objects do not exist.

Exercise 2

Show that any singleton set is both initial and final in $\mathsf{Set}_\perp$ (and, therefore, called a *zero* object). Can you think of another familiar category with a zero object?

Exercise 3

Let Rng be the category of rings and consider $Z = \langle \mathbb{Z}, +, 0, -, \cdot, 1 \rangle$ the ring of integer numbers. Show that there is a unique ring homomorphism h from Z to any other ring $\langle S, +', 0', -', \cdot', 1' \rangle$ given by

$$h(n) \mathrel{\widehat{=}} \begin{cases} 0' & \Leftarrow \; n = 0 \\ -'h(-n) & \Leftarrow \; n < 0 \\ \underbrace{1' +' 1' +' \cdots +' 1'}_{n} & \Leftarrow \; n > 0 \end{cases}$$

Exercise 4

Based on the previous exercise, conclude that Z is the initial object in Rng, showing that any other ring satisfying the universal property is isomorphic to Z. Appreciate that for the proof it does not matter ... what a ring is (just as, in the example discussed in the introduction to this Lecture, the meaning of bilinear map or vector space is indeed irrelevant to establish the uniqueness of the tensor product).

Exercise 5

Show that any map from a final object in a category to an initial one is an isomorphism.

Exercise 6

Coalgebras are a generic way represent transition systems, as the ones studied earlier in this course. Formally, a *coalgebra* for a functor $F : C \longrightarrow C$, thought of as the type of the allowed transitions, is an object U, called its carrier, or state space, and an arrow $c : U \longrightarrow T(U)$ of C. A morphism between coalgebras c and c' is an arrow $h : U \longrightarrow V$ in C making the following diagram comute:

$$\begin{tikzcd}
U \arrow[r, "h"] \arrow[d, "c"'] & V \arrow[d, "c'"] \\
F(U) \arrow[r, "F(h)"'] & F(V)
\end{tikzcd}$$

1. Instantiate the definition for $C = Set$ and $F(X) = \mathcal{P}(L \times X)$, where $\mathcal{P}$ is the finite powerfunctor and L an arbitrary set (of labels, say). What sort of transition systems correspond to this type of coalgebras?
2. Show that coalgebras and their morphisms form a category.
3. Prove that, if coalgebra $(W, \omega : W \longrightarrow F(W))$ is final in the category of F-coalgebras, ω is an isomorphism.

Exercise 7

Dualise the definition of a coalgebra given above to arrive to the dual concept of a F-*algebra*, $(A, a : F(A) \longrightarrow A)$. Show that an initial algebra in the corresponding category is also an isomorphism — notice the proof strucuture is exactly the same used in the last question of the previous exercise.

Exercise 8

Characterise product and coproduct in a poset regarded as a category. Do the same for the category Pos whose objects are posets and arrows are monotone functions.

Exercise 9

Resorting to the corresponding universal property, show that the product (respectively, coproduct) construction in a category is functorial. Show, in particular that, given two arrows $f : A \longrightarrow B$ and $g : C \longrightarrow D$, $f \times g : A \times C \longrightarrow B \times D = \langle f \cdot \pi_1, g \cdot \pi_2 \rangle$. What about $f + g$?

Exercise 10

Derive, from the universal property of products, the equality $\langle f, g \rangle \cdot h = \langle f \cdot h, g \cdot h \rangle$, for f, g and h

suitably typed, and $\langle \mathsf{id}_A, \mathsf{id}_B \rangle = \mathsf{id}_{A \times B}$. These results are known in classical program calculi [5], as the product *fusion* and *reflection* laws, respectively.

Exercise 11

A coproduct in Rel is given by disjoint union, with the universal arrow in the diagram below defined as

$$[R, S] \ \widehat{=} \ \ R \cdot \iota_1^\circ \cup S \cdot \iota_1^\circ$$

$$\begin{tikzcd}
A \arrow[r, "\iota_1"] \arrow[dr, "R"'] & A + B \arrow[d, "{[R,S]}"] & B \arrow[l, "\iota_2"'] \arrow[dl, "S"] \\
& C &
\end{tikzcd}$$

Define product in Rel by dualising this construction. Recall that Rel is a self-dual category.

Exercise 12

The product of two vector spaces U, V over a field K, in Vect_K usually represented as $U \oplus V$, is given by $U \times V = \{(u, v) \mid a \in U, v \in V\}$ made into a vector space by defining addition and scalar multiplication as follows:

$$(x, y) + (x', y') \ = \ (x + x', y + y') \ \text{ and } \ k\,(x, y) \ = \ (k\,x, k\,y)$$

Projections and the universal arrow are as in Set but required to be linear. Show such is the case indeed.

Exercise 13

The disjoint sum $U \oplus V$ of two vector spaces U, V over a field K, in Vect_K, is simultaneously their product (as discussed in the previous exercise) and coproduct. Define the embeddings $\iota_1 : U \longrightarrow U \oplus V$ and $\iota_2 : V \longrightarrow U \oplus V$ as

$$\iota_1(x) \ = \ (x, 0_V) \ \text{ and } \ \iota_2(y) \ = \ (0_U, y)$$

where $0_U, 0_V$ are the additive identities in U and V, respectively. For $f : U \longrightarrow Z$ and $g : V \longrightarrow Z$, define the universal arrow $[f, g] : U \oplus V \longrightarrow Z$ by

$$[f, g](x, y) \ = \ f(x) + g(y)$$

and prove that the relevant arrows are linear and this construction defines indeed a coproduct.

Lecture 4: Limits and Colimits

Summary.
(1) Equalisers and pullbacks: definition by a universal property. Examples. The dual constructions: coequalisers and pushouts. Examples.

(3) Going general: revisiting products, equalisers and pullbacks as final objects in certain categories of diagrams. Very brief introduction to limits and colimits.

Opening.

Products and final objects studied in the previous lecture, are examples of *limits*. Actually, many familiar constructions in mathematics are instances of limits: from the Cartesian product of two sets, to the inverse image of a function, from the disjoint sum of vector spaces to the greatest lower bound in a poset. Most examples of universal properties found so far in this course define a limit or a *colimit*, the dual concept. Informally, and taking advantage of an analogy with what happens in Set, limits can be thought of as subsets of products, and colimits as quotients of sums.

In this lecture, limits are systematised as universal cones over diagrams in a category. Roughly speaking, a cone is an object X and a family of arrows from it to every vertex in the diagram; as usual, it is *universal* if each other cone factorizes uniquely through it. Going to the dual category, i. e. reversing arrows, we arrive at the dual notions of cocone and colimit. The following table sums up the most usual limit and colimit objects in a category.

Diagram	Limit	Colimit
$\circ \qquad \circ$	product	coproduct
$\circ \rightrightarrows \circ$	equaliser	coequaliser
$\begin{tikzcd} & \circ \arrow[d] \\ \circ \arrow[r] & \circ \end{tikzcd}$	pullback	
$\begin{tikzcd} \circ \arrow[r] \arrow[d] & \circ \\ \circ & \end{tikzcd}$		pushout

A category is called *complete* (respectively, *cocomplete*) if any diagram indexed by a small category has both a limit (respectively, a colimit).

Exercise 1

Build the equaliser in Set of functions $p, q : \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined as $p(x, y, z) = x^2 + y^2 + z^2$ and $q(x, y, z) = 1$. What is the geometric interpretation?

$\boxed{\textbf{Exercise } 2}$

Show, by a suitable counterexample, that not all pairs of relations have equalisers in Rel.

$\boxed{\textbf{Exercise } 3}$

The following are two popular examples of a pullback in Set. The diagram in the left, where i, j are set inclusions and $f|Z$ denotes the restriction of $f : A \longrightarrow B$ to set Z, defines the inverse image of a function, i. e. $f^{-1}(Z) = \{a \in A \mid f(a) \in Z\}$. The other diagram, in which all arrows are inclusions, captures set intersection (actually, the intersection of two inclusions). Verify they both form pullbacks indeed and explain the constructions in your own words. Show, in particular, that the second diagram is an instance of the first (cf, $A \cap B = \{a \in A \mid a \in B\}$).

$$\begin{tikzcd}
f^{-1}(Z) \arrow[r, "f|Z"] \arrow[d, "j"'] & Z \arrow[d, "i"] \\
A \arrow[r, "f"] & B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \cap B \arrow[r] \arrow[d] & B \arrow[d] \\
A \arrow[r] & C
\end{tikzcd}$$

$\boxed{\textbf{Exercise } 4}$

Show that the pullback of two arrows in C to an object X is a product in the slice category C/X.

$\boxed{\textbf{Exercise } 5}$

What is the limit and the colimit of the empty diagram?

$\boxed{\textbf{Exercise } 6}$

In Set the coequaliser of two parallel functions $f, g : A \longrightarrow B$ can be built as the quotient $B/\equiv$ of B by the equivalence relation $\equiv$ generated by $R = \{(f(a), g(a)) \mid a \in A\}$. Figure out a concrete example and convince yourself this is indeed a coequaliser.

$\boxed{\textbf{Exercise } 7}$

In a category with initial object $\emptyset$, identify the pushout of diagram
$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & Y \\
Z &
\end{tikzcd}$$

Exercise 8

Compute in Set the pushout of the two inclusions $i : \{1\} \longrightarrow [0,6]$ and $j : \{1\} \longrightarrow [1,3]$.

Exercise 9

Show that every subset $S \subseteq X$ can be specified through the following equaliser diagram, where $\underline{\mathsf{true}}$ is the constant function always returning true, and ξ_S denotes the characteristic function of $S \subseteq X$ (i.e. $S = \{x \in X \mid \xi_S(x) = \mathsf{true}\}$),

$$\begin{tikzcd} S \arrow[r, "s"] & X \arrow[r, shift left, "\xi_S"] \arrow[r, shift right, "\underline{\mathsf{true}}\cdot !"'] & \mathbf{2} \end{tikzcd}$$

Observe that, on the other hand, any function $f : X \longrightarrow \mathbf{2}$ defines a subset of X given by $V_f = \{x \in X \mid f(x) = \mathsf{true}\}$, therefore establishing isomorphism $\mathsf{Hom}_{\mathsf{Set}}(X, \mathbf{2}) \cong \mathcal{P}(X)$.

Exercise 10

Prove that every equaliser is a monic arrow.

Exercise 11

Show that the diagram below is a pullback if the two inner squares are themselves pullback:

$$\begin{tikzcd} \circ \arrow[r] \arrow[d] & \circ \arrow[r] \arrow[d] & \circ \arrow[d] \\ \circ \arrow[r] & \circ \arrow[r] & \circ \end{tikzcd}$$

Exercise 12

The notion of a *span* can be defined in any category C with pullbacks. Formally, a span is a (X, f, g), *i.e.*, a pair of C-arrows, $f : X \longrightarrow A$ and $g : X \longrightarrow B$, with a common domain. Partial functions from a set A to a set B may be regarded as isomorphism classes of spans on A and B, where the first component (the domain map) is monic, i.e.

$$\begin{tikzcd} & U \arrow[dl, hook', "\text{dom}"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$

In general, spans generalise binary relations to an arbitrary category.

Spans compose by pullbacking: $(f, g) \mathbin{;} (f', g') = (f \cdot x, g' \cdot y)$, where (P, x, y) is an arbitrarily specified pullback of (g, f') as illustrated in the following diagram:

$$\begin{tikzcd}
 & & P \arrow[dl, "x"'] \arrow[dr, "y"] & & \\
 & X \arrow[dl, "f"'] \arrow[dr, "g"] & & X' \arrow[dl, "f'"'] \arrow[dr, "g'"] & \\
A & & B & & C
\end{tikzcd}$$

Note that, as pullbacks are defined up to isomorphism, and in the definition of ; they are arbitrarily chosen, associativity of composition also holds only up to isomorphism.

Show that, given a category C, a new category can be obtained over the same objects of C but taking spans as its arrows. The identity on object A is, of course, (id_A, id_A), where id_A is the identity on A in C.

Exercise 13

Another category may be defined taking spans as objects and morphisms between spans on the same objects as arrows. An arrow in this category is a C-arrow $k : X \longrightarrow Y$ making the following diagram to commute.

$$\begin{tikzcd}
 & X \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "k"] & \\
A & & B \\
 & Y \arrow[ul, "f'"] \arrow[ur, "g'"'] &
\end{tikzcd}$$

Show this forms a category. Discuss whether product $A \times B$ can be considered the final object in this category.

Exercise 14

Discuss the following observation: In a preorder regarded as a category, the limit (respectively, the colimit) of a diagram is the infimum (respectively, the supremum) of the involved objects.

Exercise 15

Show that the equaliser of an arrow $f : A \longrightarrow B$ with itself is the identity on A.

Exercise 16

Show that the square

$$\begin{tikzcd} A \arrow[r, "id"] \arrow[d, "id"'] & A \arrow[d, "m"] \\ A \arrow[r, "m"'] & B \end{tikzcd}$$

defines a pullback iff arrow $m : A \longrightarrow B$ is monic. Use this fact to show that $e : E \longrightarrow A$ being a equaliser of $f, g : A \longrightarrow B$ does not imply that diagram

$$\begin{tikzcd} E \arrow[r, "e"] \arrow[d, "e"'] & A \arrow[d, "f"] \\ A \arrow[r, "g"'] & B \end{tikzcd}$$

defines a pullback.

Exercise 17

It is well known that a category with pullbacks and a final object has all finite limits. Show how products can be constructed from the other two limits. And what about equalisers?

Exercise 18

Let $D : J \longrightarrow C$ be a diagram of type J in a category C. Recall that a cone to D is an object X in C and a family of arrows $x_j : X \longrightarrow D(j)$ from X for every node $D(j)$ in the image of the diagram in C, such that for every arc $e : i \longrightarrow j$ in J the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "x_i"'] \arrow[dr, "x_j"] & \\ D(i) \arrow[rr, "D(e)"] & & D(j) \end{tikzcd}$$

Given two cones, $(X, \{x_j \mid j \text{ an object of } J\})$, $(X', \{x'_j \mid j \text{ an object of } J\})$, define a morphism between them as an arrow $h : X \longrightarrow X'$ such that

$$\begin{tikzcd} & D(j) & \\ X \arrow[ur, "x_j"] \arrow[rr, "h"] & & X' \arrow[ul, "x'_j"'] \end{tikzcd}$$

Show that cones to a diagram D and their morphisms define a category. Explain why a limit for D is just the final object in that category.

$\boxed{\textbf{Exercise } 19}$

A subobject of an object X in a category C is a monomorphism $m : S \longrightarrow X$. Indeed, monic arrows can be thought as *generalised subsets*: recall that every subset of a set may be defined as an equaliser and that equalisers are always monomorphisms. A morphism between subobjects S_1 and S_2 of X is a an arrow in C making the following diagram to commute:

$$\begin{tikzcd} & X & \\ S_1 \arrow[ur, "m"] \arrow[rr, "h"] & & S_2 \arrow[ul, "n"'] \end{tikzcd}$$

Show that the class of subobjects of an object X in C and their morphisms form a preorder category. Observe that a morphism in this category is just an arrow in the slice category C/X.

$\boxed{\textbf{Exercise } 20}$

Show that in any category an equaliser is always monic. The converse does not hold in general, but it does in Set^4. Show that a monic function $m : S \longrightarrow X$ can be expressed as the following equaliser:

$$\begin{tikzcd} S \arrow[r, "m"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "i"'] & B_\perp \end{tikzcd}$$

where i is the inclusion of B in $B_\perp = B \cup \{\perp\}$ and

$$f(b) \mathrel{\widehat{=}} \begin{cases} b & \Leftarrow \text{ b is in the image of } m \\ \perp & \Leftarrow \text{ otherwise} \end{cases}$$

---

[4] This is in fact a special case of a more general result: in any topos monomorphisms and equalizers coincide.

Lecture 5: Naturality

Summary.
(1) Natural transformation: motivation and formal definition. Naturality as a source of genericity. Examples: parametric operators in programming.

(2) Vertical and horizontal composition of natural transformations. Functor categories.

(3) Natural isomorphisms and equivalence of categories.

(4) Small case study: revisiting Hom-functors and natural transformation between them. Brief mention to the Yoneda lemma.

Opening.

If functors are arrows between categories, natural transformations can be regarded as arrows between functors. Historically, the concept seems to predate those of a functor or a category, to describe structural transformations which are canonical in the sense of being built without resorting to any sort of arbitrary choices. As T. Leinster puts it [9]:

> *In fact, it was the desire to formalize the notion of natural transformation that led to the birth of category theory. By the early 1940s, researchers in algebraic topology had started to use the phrase 'natural transformation', but only in an informal way. Two mathematicians, Samuel Eilenberg and Saunders Mac Lane, saw that a precise definition was needed. But before they could define natural transformation, they had to define functor; and before they could define functor, they had to define category. And so the subject was born..*

In programming natural transformations model parametric operators, i. e. operations defined in a way which does not depend on the argument basic types but only on the 'shape' used to organise them. For example, function $\mathsf{elems} : X^* \longrightarrow \mathcal{P}(X)$ which maps a sequence into the set of its elements does not depend on what X actually is. In the language of categories, elems can be regarded as a family of arrows $(\mathsf{elems}_A)_{A \,\mathrm{in}\, \mathsf{Set}}$ such that $\mathsf{elems}_A : A^* \longrightarrow \mathcal{P}(A)$ is uniformly defined, i.e. the following diagram commutes for any set A and function $f : A \longrightarrow B$:

$$\begin{tikzcd}
A^* \arrow[r, "\mathsf{elems}_A"] \arrow[d, "f^*"'] & \mathcal{P}(A) \arrow[d, "\mathcal{P}(f)"] \\
B^* \arrow[r, "\mathsf{elems}_B"] & \mathcal{P}(B)
\end{tikzcd}$$

The corresponding equation — $\mathcal{P}(f) \cdot \mathsf{elems}_A = \mathsf{elems}_B \cdot f^*$ — is the *naturality* condition.

Exercise 1

Recall from the Functional Programming course, functions elems, mentioned above, and $\mathsf{merge} : X^* \times X^* \longrightarrow X^*$ which merges two sequences. Prove their naturality.

Exercise 2

Show that $\eta : \mathrm{Id}_{\mathsf{Set}} \Longrightarrow \mathcal{P}$ mapping x to $\{x\}$ is a natural transformation. Draw the corresponding diagram.

Exercise 3

Let Mon be the category of monoids and monoid homomorphisms, and consider a functor $F : \mathsf{Set} \longrightarrow \mathsf{Mon}$ which builds a monoid freely from a set S, i. e.

$$S \;\mapsto\; (S^*, \frown, \epsilon)$$

where $\frown$ is word (sequence) concatenation and ϵ denotes the empty word. What is the action of F on functions?

Let $U : \mathsf{Mon} \longrightarrow \mathsf{Set}$ be the forgetful functor (which, as the name says, 'forgets' the monoid structure). Show that $\zeta : FU \Longrightarrow \mathrm{Id}_{\mathsf{Mon}}$ and $\eta : \mathrm{Id}_{\mathsf{Set}} \Longrightarrow UF$ defined by

$$\begin{tikzcd}
FU(M,\times,1) = (M^*, \frown, \epsilon) \arrow[r, "\zeta_{(M,\times,1)}"] \arrow[d, "h^*"'] & (M,\times,1) \arrow[d, "h"] & S \arrow[r, "\eta_S"] \arrow[d, "f"'] & UF(S) = S^* \arrow[d, "f^*"] \\
FU(N,+,0) = (N^*, \frown, \epsilon) \arrow[r, "\zeta_{(N,+,0)}"] & (N,+,0) & T \arrow[r, "\eta_T"] & UF(T) = T^*
\end{tikzcd}$$

$$\zeta_{(M,\times,1)}(m_1 m_2 \cdots m_n) \;\widehat{=}\; m_1 \times m_2 \times \cdots \times m_n \qquad\qquad \eta_S(s) \;\widehat{=}\; s$$

are indeed natural transformations.

Exercise 4

Define an endofunctor $\Delta_{\otimes}$ in the category Vect_K of vector spaces over a field K mapping a vector space U to $U \otimes U$, for $\otimes$ denoting the tensor product. Show that there is a natural transformation $\zeta : \mathrm{Id}_{\mathsf{Vect}_K} \Longrightarrow \Delta_{\otimes}$ whose components map each vector to 0 (the additive identity).

Observe this is the only natural transformation that can be defined between the two functors: actually, there is no basis-independent way to define a linear map from U to $U \otimes U$[5]

---

[5] This observation, which also holds in the category of Hilbert spaces, the classical semantic universe for quantum computing, is related to what is called the no-cloning theorem in that setting.

Exercise 5

Let $\mathbf{2}$ be the discrete category with two objects. Observe that a functor from $\mathbf{2}$ to a category C is a pair of objects of C, and a natural transformation is a pair of maps. Show that the functor category $C^{\mathbf{2}}$ is therefore isomorphic to the product category $C \times C$.

Exercise 6

Functors between posets regarded as categories are monotone functions. Show that between two such functions f and g, between posets P and Q, there exists at most a natural transformation iff, for all $x \in P$, $f(x) \leq g(x)$.

Exercise 7

Let T, S and R be functors from category C to D, and consider the following natural transformations $\sigma : T \Longrightarrow S$ and $\sigma' : S \Longrightarrow R$. Then, σ and σ' can be composed originating $\sigma' \cdot \sigma : T \Longrightarrow R$, by defining

$$(\sigma' \cdot \sigma)_X \;=\; \sigma'_X \cdot \sigma_X$$

as illustrated in the diagram

$$\begin{tikzcd}
C \arrow[r, bend left=50, "T"{name=T}] \arrow[r, "S"'{name=S}] \arrow[r, bend right=50, "R"'{name=R}] & D
\arrow[Rightarrow, from=T, to=S, shorten=2pt, "\sigma"]
\arrow[Rightarrow, from=S, to=R, shorten=2pt, "\sigma'"]
\end{tikzcd}$$

This is known as the *vertical* composition of natural transformations.

Show that the *functor category*, D^C, of functors from C to D and natural transformations is a category indeed. Notice that, for each functor T in D^C, the family of identity arrows on $T(X)$ in D gives rise to a trivial natural transformation denoted by $\mathtt{id}_T$, which acts as an identity in D^C.

Exercise 8

There is also a notion of composition for natural transformations between pairs of composable functors. It will be denoted by by ; and used in diagrammatic order. Suppose T and S are functors from C to D and T' and S' are functors from D to E. If there exist natural transformations $\sigma : T \Longrightarrow S$ and $\sigma' : T' \Longrightarrow S'$, their *horizontal* composite is $\sigma\,;\sigma' : T'\,T \Longrightarrow S'\,S$ whose component at X is given by

$$(\sigma\,;\sigma')_X \;= S'\,\sigma_X \cdot \sigma'_{T\,X} \;\;=\; \sigma'_{S\,X} \cdot T'\,\sigma_X$$

as, by definition of σ and σ', the following diagram commutes:

$$\begin{tikzcd}
\mathsf{T'T(X)} \arrow[r, "\sigma'_{\mathsf{T(X)}}"] \arrow[d, "\mathsf{T'}(\sigma_{\mathsf{X}})"'] \arrow[dr, "(\sigma;\sigma')_{\mathsf{X}}"] & \mathsf{S'S(X)} \arrow[d, "\mathsf{S'}(\sigma_{\mathsf{X}})"] \\
\mathsf{T'S(X)} \arrow[r, "\sigma'_{\mathsf{S(X)}}"'] & \mathsf{S'S(X)}
\end{tikzcd}$$

Particular cases of this situation occur when σ or σ' are the identity $\mathtt{id}_{\mathsf{R}}$ on a functor R. We may, then, pre- or post-compose σ with $\mathtt{id}_{\mathsf{R}}$, yielding

$$\mathsf{R}\sigma \overset{\text{abv}}{=} \sigma\,;\mathtt{id}_{\mathsf{R}} : \mathsf{RT} \longrightarrow \mathsf{RS} \qquad \text{with} \quad (\mathsf{R}\sigma)_{\mathsf{X}} = \mathsf{R}(\sigma_{\mathsf{X}})$$

$$\sigma\mathsf{R} \overset{\text{abv}}{=} \mathtt{id}_{\mathsf{R}}\,;\sigma : \mathsf{TR} \longrightarrow \mathsf{SR} \qquad \text{with} \quad (\sigma\mathsf{R})_{\mathsf{X}} = \sigma_{\mathsf{R(X)}}$$

where $\mathsf{T}, \mathsf{S} : \mathsf{C} \longrightarrow \mathsf{D}$, $\sigma : \mathsf{T} \Longrightarrow \mathsf{S}$ and R is a functor from D to E, in the first case, and from B to C, in the second.

Show that the horizontal composition of two natural transformations still is a natural transformation.

Observe that vertical and horizontal composition of natural transformations interact via the *interchange law*:

$$(\sigma\,;\sigma') \cdot (\gamma\,;\gamma') \;=\; (\gamma \cdot \sigma)\,;(\sigma' \cdot \gamma')$$

which gives an unambiguous meaning to the diagram

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left=50, "\mathsf{T}"] \arrow[r, "\mathsf{S}"'] \arrow[r, bend right=50, "\mathsf{R}"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3, "\Downarrow\sigma"] & \mathsf{D} \arrow[r, bend left=50, "\mathsf{T'}"] \arrow[r, "\mathsf{S'}"'] \arrow[r, bend right=50, "\mathsf{R'}"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma'"] \arrow[r, phantom, shift right=3, "\Downarrow\sigma'"] & \mathsf{E}
\end{tikzcd}$$

This pattern often occurs in Computer Science, namely to relate temporal (parallel) and spatial (sequential) composition of a system's behaviour.

Exercise 9

Two categories C and D are *equivalent* if there exists functors $\mathsf{F} : \mathsf{D} \longrightarrow \mathsf{C}$ and $\mathsf{E} : \mathsf{C} \longrightarrow \mathsf{D}$, and natural isomorphisms $\sigma : \mathsf{Id}_{\mathsf{E}} \Longrightarrow \mathsf{FE}$ and $\gamma : \mathtt{id}_{\mathsf{D}} \Longrightarrow \mathsf{EF}$, called the *pseudo inverses*.

Compare with the notion of isomorphism of categories and explain why an equivalence of categories is often called an *isomorphism up to an isomorphism*.

Exercise 10

Show that the notion of equivalence of categories is in fact an equivalence relation.

Exercise 11

Consider the following two categories suitable to represent the universe of partial functions:

- Pfn is the category of partial functions: objects are sets; an arrow f from A to B is a pair $(\mathrm{dom}\, f \subseteq A, |f| : \mathrm{dom}\, f \longrightarrow B)$. The composition of two arrows f and g suitably typed is the pair $(\mathrm{dom}\,(g \cdot f) = f^{-1}(\mathrm{dom}\, g) \subseteq A, |g \cdot f| : \mathrm{dom}\,(g \cdot f) \longrightarrow C)$.
- $\mathsf{Set}_\perp$ is the category of pointed sets (A, a), with $a \in A$, whose arrows $f : (A, a) \longrightarrow (B, b)$ are functions such that $f(a) = b$. Composition and identities are as is Set.

Pfn and $\mathsf{Set}_\perp$ are related through the functors $S : \mathsf{Pfn} \longrightarrow \mathsf{Set}_\perp$ and $T : \mathsf{Set}_\perp \longrightarrow \mathsf{Pfn}$ defined by

$$\begin{aligned} S(A) &\mathrel{\widehat{=}} (A \cup \{\perp\}, \perp) \\ S(f)(x) &\mathrel{\widehat{=}} \begin{cases} f(x) & \Leftarrow\ x \in \mathrm{dom}\, f \\ \perp & \Leftarrow\ \text{otherwise} \end{cases} \end{aligned}$$

and

$$\begin{aligned} T((A, a)) &\mathrel{\widehat{=}} A - \{a\} \\ T(f : (A, a) \longrightarrow (B, b)) &\mathrel{\widehat{=}} \begin{cases} \mathrm{dom}\, T(f) & = A - f^{-1}(b) \\ T(f)(x) & = f(x) \end{cases} \end{aligned}$$

Show that Pfn and $\mathsf{Set}_\perp$ are equivalent categories. Discuss why the notion of isomorphism of categories does not apply in this case.

Exercise 12

Let C be a locally small category and W an object of C. C can be *represented* in Set through the so-called hom-functors, defined as follows:

$$C \xrightarrow{\mathrm{Hom}_C(W,-)} \mathsf{Set} \qquad\qquad C^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_C(-,W)} \mathsf{Set}$$

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & \mathrm{Hom}_C(W, X) \arrow[d, "\mathrm{Hom}_C(W,f)=f_*"] & X \arrow[r, mapsto] \arrow[d, "f"'] & \mathrm{Hom}_C(X, W) \\
Y \arrow[r, mapsto] & \mathrm{Hom}_C(W, Y) & Y \arrow[r, mapsto] & \mathrm{Hom}_C(Y, W) \arrow[u, "\mathrm{Hom}_C(f,W)=f^*"']
\end{tikzcd}$$

where $f_*(g) = f \cdot g$ and $f^*(g) = g \cdot f$. Verify that both constructions are indeed functors. Explain why the restriction to locally small categories is necessary.

Exercise 13

The lemma of Yoneda is one of the most useful results in Category Theory. This exercise invites the reader to approach its core while playing with natural transformations. Informally, the message is that every natural transformation between a Hom-functor and another functor also valued in Set can be determined by a single object in the source category. More generally, its relevance comes from making explicit a representation of generic mathematical constructions through functors valued in the category of sets, and therefore reducing proofs of isomorphisms between categorial constructions to the definition of bijections between their set-theoretical analogs[6].

For any locally small category C, the lemma establishes a bijective correspondence between the set of natural transformations from $\mathsf{Hom}_\mathsf{C}(W,-) : \mathsf{C} \longrightarrow \mathsf{Set}$ and an arbitrary functor $F : \mathsf{C} \longrightarrow \mathsf{Set}$, and set $F(W)$. Let us describe the two components of such a bijection.

For each element $\omega \in F(W)$ define a natural transformation $\gamma^\omega : \mathsf{Hom}_\mathsf{C}(W,-) \Longrightarrow F$ by

$$\gamma^\omega_X\,(h : W \longrightarrow X) \mathrel{\hat{=}} F(h)(\omega)$$

It is easy to verify the naturality of γ^ω by showing that the diagram below commutes, for any C-arrow $f : X \longrightarrow Y$:

$$\begin{tikzcd}
\mathsf{Hom}_\mathsf{C}(W,X) \arrow[r, "\gamma^\omega_X"] \arrow[d, "\mathsf{Hom}_\mathsf{C}(W,f)"'] & F(X) \arrow[d, "F(f)"] \\
\mathsf{Hom}_\mathsf{C}(W,Y) \arrow[r, "\gamma^\omega_Y"] & F(Y)
\end{tikzcd}$$

Actually,

$$\gamma^\omega_Y \cdot \mathsf{Hom}_\mathsf{C}(W,f)\,(h : W \longrightarrow X) \;=\; \gamma^\omega_Y \cdot (f \cdot h) \;=\; F(f \cdot h)(\omega) \;=\; F(f) \cdot \gamma^\omega_X\,(h)$$

On the other hand, for each natural transformation $\eta : \mathsf{Hom}_\mathsf{C}(W,-) \Longrightarrow F$, the diagram

$$\begin{tikzcd}
\mathsf{Hom}_\mathsf{C}(W,W) \arrow[r, "\eta_W"] \arrow[d, "\mathsf{Hom}_\mathsf{C}(W,f)"'] & F(W) \arrow[d, "F(f)"] \\
\mathsf{Hom}_\mathsf{C}(W,X) \arrow[r, "\eta_X"] & F(X)
\end{tikzcd}$$

which, by naturality, commutes for any $f : W \longrightarrow X$, defines the (arbitrary) component of η at X as

$$\eta_X(f) \;=\; \eta_X(f \cdot id_W) \;=\; \eta_X(\mathsf{Hom}_\mathsf{C}(W,f)(id_W) \;=\; F(f)(\eta_W(id_W))$$

and, of course, $\eta_W(id_W) \in F(W)$.

1. Show that the correspondences $\omega \in FW \;\mapsto\; \eta^\gamma$ and $\eta \;\mapsto\; \eta_W(id_W)$ are mutually inverse, thus establishing the isomorphism

   $$\mathsf{Hom}_{\mathsf{Set}^\mathsf{C}}(\mathsf{Hom}_\mathsf{C}(W,-), F) \;\cong\; F(W)$$

   and completing the proof.

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[6] In its essence, this is similar to the classical representation of an arbitrary abstract group by a subgroup of a permutation group.

2. The isomorphism above is natural. Write down the assertions that need be verified to establish the fact.

3. Formulate (possibly resorting to some help from a text book) the lemma of Yoneda for the contravariant Hom functor.

Lecture 6: Adjunctions

Summary.

(1) Motivation: ‘Free’ and ‘forgetful’ transformations.

(2) Adjunctions: Definition and examples. Relation to universal properties. Properties.

(3) Adjunctions on ordered structures: Galois connections.

Opening.

If categories can be thought of as particular mathematical spaces and functors as structure-preserving translations between them, an *adjunction* between, say, two functors $F : C \longrightarrow D$ and $G : D \longrightarrow C$, can be regarded as a source of *universals* in C and D. In fact, products and coproducts, final and initial objects and, in general, any universal construction arise in such a context. The notion of an adjunction pervades category theory and, in a sense, Mathematics as a whole.

As a motivation, recall the free monoid construction discussed in Lecture 5 (exercise 3), captured by the free functor $F : \mathit{Set} \longrightarrow \mathit{Mon}$ which builds a ‘syntactic’ monoid of ‘words’ over a set S. The forgetful functor $U : \mathit{Mon} \longrightarrow \mathit{Set}$ ‘undoes’ this construction returning the set of words over S. It is not difficult to verify that there exists a sort of symmetry between arrows involving these two functors. In detail, giving a set S and a monoid M, for each function $f : S \longrightarrow U(M)$ there is a unique monoid homomorphism $f^\bullet : F(S) \longrightarrow M$ making the diagram below left to commute. Or, starting from the other end, for each monoid homomorphism $h : F(S) \longrightarrow M$, there is a unique function $h_\bullet : S \longrightarrow U(M)$ so that the diagram in the right commutes, where η and ϵ are the natural transformations defined in the exercise mentioned above.

$$\begin{tikzcd}
UF(S) \arrow[r, "U(f^\bullet)"] & U(M) \\
S \arrow[u, "\eta_S"] \arrow[ur, "f"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
F(S) \arrow[r, "U(h_\bullet)"] \arrow[dr, "h"'] & FU(M) \arrow[d, "\epsilon_M"] \\
& M
\end{tikzcd}$$

$$F(S) \xrightarrow{\quad f^\bullet \quad} M \qquad\qquad S \xrightarrow{\quad h_\bullet \quad} U(M)$$

We’ve just captured a universal property. Recall that, by an entity being universal among a collection of similar ones, it is understood that there exists a unique way in which every other entity in the collection can be reduced to (or factored through). What we’ve just observed is that each component η_S of natural transformation η is universal among the arrows $f : S \longrightarrow U(M)$ in the sense that, for each such arrow, there exists a unique arrow which factors uniquely through η_S. And similarly for ϵ.

The notion of an *adjunction* captures a sort of symmetry and is therefore a source of universality, as discussed in the exercises below. We write $F \dashv U$, calling F the *left* and U the *right* adjoint functor. Natural transformations $\eta : \mathrm{Id} \Longrightarrow GF$ and $\epsilon : FG \Longrightarrow \mathrm{Id}$ are called the *unit* and *counit* of the adjunction, respectively.

Universality entails the existence of a natural isomorphism

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Mon}}(F(S), M) \arrow[rr, bend right, "i"'] \arrow[rr, phantom, "\cong"] & & \mathrm{Hom}_{\mathrm{Set}}(S, U(M)) \arrow[ll, bend right, "j"']
\end{tikzcd}$$

Exercise 1

Let $F \dashv G$. Compute $\eta : \mathrm{Id} \Longrightarrow GF$ and $\epsilon : FG \Longrightarrow \mathrm{Id}$ from the underlying natural isomorphism between homsets.

Exercise 2

Consider functors $! : \mathbf{1} \longrightarrow C$ and $\triangle: C \longrightarrow C \times C$, where $\mathbf{1}$ is the final object in Cat and $\triangle(A) = (A, A)$. Derive, for each of them, a right and a left adjoint. Comment the following statement: *all limits come from the right adjoints; all colimits from the left ones*.

Exercise 3

Suppose functors T and S compose and both have a left adjoint. Show that their composition TS has a left adjoint as well.

Exercise 4

Show that the unit and counit of an adjunction $F \dashv G$ satisfy the following conditions, known as the *triangle equalities*:

$$\begin{aligned} \epsilon F \cdot F\eta &= id_F \\ G\,\epsilon \cdot \eta G &= id_G \end{aligned}$$

Draw the relevant diagrams.

Exercise 5

An adjunction $f \dashv g$ between posets regarded as categories, say $P = (P, \leq)$ and $Q = (Q, \sqsubseteq)$ is a Galois connection:

$$\underbrace{f}_{\text{left adjoint}}(b) \leq a \quad \Leftrightarrow \quad b \sqsubseteq \underbrace{g}_{\text{right adjoint}}(a)$$

Draw the corresponding diagrams and explain why the adjunction unit and counit boil down to inequalities

$$f(g\ a) \leq a \quad \text{and} \quad b \sqsubseteq g(f\ b)$$

Exercise 6

In a Galois connection $f(b) \leq a \Leftrightarrow b \sqsubseteq g(a)$ the adjuncts determine each other uniquely: for example $f(b)$ is the greatest lower bound of all elements a such that $b \sqsubseteq g(a)$. Thus,

$$f\ b = \bigwedge\{a \mid b \sqsubseteq g\ a\} \quad \text{and} \quad g\ a = \bigsqcup\{b \mid f\ b \leq a\}$$

Using this fact, show that $f(b \sqcup b') = (f\ b) \vee (f\ b')$ and $g(a' \wedge a) = (g\ a') \sqcap (g\ a)$. Relate this result to the general fact that left adjoints preserve colimits and right adjoints preserve limits.

Exercise 7

Ler Rel be the poset of binary relations ordered by set inclusion, and consider the *converse* operation which computes the converse of a given relation. The usual relational laws

$$\begin{aligned}(R^\circ)^\circ &= R \\ (R \cap S)^\circ &= R^\circ \cap S^\circ \\ (R \cup S)^\circ &= R^\circ \cup S^\circ\end{aligned}$$

correspond to a particular adjunction over Rel. Can you identify it?

Exercise 8

Consider the following Galois connections in Rel where f and g are functions (thus, special cases of relations): $(f\cdot) \dashv (f^\circ\cdot)$ and $(\cdot f^\circ) \dashv (\cdot f)$. Write down the corresponding isomorphisms, known in the relational calculus as the *shunting* laws, and use them to conclude that

$$f \subseteq g \;\Leftrightarrow\; f = g \;\Leftrightarrow\; f \supseteq g$$

Exercise 9

Several laws in the calculus of binary relations are consequences of specific Galois connections. Consider the following operators, called the *right* and *left division*, respectively, and often useful to compute with relational data:

$$\begin{array}{rcl} a(R \setminus S)c & \Leftrightarrow & \forall_b \,.\, (bRa) \Rightarrow (bSc) \\ c(S \,/\, R)a & \Leftrightarrow & \forall_b \,.\, (aRb) \Rightarrow (cSb) \end{array}$$

To quickly grasp the meaning of the right division, observe that if R relates flights with passengers, and S flights to the air-companies in charge of them, assertion $a\ (R \setminus S)\ c$ states that passenger a only flies with company c. Give a similar explanation for the meaning of left division.

Both divisions can be actually defined through Galois connections $(R\cdot) \dashv (R\setminus)$ and $(\cdot R) \dashv (/R)$, i.e.

$$\begin{array}{c} R \cdot X \subseteq S \ \Leftrightarrow\ X \subseteq R \setminus S \\ X \cdot R \subseteq S \ \Leftrightarrow\ X \subseteq S \,/\, R \end{array}$$

and related to each other by still another adjunction: $(R/) \dashv (\setminus R)$. Show that the following laws are immediate consequences of these facts:

$$\begin{array}{rcl} R \cdot (S \cup T) & = & (R \cdot S) \cup (R \cdot T) \\ (S \cup T) \cdot R & = & (S \cdot R) \cup (T \cdot R) \\ R \setminus (S \cap T) & = & (R \setminus S) \cap (R \setminus T) \\ (S \cap T) \,/\, R & = & (S \,/\, R) \cap (T /\, R) \\ R \,/\, (S \cup T) & = & (R \,/\, S) \cap (R \,/\, T) \\ (S \cup T) \setminus R & = & (S \setminus R) \cap (T \setminus R) \\ R \setminus (S \setminus T) & = & (S \cdot R) \setminus T \end{array}$$

Exercise 10

Every binary relation $R : A \longrightarrow B$ induces a function $\mathrm{Im}_R : \mathcal{P}(A) \longrightarrow \mathcal{P}(B)$ mapping each $S \subseteq A$ to $\{b \in B \mid \exists_{a \in A}.\ (a, b) \in R\}$. This relation has a right adjoint: $[R] : \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$. Draw the diagram corresponding to $\mathrm{Im}_R \dashv [R]$ and show that a possible definition for $[R]$ is

$$[R](S') \,\widehat{=}\, \{a \in A \mid \forall_{b \in B}.\ (a, b) \in R \Rightarrow b \in S'\}$$

Observe that, in the context of transition systems, where R is an accessibility relation over a set of states, $[R]$ gives the semantics of the modal logic operator $\Box$ discussed in the first module of this course.

Lecture 7: Exponentials: The $- \times \mathsf{C} \dashv -^{\mathsf{C}}$ case study

Summary.

(1) Case study on adjunctions: internalising 'arrow spaces' — exponentiation $(-)^{\mathsf{C}}$ as a right adjoint to functor $- \times \mathsf{C}$. Cartesian closed categories.

Opening.

This last lecture is a case study on an adjunction defining a fundamental universal construction which turns out not to be neither a limit nor a colimit. Actually, the categorical version of the usual notion of a function space in Set arises, as one could expect, from an adjunction. Let us briefly detail this construction.

Let C be an object of C and suppose that functor $- \times \mathsf{C}$ has a right adjoint which we shall denote by $-^{\mathsf{C}}$. This means that for all $f : X \times C \longrightarrow Y$, there exists a unique $f_\bullet : X \longrightarrow Y^C$ such that $f = \epsilon_Y \cdot (f_\bullet \times C)$, both the object Y^C and the universal ϵ_Y being uniquely determined up to isomorphism. Diagrammatically,

$$\begin{tikzcd}
X \times C \arrow[r, "f_\bullet \times id"] \arrow[dr, "f"'] & Y^C \times C \arrow[d, "\epsilon_Y"] \\
 & Y
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "f_\bullet"] & Y^C
\end{tikzcd}$$

Construction $-^{\mathsf{C}}$ extends to a functor, the covariant *exponential* functor, by defining

$$h^C : A^C \longrightarrow B^C \;=\; (h \cdot \epsilon_A)_\bullet$$

for $h : A \longrightarrow B$. Note that Y^C has exactly the characteristic properties of the set of functions from C to Y in Set. Bijection $f \leftrightsquigarrow f_\bullet$ corresponds, in this particular context, to *currying*: the well-known isomorphism between (binary) functions from $X \times C$ to Y and (unary) functions from X to the set of functions from C to Y. Being so popular, this terminology is also adopted in an arbitrary category: $f_\bullet$ is called the *curry* of f and written $\overline{f}$.

The family $\epsilon_X : X^C \times C \longrightarrow X$ is, of course, the counit of the adjunction

$$- \times C \dashv -^C$$

On the other hand, its unit has $\eta_X : X \longrightarrow (X \times C)^C$ as components. In Set, ϵ corresponds to function evaluation and η to a function constructor:

$$\begin{aligned}
\epsilon_Y\,(g, c) &= g\,(c) \qquad \text{(for } g : X \longrightarrow Y\text{)}\\
\eta_X\,(x)(c) &= (x, c)
\end{aligned}$$

Counit ϵ is more commonly named ev, after *evaluation*. We shall also refer to η as sp, after *stamping*, and, again, such designations will carry over to general case.

The universal property captured by the $- \times C \dashv -^C$ adjunction diagram above can be written as the following equivalence (the concrete component of ev being of course determined by the type of f):

$$k = \overline{f} \quad \Leftrightarrow \quad f = ev \cdot (k \times id)$$

Note that the left to right implication expresses existence, while the converse one entails uniqueness (why?).

In an arbitrary category with exponentials C, X^C represents, as a C-object, the arrows from C to X. Consequently, the action of $-^C$ on each arrow $f : X \longrightarrow Y$ should *internalise* composition with f. In Set it is easy to verify that this is indeed the case. For $g : C \longrightarrow X$ and $c \in C$, a simple calculation yields,

$$\begin{array}{cl}
 & f^C(g)(c) \\
= & \quad \{ \; f^C = \overline{(f \cdot ev)}, \text{ as discussed above} \} \\
 & \overline{(f \cdot ev)}(g)(c) \\
= & \quad \{ \text{ uncurrying } \} \\
 & f \cdot ev \; (g, c) \\
= & \quad \{ \text{ function composition } \} \\
 & f(ev(g, c)) \\
= & \quad \{ \; ev \text{ definition} \} \\
 & f(g(c)) \\
= & \quad \{ \text{ function composition } \} \\
 & (f \cdot g) \; (c)
\end{array}$$

In an arbitrary category, however, we cannot talk about ‘applying’ a morphism to an ‘element’ of an object. We have, then, to state the intended result in the language of generalised elements (see Lecture 1, exercise 21). A generalised element of an exponential X^C is an arrow $\overline{g} : T \longrightarrow X^C$, which corresponds uniquely, under the adjunction, to $g : T \times C \longrightarrow X$. Keeping in mind that, in the generalised elements notation, $f^C \; (\overline{g})$ corresponds to $f^C \cdot \overline{g}$, the ‘internalisation’ result takes the form of an ‘absorption’ property for exponentials:

$$\overline{f \cdot g} \;\; = f^C \cdot \overline{g}$$

Taking g as a *point*, i.e. $\overline{g} : 1 \longrightarrow X^C$, $f^C \; (\overline{g})$ equals $\overline{f \cdot g}$ as proved above, but now $\overline{f \cdot g}$ is itself a point of B^C, which corresponds to morphism $f \cdot g$. In other words,

$$f^C \;\; = \;\; f \cdot \_$$

Furthermore, the exponential functor above can be made into a *bifunctor* by defining, for each $h : C \longrightarrow D$, an arrow $X^h : X^D \longrightarrow X^C$ as follows:

$$X^h \;\widehat{=}\; X^D \xrightarrow{sp} (X^D \times C)^C \xrightarrow{(id_{X^D} \times h)^C} (X^D \times D)^C \xrightarrow{ev^C} X^C$$

Note that the exponential bifunctor is *contravariant* in its second argument. Moreover, X^h can be proved equal to post-composition with h, i. e. $X^h = \_ \cdot h$.

A category with finite products is called Cartesian and provides the right setting for discussing the existence of exponentials. When they exist, the category is called Cartesian closed.

$\boxed{\textbf{Exercise } 1}$

Using the universal property entailed by the adjunction $- \times C \dashv -^C$, show that

$$\overline{ev} = id_{X^C} \quad \text{and} \quad sp = \overline{id_{X \times C}}$$

$\boxed{\textbf{Exercise } 2}$

In the context of the previous exercise, derive the following results, known in the Bird and Moor algebra of programs [5] as the exponential *cancellation* and *fusion* laws, respectively.

$$f = ev \cdot (\overline{f} \times id) \quad \text{and} \quad \overline{g} \cdot f = \overline{g \cdot (f \times id)}$$

$\boxed{\textbf{Exercise } 3}$

Consider the diagram below. Why do the left triangle and right square commute?

$$\begin{tikzcd}
T \times C \arrow[r, "\overline{g} \times C"] \arrow[dr, "g"'] & A^C \times C \arrow[r, "f^C \times C"] \arrow[d, "ev_A"] & B^C \times C \arrow[d, "ev_B"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}$$

Fill in the explanations in the following conclusion of the proof that $\overline{f \cdot g} = f^C \cdot \overline{g}$:

$$\begin{aligned}
& f \cdot g = ev_B \cdot (f^C \times C) \cdot (\overline{g} \times C) \\
\equiv \quad & \quad \{ \cdots \} \\
& f \cdot g = ev_B \cdot (f^C \cdot \overline{g} \times C) \\
\equiv \quad & \quad \{ \cdots \} \\
& \overline{f \cdot g} = f^C \cdot \overline{g}
\end{aligned}$$

Exercise 4

Exponentials can be defined in any category with products such that, for every object X, the functor $(- \times X)$ is a left adjoint. Consider the category Graph of finite graphs. An object T in Graph is a pair of parallel functions $s_T, t_T : T_e \longrightarrow T_v$ from the set of edges T_e to the set of vertices T_v specifying the source and target of each edge, respectively. An arrow $h : T \longrightarrow R$ is a homomorphism of graphs defined as a pair of functions (h_v, h_e) such that the following diagram commutes:

$$\begin{tikzcd}
T_e \arrow[r, "h_e"] \arrow[d, shift right, "T_e"'] \arrow[d, shift left, "T_v"] & R_e \arrow[d, shift right, "R_e"'] \arrow[d, shift left, "R_v"] \\
T_v \arrow[r, "h_v"] & R_v
\end{tikzcd}$$

The category has products defined pointwise: in particular, an object $T \times R$ of $\mathrm{Graph} \times \mathrm{Graph}$ is given by $s_T \times s_R, t_T \times t_R : T_e \times R_e \longrightarrow T_v \times R_v$. The exponential object T^R is defined in [3] as a graph whose vertices are maps $\phi : T_v \longrightarrow R_v$. An edge θ connecting vertices ϕ to ψ is a map $\theta : T_e \longrightarrow R_e$ making the following diagram commute:

$$\begin{tikzcd}
T_v \arrow[d, "\phi"'] & T_e \arrow[l, "s_T"'] \arrow[r, "t_T"] \arrow[d, "\theta"] & T_e \arrow[d, "\psi"] \\
R_v & R_e \arrow[l, "s_R"'] \arrow[r, "t_R"] & R_e
\end{tikzcd}$$

i.e. a family $(\theta_e)_{e \in T_v}$ such that $s_R(\theta_e) = \phi(s_T(e))$ and $t_R(\theta_e) = \psi(t_T(e))$.

Thinking about maps ϕ and ψ as two different images of the vertices of graph T in graph R, θ is a family of edges in T, labeled by the edges of R, each connecting the source vertex in ϕ to the corresponding target one in ψ.

The evaluation arrow $ev : T^R \times R \longrightarrow T$ maps a vertex (ϕ, r) to the vertex $\phi(r)$, and an edge (θ, e) to the edge θ_e. On the other hand, the curry $\overline{h} : S \longrightarrow T^R$ of a graph homomorphism $h : S \times R \longrightarrow T$ takes a vertex $a \in T_v$ to the map $h(a, -) : R_v \longrightarrow T_v$, and an edge $c : a \longrightarrow b \in T_e$ to the map $h(c, -) : R_e \longrightarrow T_e$.

Verify that these data defines exponentials in Graph. Draw all necessary diagrams.

Exercise 5

Products are defined pointwise in the category Pos of partially ordered sets, i.e. given $(P, \leq)$ and $Q = (Q, \sqsubseteq)$,

$$(P, \leq) \times (Q, \sqsubseteq) \;=\; (P \times Q, \leq \times \sqsubseteq)$$

The exponential Q^P is defined as

$$(\{h : P \longrightarrow Q \mid f \text{ is monotone}\}, \dot{\leq})$$

where $h \mathrel{\dot{\leq}} h' \mathrel{\widehat{=}} \forall_{x \in P}.\ h(x) \leq h'(x)$. The ev natural transformation and the curry $\overline{f} : X \longrightarrow Q^P$ of $f : X \times P \longrightarrow Q$ are defined as in Set.

Complete the exponential construction in Pos showing that all functions involved are indeed monotone.

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