Quantum error correction - A practical introduction

Introduction

- Quantum error correction is **advancing extremely fast** right now
- First quantum computers emerged in 2016-2019
- Good **QLDPCs** were discovered in **2022**
- A “good” General **logical operations** framework for QLDPCs was discovered in **2025/6**

Introduction

- My **name** is Bruno
  Contact **me anytime**: bruno.coelhocoutinho@dlr.de
  all will be here the **entire week**

- I'm a **scientist** at the DLR (German Aerospace agency)

- **Mostly** work on **quantum error correction**

  - Some work on quantum **communications** and quantum **computation**

- Part of my work involves **PhD** and **master student** supervision

  - I'm looking for one more good master student this semester

Class 1

- The problem of quantum error correction
- Historical codes
- Stabilizer formalism

- Bibliography
  - Nielsen & Chuang, Quantum Computation and Quantum Information (Cambridge, 2010)
  - Devitt, Munro & Nemoto, "Quantum error correction for beginners," Rep. Prog. Phys. 76, 076001 (2013)

Error correction

- Quantum technologies are **particularly** susceptible to **noise**

  $|0\rangle \otimes |0\rangle \otimes |0\rangle \rightarrow_e |1\rangle \otimes |0\rangle \otimes |0\rangle$

- Error correction allows us to **detect** and **correct errors** in a state

- But, unlike in the classical case

- **We can't measure** the **state**

- Needs to correct **two types** of **errors**: bit flips and phase flips

  $\sigma_x|0\rangle \rightarrow |1\rangle, \sigma_x|1\rangle \rightarrow |0\rangle$

  - $\sigma_z|0\rangle \rightarrow |0\rangle, \sigma_z|1\rangle \rightarrow -|1\rangle$

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Classical codes - Three bit flip code

- We make **quantum** states **resilient against noise** the same way we make **classical** states resilient to noise
  - We **add redundancy**

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Reversible logic and quantum computers

Asher Peres
Department of Physics, Technion—Israel Institute of Technology, 32000 Haifa, Israel
(Received 27 March 1985)

This article is concerned with the construction of a quantum-mechanical Hamiltonian describing a computer. This Hamiltonian generates a dynamical evolution which mimics a sequence of elementary logical steps. This can be achieved if each logical step is *locally* reversible (*global* reversibility is insufficient). Computational errors due to noise can be corrected by means of redundancy. In particular, reversible error-correcting codes can be embedded in the Hamiltonian itself. An estimate is given for the minimum amount of entropy which must be dissipated at a given noise level and tolerated error rate.

- The **three-bit flip code** is the first quantum error correction code proposed by Asher Peres in 1985

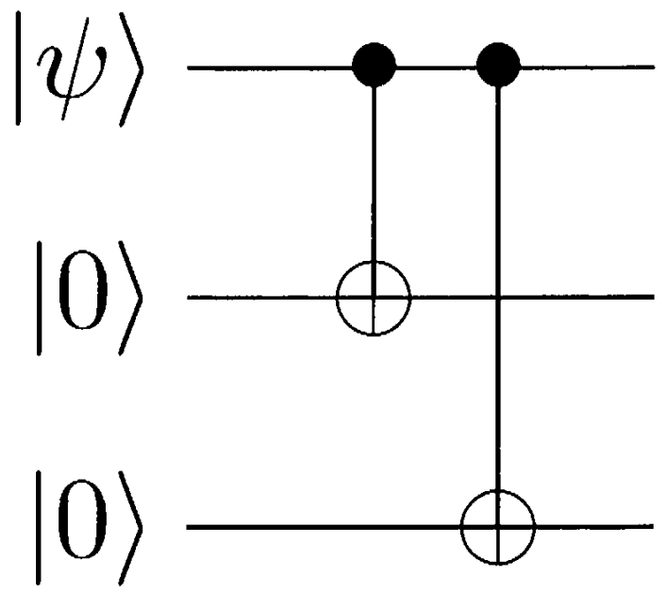
- It only protects against **bit-flips, X error**
  - But shows that quantum error correction is possible

Three-qubit bit-flip code

- First we need to encode a logical state $|\psi_L\rangle = \alpha|0_L\rangle + \beta|1_L\rangle$ into multiple physical states

$$|0_L\rangle := |000\rangle$$

$$|1_L\rangle := |111\rangle$$



- Modern quantum error correction does not use this definition of logical states, or encode states like this

Three-qubit bit-flip code

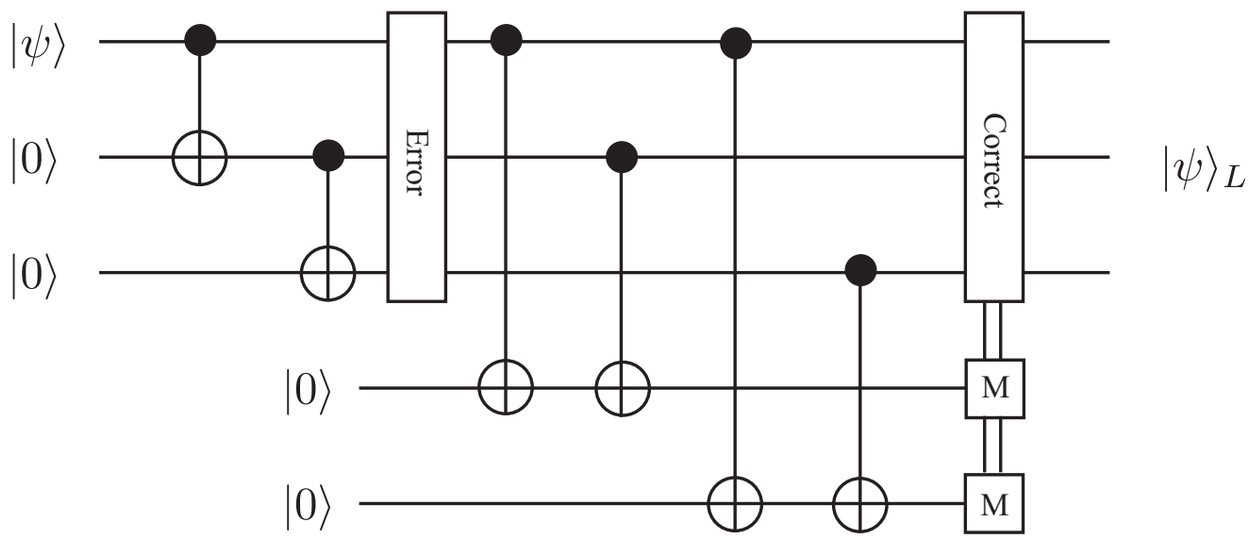
$$|\psi_L\rangle = \alpha|000\rangle + \beta|111\rangle$$

- Let us look at the effect of different bit-flip
- $\sigma_x^1|\psi\rangle = \alpha|100\rangle + \beta|011\rangle$
- $\sigma_x^2|\psi\rangle = \alpha|010\rangle + \beta|101\rangle$
- $\sigma_x^3|\psi\rangle = \alpha|001\rangle + \beta|110\rangle$

Three-qubit bit-flip code

$$|\psi\rangle_L = \alpha|000\rangle + \beta|111\rangle$$

$$P_0 \equiv |000\rangle\langle 000| + |111\rangle\langle 111| \text{ no error}$$
$$P_1 \equiv |100\rangle\langle 100| + |011\rangle\langle 011| \text{ bit flip on qubit one}$$
$$P_2 \equiv |010\rangle\langle 010| + |101\rangle\langle 101| \text{ bit flip on qubit two}$$
$$P_3 \equiv |001\rangle\langle 001| + |110\rangle\langle 110| \text{ bit flip on qubit three.}$$



Three-qubit bit code

- Now let us assume we have two bit-flip errors on different qubits? What happens?

$P_0 \equiv |000\rangle\langle 000| + |111\rangle\langle 111|$ no error

$P_1 \equiv |100\rangle\langle 100| + |011\rangle\langle 011|$ bit flip on qubit one

$P_2 \equiv |010\rangle\langle 010| + |101\rangle\langle 101|$ bit flip on qubit two

$P_3 \equiv |001\rangle\langle 001| + |110\rangle\langle 110|$ bit flip on qubit three.

Error Location	Final State, $\|\text{data}\rangle\,\|\text{ancilla}\rangle$
No Error	$\alpha\,\|000\rangle\,\|00\rangle + \beta\,\|111\rangle\,\|00\rangle$
Qubit 1	$\alpha\,\|100\rangle\,\|11\rangle + \beta\,\|011\rangle\,\|11\rangle$
Qubit 2	$\alpha\,\|010\rangle\,\|10\rangle + \beta\,\|101\rangle\,\|10\rangle$
Qubit 3	$\alpha\,\|001\rangle\,\|01\rangle + \beta\,\|110\rangle\,\|01\rangle$

Three-qubit phase-flip code

- How do we do the same to protect against phase-flips?
- $\sigma_z|0\rangle = |0\rangle$
- $\sigma_z|1\rangle = -|1\rangle$
- Anyone?

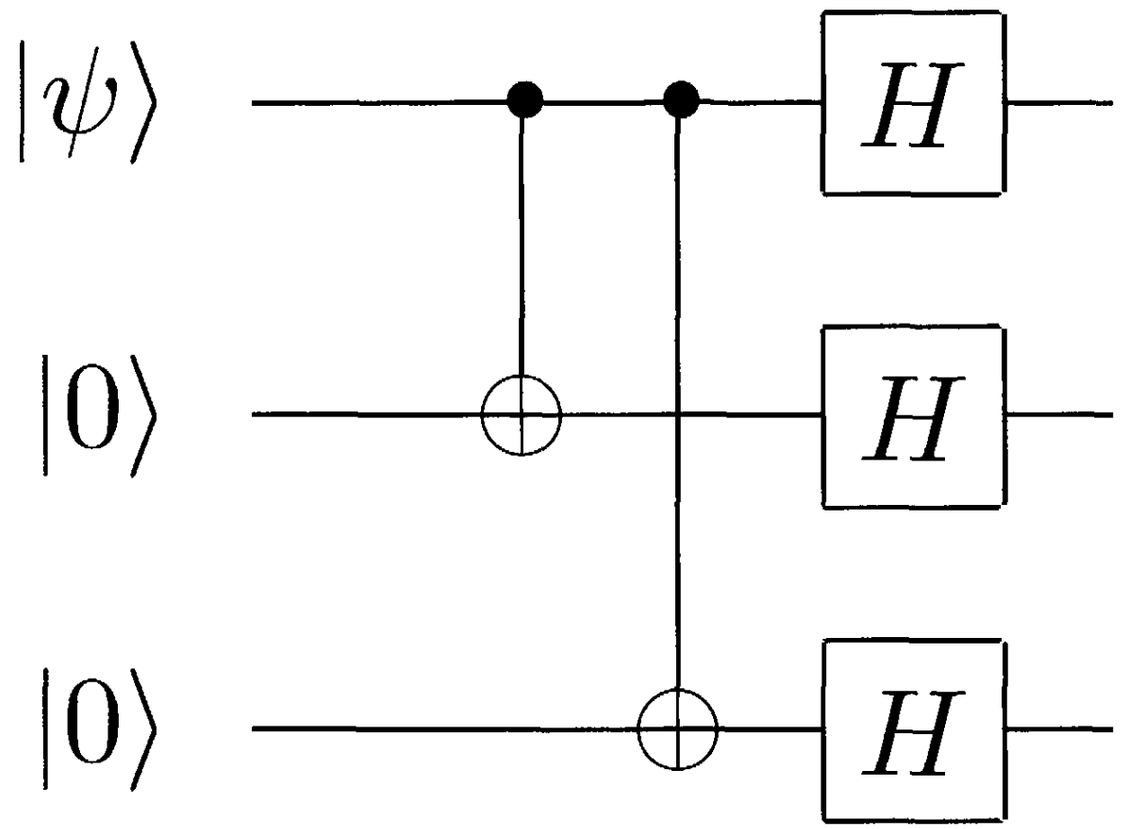
Three-qubit phase-flip code

- Let use consider the states $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$
- Notice that
- $\sigma_z |+\rangle = |-\rangle$
- $\sigma_z |-\rangle = |+\rangle$
- So we can encode our state as

$$|\psi\rangle_L = \alpha |+++\rangle + \beta |---\rangle$$

Three-qubit phase-flip code

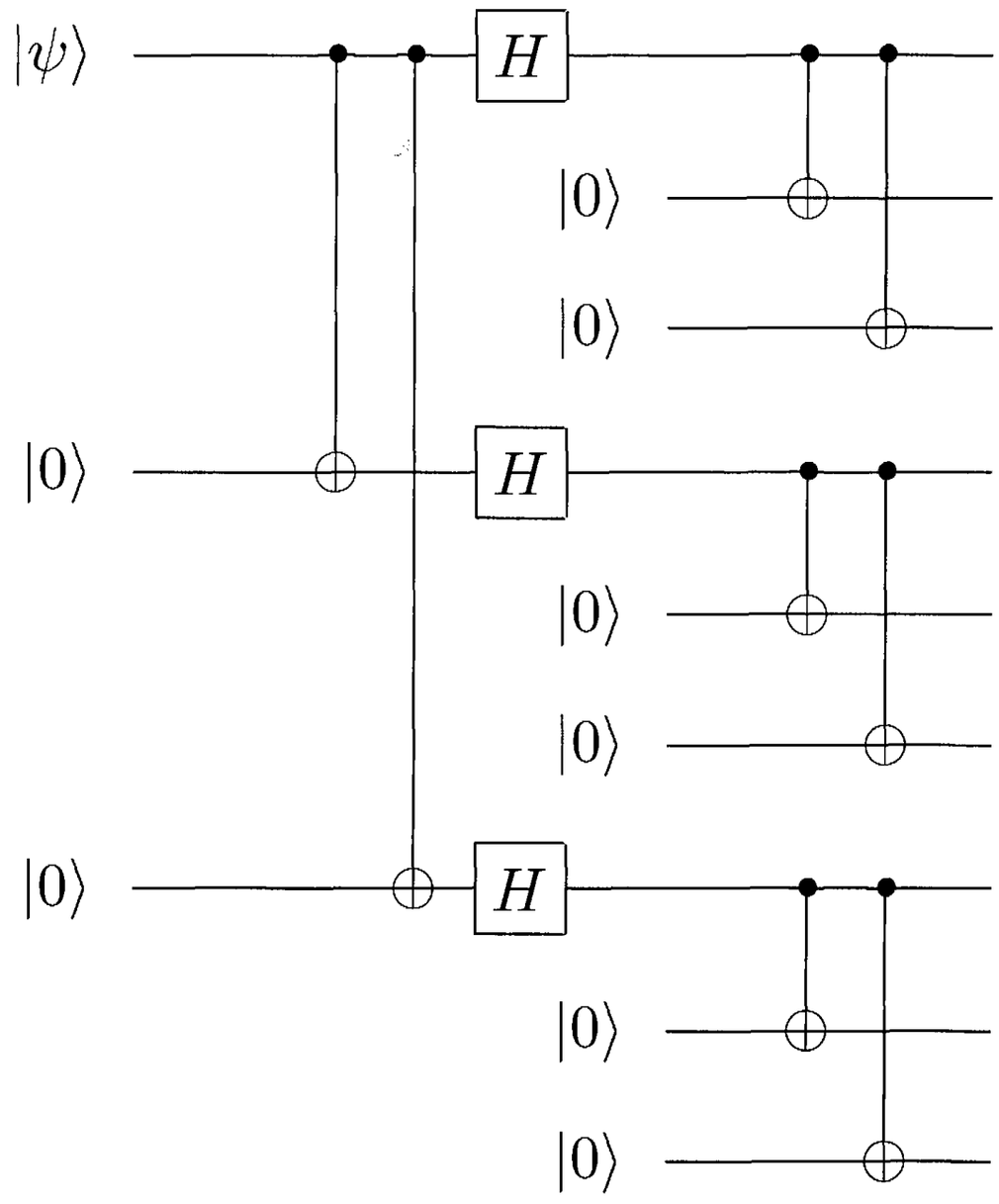
$$|\psi\rangle_L = \alpha|+++\rangle + \beta|---\rangle$$



The Shor code

- The Shor code was designed by Shor in **1995** and marks the **first** code that can correct any **bit-flip** and **phase-flip** affecting only one qubit

- **9** qubits encoding 1 **single logical** state



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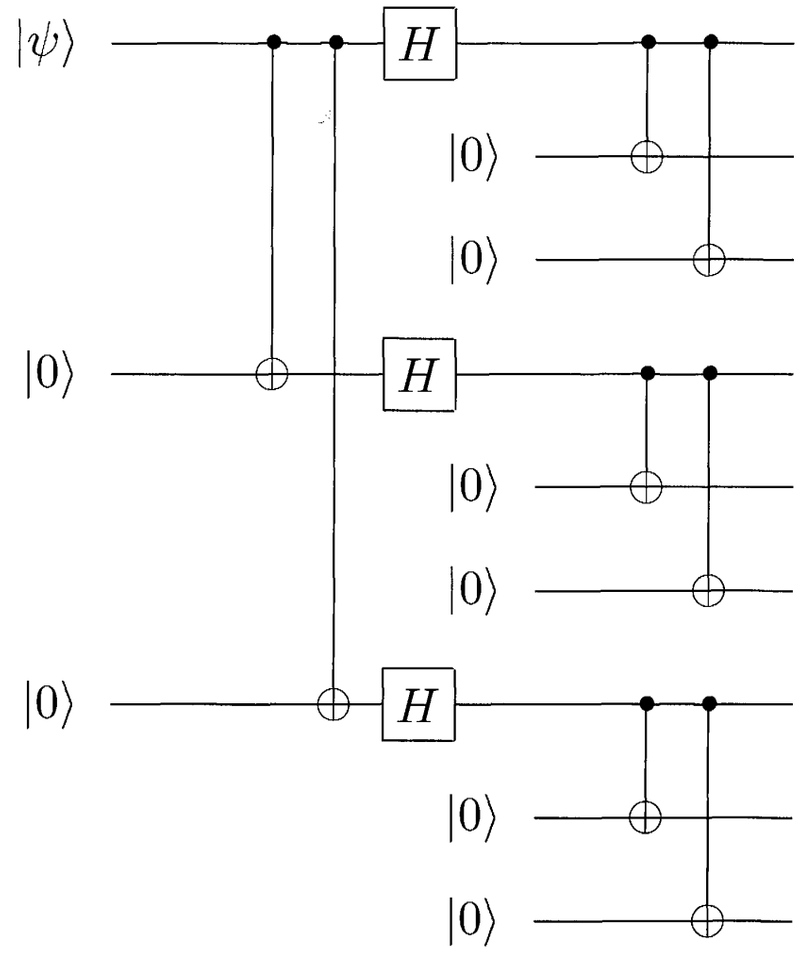


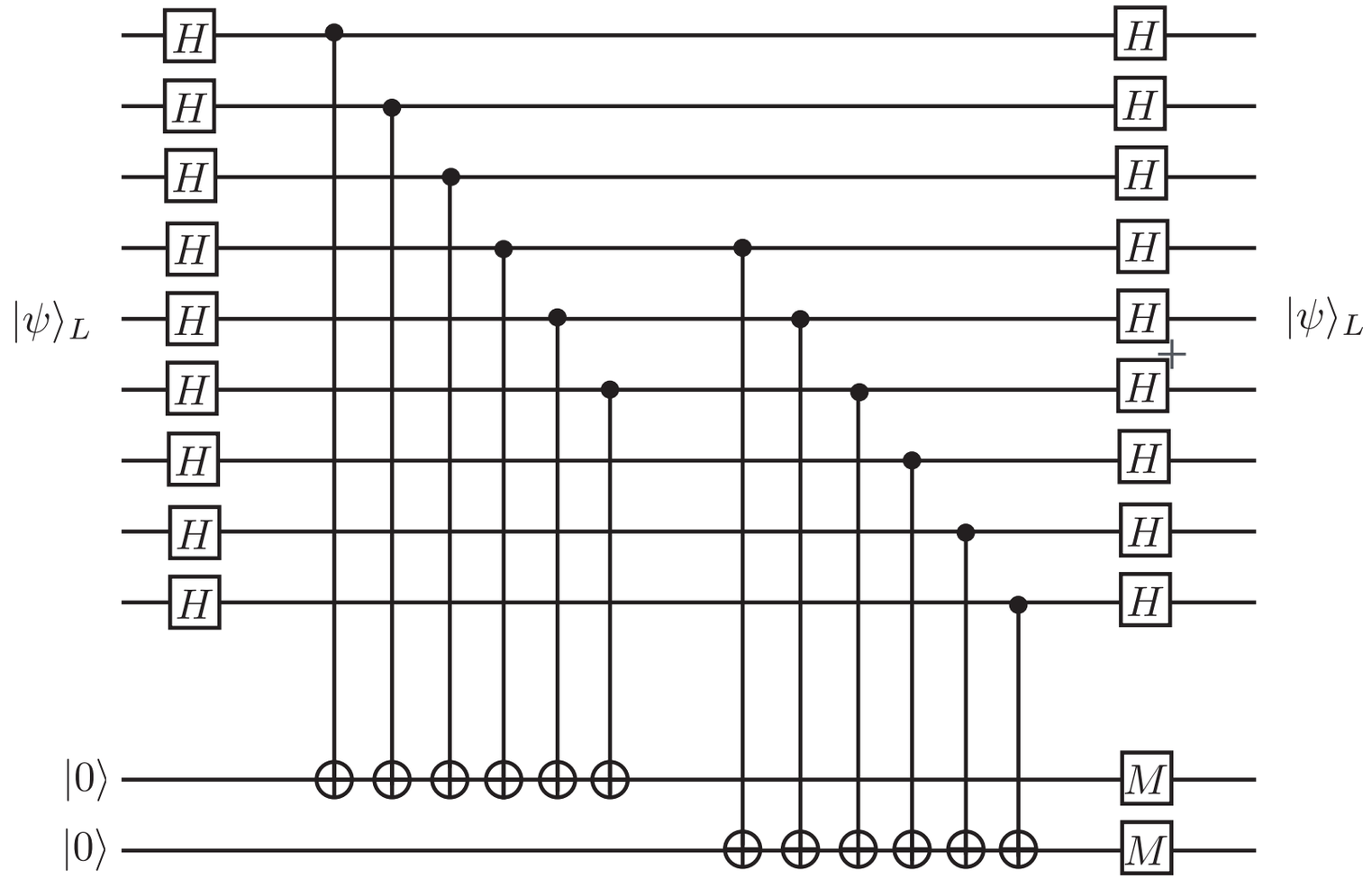
$$|0\rangle \rightarrow |0_L\rangle \equiv \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}}$$

$$|1\rangle \rightarrow |1_L\rangle \equiv \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}$$

The Shor code

$$|0\rangle \rightarrow |0_L\rangle \equiv \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}}$$

$$|1\rangle \rightarrow |1_L\rangle \equiv \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}$$



Circuit required to perform Z-error correction for the 9-qubit code.

Error correction

- What we need to do for error correction
  - Encode a lower-dimensional state in a larger one (we need redundancy)
  - We need a membership test
    - **Know if the code word is valid without measuring the state**
  - When a word is not valid, we need to be able to guess the likely error that occurred

Valid codewords

$|000\rangle$

$|111\rangle$

Non-valid codeword

$|100\rangle$

The stabilizer formalism

- Stabilizer formalism is an important tool for error correction
- It is a general formalism used to test if a state is of a certain form without measuring it
- With it we can create and understand error correction codes easily

A state $|\psi\rangle$ is defined to be stabilized by some operator K,
if it is a eigenstate of , i.e.

$$K|\psi\rangle = |\psi\rangle$$

The stabilizer formalism

- Examples:
  - The qubit state $|0\rangle$ is stabilized by $K = \sigma_{z'}\, \sigma_z |0\rangle = |0\rangle$
  - The qubit state $|1\rangle$ is stabilized by $K = -\sigma_{z'} - \sigma_z |1\rangle = |1\rangle$
  - The qubit state $|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$ is stabilized by $K = \sigma_x$
  - The qubit state $|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$ is stabilized by $K = -\sigma_x$

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The stabilizer formalism

- Let us suppose we have the state, $|\psi\rangle = |0\rangle$ and the σ_z

$$\sigma_z|0\rangle = |0\rangle$$

- And there is an error $\sigma_{x'}$ $\sigma_x|\psi\rangle = |1\rangle$

- Now

$$\sigma_z|\psi\rangle = -|\psi\rangle$$

The stabilizer formalism

- Let us suppose we have the state, $|\psi\rangle = |+\rangle$ and the stabilizer σ_x

$$\sigma_x |+\rangle = |+\rangle$$

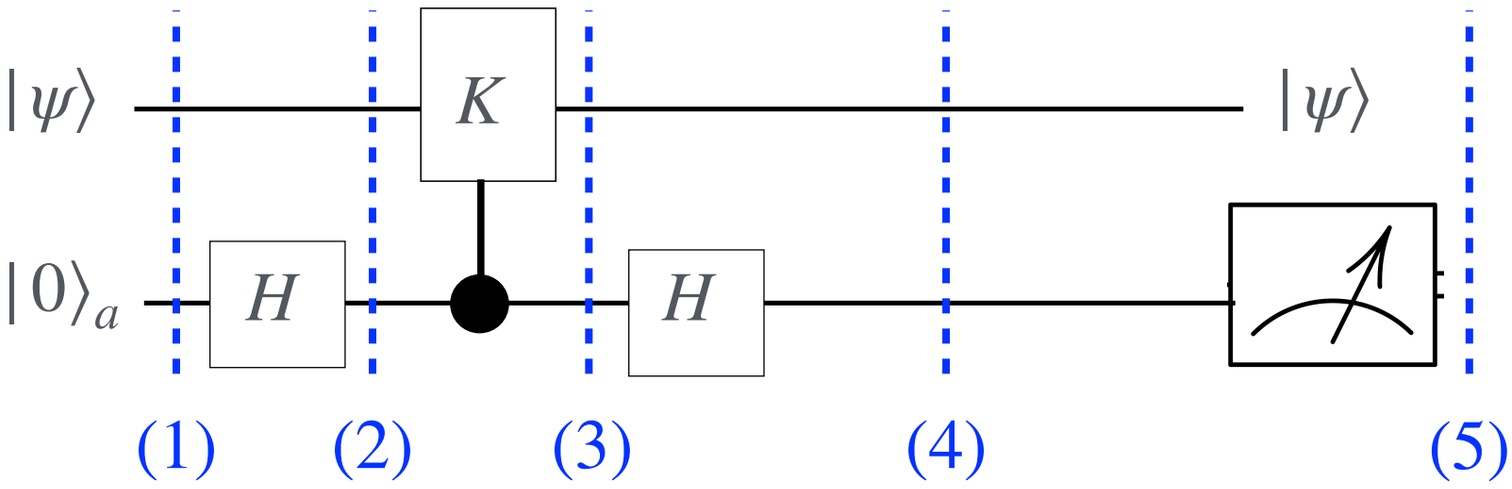
- And there is an error σ_z, $\sigma_z |\psi\rangle = |1\rangle$

- Now

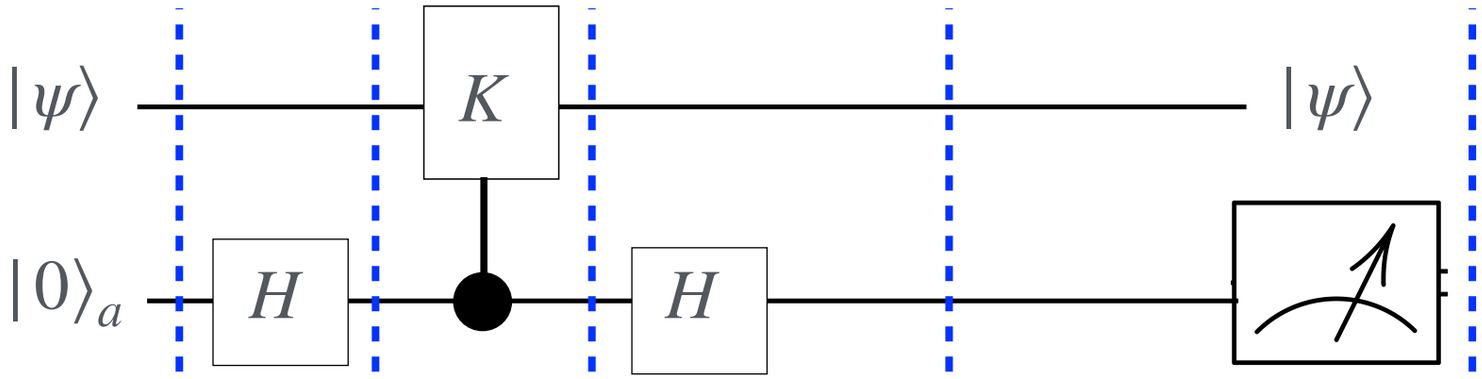
$$\sigma_x |\psi\rangle = -|\psi\rangle$$

Test

- We can test the phase shift without directly measuring the state



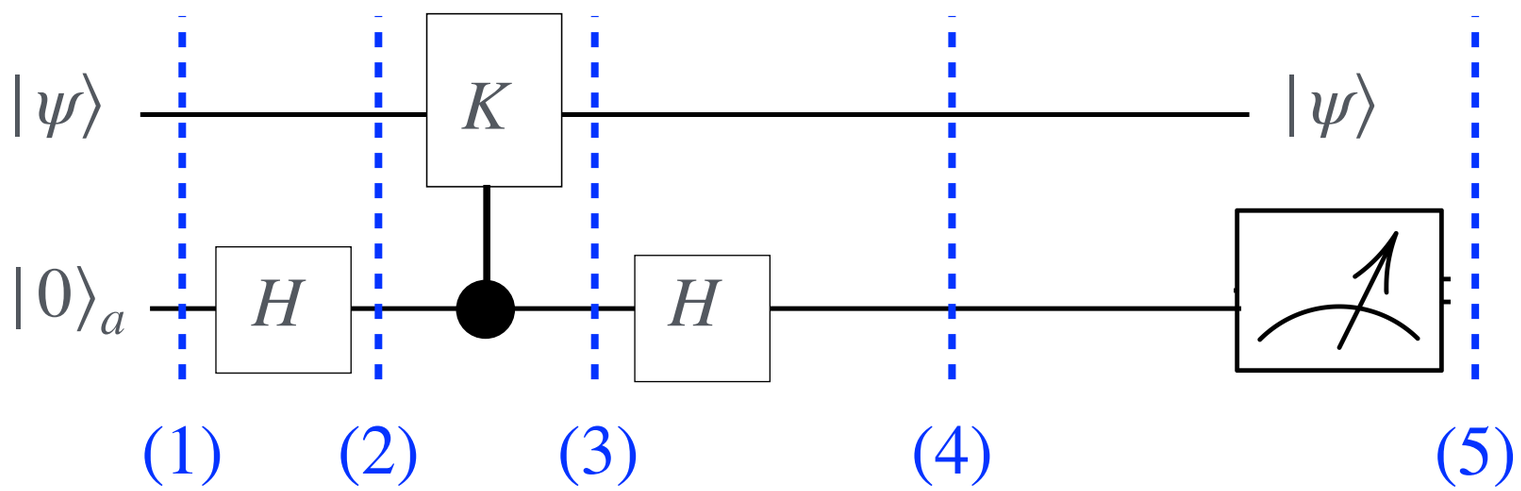
Test



(1) $|\psi\rangle|0\rangle_a$

(2) $\frac{1}{\sqrt{2}}|\psi\rangle(|0\rangle_a+|1\rangle_a)=\frac{1}{\sqrt{2}}\left(|\psi\rangle|0\rangle_a+|\psi\rangle|1\rangle_b\right)$

Test



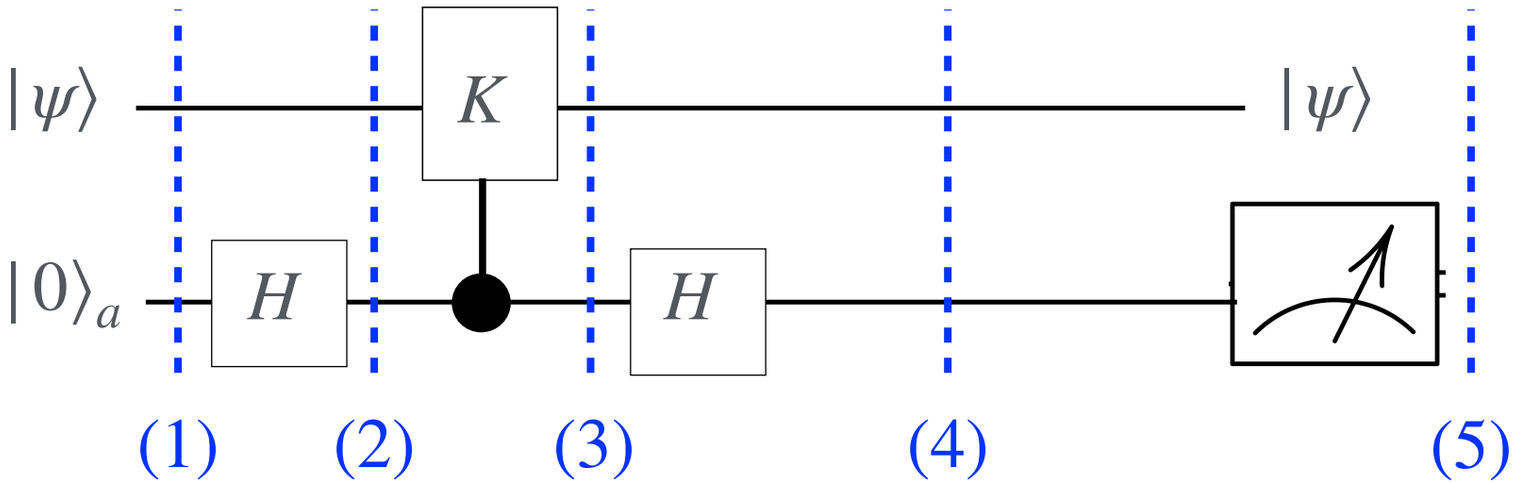
(2) $$\frac{1}{\sqrt{2}}|\psi\rangle(|0\rangle_a+|1\rangle_a)=\frac{1}{\sqrt{2}}\left(|\psi\rangle|0\rangle_a+\psi\rangle|1\rangle_b\right)$$

(3) $$\frac{1}{\sqrt{2}}\left(|\psi\rangle|0\rangle_a+K|\psi\rangle|1\rangle_b\right)=\left(|\psi\rangle|0\rangle_a\pm|\psi\rangle|1\rangle_b\right)=|\psi\rangle|\pm\rangle$$

Test

- But if $K|\psi\rangle$ is not either -1 or 1?
  - Because we are measuring the ancilla, the state always collapses to one of the two subspaces associated with -1 and 1
- Let us consider
- the state $|\psi\rangle = a|0\rangle + b|1\rangle$
- And the stabilizer σ_z

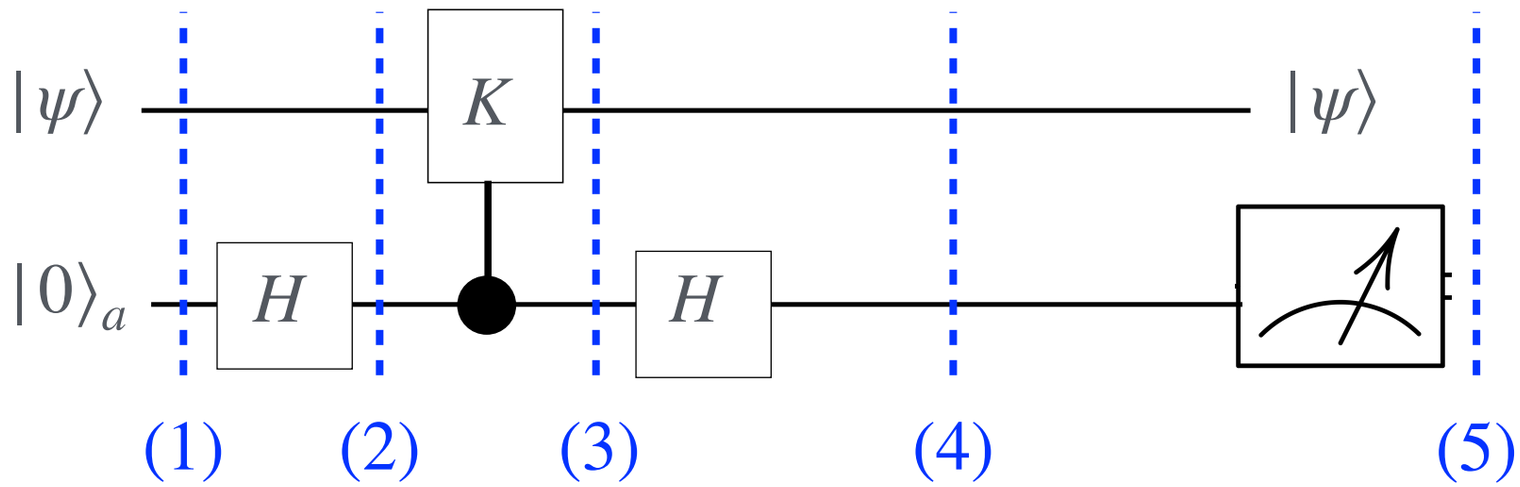
Test



(1) $\left(a|0\rangle + b|1\rangle\right)|0\rangle_a$

(2) $\frac{a}{\sqrt{2}}|0\rangle\left(|0\rangle_a + |1\rangle_a\right) + \frac{b}{\sqrt{2}}|1\rangle\left(|0\rangle_a + |1\rangle_a\right)$

Test

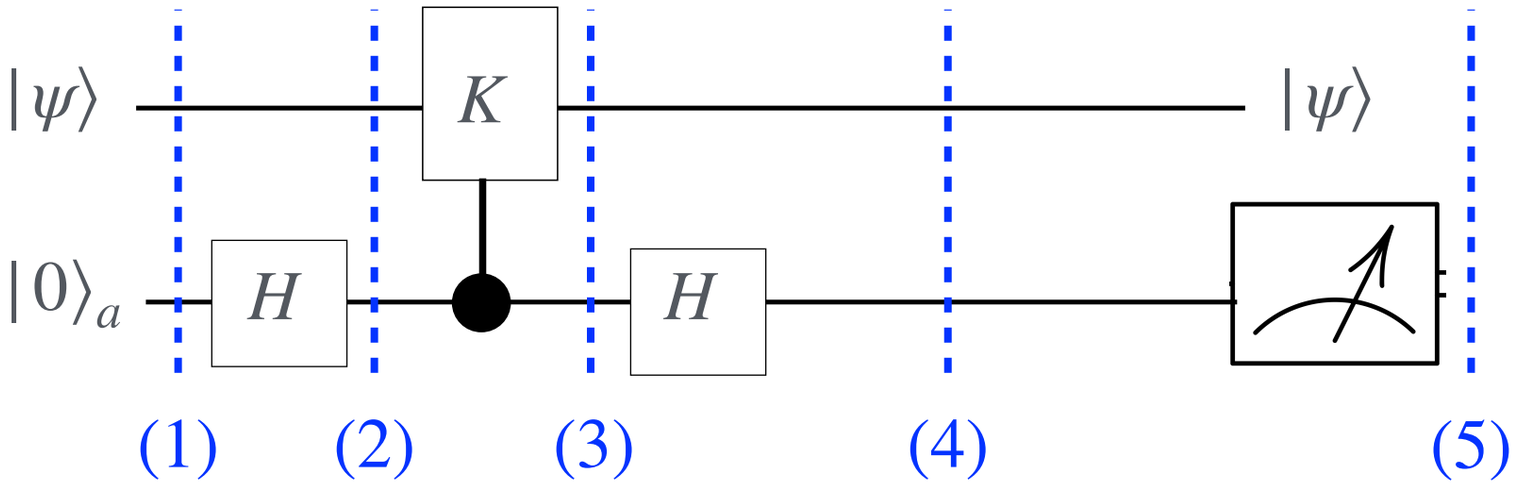


(2) $$\frac{a}{\sqrt{2}}|0\rangle\left(|0\rangle_a+|1\rangle_a\right)+\frac{b}{\sqrt{2}}|1\rangle\left(|0\rangle_a+|1\rangle_a\right)$$

$$=\frac{a}{\sqrt{2}}\left(|0\rangle|0\rangle_a+|0\rangle|1\rangle_a\right)+\frac{b}{\sqrt{2}}\left(|1\rangle|0\rangle_a+|1\rangle|1\rangle_a\right)$$

(3) $$\frac{a}{\sqrt{2}}\left(|0\rangle|0\rangle_a+\sigma_z|0\rangle|1\rangle_a\right)+\frac{b}{\sqrt{2}}\left(|1\rangle|0\rangle_a+\sigma_z|1\rangle|1\rangle_a\right)$$

Test



(3) $$\frac{a}{\sqrt{2}}|0\rangle|+\rangle_a+\frac{b}{\sqrt{2}}|1\rangle|-\rangle_a$$

(4) $$\frac{a}{\sqrt{2}}|0\rangle|0\rangle_a+\frac{b}{\sqrt{2}}|1\rangle|1\rangle_a$$

Test

- In this case, the **superposition is lost because** we know exactly the state associated with each measurement of the ancilla qubit
  - **Not very useful**
- But we can use the same trick to **reduce** the **dimensionality** of a higher-dimensional state to a lower-dimensional one

Stabilizers examples

- Let us consider a two-qubit system. The following states form a complete basis

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \quad |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \quad |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

- That are stabilized by

$$\Phi^+ \equiv \begin{pmatrix} K^1 = XX \\ K^2 = ZZ \end{pmatrix} \quad \Phi^- \equiv \begin{pmatrix} K^1 = -XX \\ K^2 = ZZ \end{pmatrix}$$
$$\Psi^+ \equiv \begin{pmatrix} K^1 = XX \\ K^2 = -ZZ \end{pmatrix} \quad \Psi^- \equiv \begin{pmatrix} K^1 = -XX \\ K^2 = -ZZ \end{pmatrix}$$

Stabilizers examples

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \quad |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \quad |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

- For the stabilizer $K = \sigma_z \otimes \sigma_z \rightarrow ZZ$
- Only the $|\Phi^+\rangle$ and $|\phi^+\rangle$ states are stabilized by K

Stabilizers examples

- All states that pass a test with stabilizer KK are a linear combination of

$$|0\rangle_L = |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|1\rangle_L = |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

- In other words, all the states such that $XX|\psi\rangle = |\psi\rangle$, are of the form
  - $|\psi\rangle = a|0\rangle_L + b|1\rangle_L$
- In also another take, we measure the parity between states using stabilizers without measuring the state

Stabilizers examples

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \quad |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \quad |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

- For the stabilizer $K_2 = \sigma_x \otimes \sigma_x \rightarrow XX$
- Only the $|\Phi^+\rangle$ and $|\psi^+\rangle$ states are stabilized by K

Stabilizers examples

- All states that pass a test with stabilizer K_2 are a linear combination of

$$|0\rangle_L = |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|1\rangle_L = |\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

- In other words, all the states such that $XX|\psi\rangle = |\psi\rangle$, are of the form
  - $|\psi\rangle = a|0\rangle_L + b|1\rangle_L$
- We measure the parity of the relative phase

Stabilizers examples - building the Shor code

$$|0\rangle \rightarrow |0_L\rangle \equiv \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}}$$

$$|1\rangle \rightarrow |1_L\rangle \equiv \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}$$

- Exercise
- What are the parities here?
- What are the stabilizers?

Stabilizers examples - building the Shor code

$$|0\rangle \rightarrow |0_L\rangle \equiv \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}}$$

$$|1\rangle \rightarrow |1_L\rangle \equiv \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}$$

- Exercise
- What are the parities here?
- What are the stabilizers?

The Shor code - revisited

- 9 qubits encode one single qubit
- Corrects all single-qubit errors

K^1	Z	Z	I	I	I	I	I	I	I
K^2	Z	I	Z	I	I	I	I	I	I
K^3	I	I	I	Z	Z	I	I	I	I
K^4	I	I	I	Z	I	Z	I	I	I
K^5	I	I	I	I	I	I	Z	Z	I
K^6	I	I	I	I	I	I	Z	I	Z
K^7	X	X	X	X	X	X	I	I	I
K^8	X	X	X	I	I	I	X	X	X

How to correct errors?

$$|0\rangle_L = |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|1\rangle_L = |\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \quad ZI|\Phi^-\rangle = |0\rangle_L$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \quad ZI|\Psi^-\rangle = |1\rangle_L$$

- Remember that $\sigma_i^2 = I$
- So if we have an z error on qubit 1,
  - that error can be corrected by applying σ_z on qubit 1

Steane code

- The Steane code was Introduced in 1

- It is one of the smallest codes that can connect any single arbitrary error in a quantum system

- Seven physical qubits, 1 logical qubit, distance 3. [7,1,3]

Steane code

$$
\begin{array}{lllllll}
K^1 & I & I & I & X & X & X \\
K^2 & X & I & X & I & X & I \\
K^3 & I & X & X & I & I & X \\
K^4 & I & I & I & Z & Z & Z \\
K^5 & Z & I & Z & I & Z & I \\
K^6 & I & Z & Z & I & Z & Z
\end{array}
\qquad
\begin{array}{c}
X \\ X \\ X \\ Z \\ Z \\ I
\end{array}
$$

The Steane code

- 7 qubits encode one single qubit
- Corrects all single-qubit errors

$$
\begin{array}{cccccccc}
K^1 & I & I & I & X & X & X & X \\
K^2 & X & I & X & I & X & I & X \\
K^3 & I & X & X & I & I & X & X \\
\hline
K^4 & I & I & I & Z & Z & Z & Z \\
K^5 & Z & I & Z & I & Z & I & Z \\
K^6 & I & Z & Z & I & Z & Z & I
\end{array}
$$

$$|0\rangle_L = \frac{1}{\sqrt{8}}(|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle),$$

$$|1\rangle_L = \frac{1}{\sqrt{8}}(|1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle + |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle).$$

The Shor code

σ_x σ_y

K^1	Z	Z	I	I	I	I	I	I	I
K^2	Z	I	Z	I	I	I	I	I	I
K^3	I	I	I	Z	Z	I	I	I	I
K^4	I	I	I	Z	I	Z	I	I	I
K^5	I	I	I	I	I	I	Z	Z	I
K^6	I	I	I	I	I	I	Z	I	Z
K^7	X	X	X	X	X	X	I	I	I
K^8	X	X	X	I	I	I	X	X	X

The Steane code

$i = S_3, S_2, S_1$

Detects

K^1	X	I	X	I	X	I	X
K^2	I	X	X	I	I	X	X
K^3	I	I	I	X	X	X	X
K^4	Z	I	Z	I	Z	I	Z
K^5	I	Z	Z	I	Z	Z	I
K^6	I	I	I	Z	Z	Z	Z

z

x

$j = S_6, S_5, S_4$

The Steane code

No error

$i = 000$

$$
\begin{array}{cccccccc}
K^1 & X & I & X & I & X & I & X \\
K^2 & I & X & X & I & I & X & X \\
K^3 & I & I & I & X & X & X & X \\
\hdots & \hdots & \hdots & \hdots & \hdots & \hdots & \hdots & \hdots \\
K^4 & Z & I & Z & I & Z & I & Z \\
K^5 & I & Z & Z & I & Z & Z & I \\
K^6 & I & I & I & Z & Z & Z & Z
\end{array}
$$

$j = 000$

The Steane code

$i = 001$

Error σ_z

$$
\begin{array}{cccccccc}
K^1 & X & I & X & I & X & I & X \\
K^2 & I & X & X & I & I & X & X \\
K^3 & I & I & I & X & X & X & X \\
\hline
K^4 & Z & I & Z & I & Z & I & Z \\
K^5 & I & Z & Z & I & Z & Z & I \\
K^6 & I & I & I & Z & Z & Z & Z
\end{array}
$$

$j = 000$ Can be corrected with $\boldsymbol{\sigma_z}$ on qubit 1

The Steane code

$i = 010$

Error σ_z

K^1	X	I	X	I	X	I	X
K^2	I	X	X	I	I	X	X
K^3	I	I	I	X	X	X	X
K^4	Z	I	Z	I	Z	I	Z
K^5	I	Z	Z	I	Z	Z	I
K^6	I	I	I	Z	Z	Z	Z

$j = 000$ Can be corrected with $\boldsymbol{\sigma}_z$ on qubit 2

The Steane code

$i = 011$

Error σ_z

K^1	X	I	X	I	X	I	X
K^2	I	X	X	I	I	X	X
K^3	I	I	I	X	X	X	X
K^4	Z	I	Z	I	Z	I	Z
K^5	I	Z	Z	I	Z	Z	I
K^6	I	I	I	Z	Z	Z	Z

$j = 000$ Can be corrected with $\boldsymbol{\sigma}_z$ on qubit 3

The Steane code

$i = 100$

Error σ_z

$$
\begin{array}{cccccccc}
K^1 & X & I & X & I & X & I & X \\
K^2 & I & X & X & I & I & X & X \\
K^3 & I & I & I & X & X & X & X \\
\hline
K^4 & Z & I & Z & I & Z & I & Z \\
K^5 & I & Z & Z & I & Z & Z & I \\
K^6 & I & I & I & Z & Z & Z & Z
\end{array}
$$

$j = 000$ Can be corrected with σ_z on qubit 4

The Steane code

$i = 101$

Error σ_z

K^1	X	I	X	I	X	I	X
K^2	I	X	X	I	I	X	X
K^3	I	I	I	X	X	X	X
K^4	Z	I	Z	I	Z	I	Z
K^5	I	Z	Z	I	Z	Z	I
K^6	I	I	I	Z	Z	Z	Z

$j = 000$ Can be corrected with σ_z one qubit 5

The Steane code

$i = 110$

Error σ_z

K^1	X	I	X	I	X	I	X
K^2	I	X	X	I	I	X	X
K^3	I	I	I	X	X	X	X
K^4	Z	I	Z	I	Z	I	Z
K^5	I	Z	Z	I	Z	Z	I
K^6	I	I	I	Z	Z	Z	Z

$j = 000$ Can be corrected with σ_z one qubit 6

The Steane code

$i = 111$

Error

							σ_z
K^1	X	I	X	I	X	I	X
K^2	I	X	X	I	I	X	X
K^3	I	I	I	X	X	X	X
K^4	Z	I	Z	I	Z	I	Z
K^5	I	Z	Z	I	Z	Z	I
K^6	I	I	I	Z	Z	Z	Z

$j = 000$

Can be corrected with $\boldsymbol{\sigma_z}$ one qubit 7

The Steane code

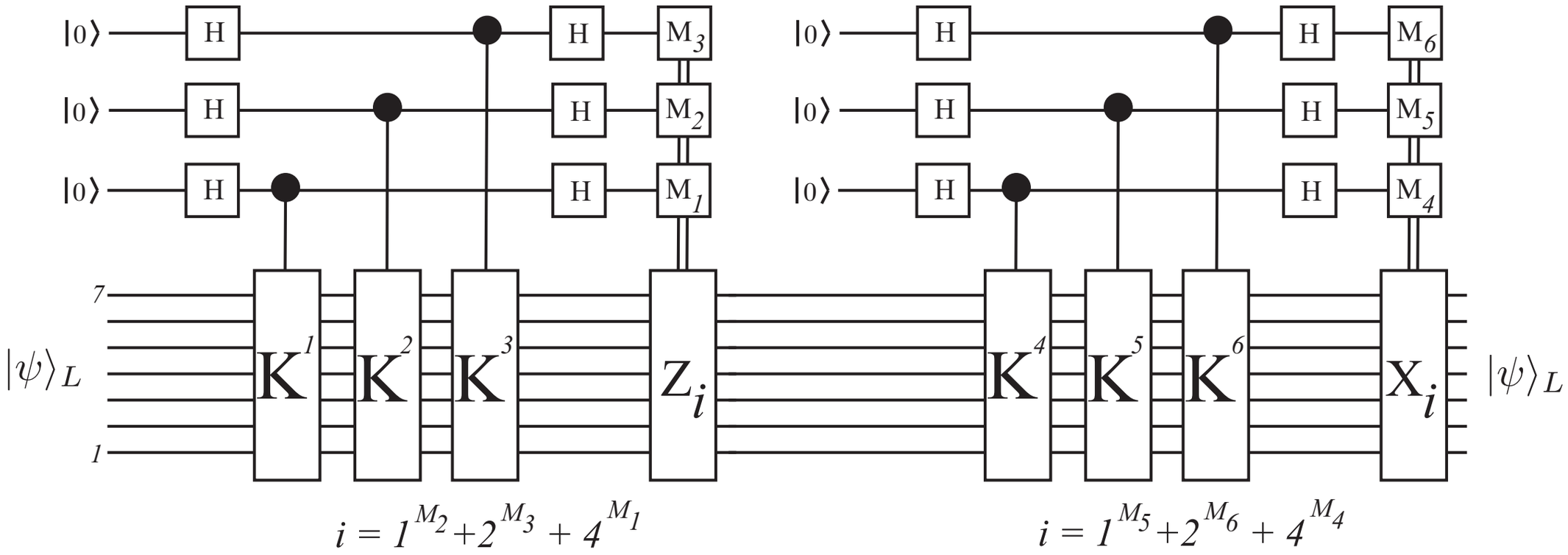
$i = 000$

Error σ_x

K^1	X	I	X	I	X	I	X
K^2	I	X	X	I	I	X	X
K^3	I	I	I	X	X	X	X
K^4	Z	I	Z	I	Z	I	Z
K^5	I	Z	Z	I	Z	Z	I
K^6	I	I	I	Z	Z	Z	Z

$j = 001$ Can be corrected with $\boldsymbol{\sigma_x}$ one qubit 1

Steane code


|0⟩
H
M3
M2
M1
M5
M4
7
1
|ψ⟩L
K1
K2
K3
K4
K5
K6
Zi
Xi
|ψ⟩L
i = 1^M2 +2^M3 + 4^M1
i = 1^M5 +2^M6 + 4^M4

Steane code

- Is this the smallest code capable of correcting all single-qubit errors?

Class 2

- CSS-codes and non-CSS codes
- The parity check matrix representation
- Surface codes
- Degeneracy
- The decoding problem
- Bibliography
  - Nielsen & Chuang, Quantum Computation and Quantum Information (Cambridge, 2010)
  - Devitt, Munro & Nemoto, "Quantum error correction for beginners," Rep. Prog. Phys. 76, 076001 (2013)

Steane code

- Is this the smallest code capable of correcting all single-qubit errors?

$$
\begin{array}{llllllll}
K^1 & I & I & I & X & X & X & X \\
K^2 & X & I & X & I & X & I & X \\
K^3 & I & X & X & I & I & X & X \\
K^4 & I & I & I & Z & Z & Z & Z \\
K^5 & Z & I & Z & I & Z & I & Z \\
K^6 & I & Z & Z & I & Z & Z & I
\end{array}
$$

$$2^{\#_s} \geq \#_E$$

Steane code

- Is this the smallest code capable of correcting all single-qubit errors?

$$
\begin{array}{lccccccc}
K^1 & I & I & I & X & X & X & X \\
K^2 & X & I & X & I & X & I & X \\
K^3 & I & X & X & I & I & X & X \\
\hline
K^4 & I & I & I & Z & Z & Z & Z \\
K^5 & Z & I & Z & I & Z & I & Z \\
K^6 & I & Z & Z & I & Z & Z & I
\end{array}
$$

z

x

Steane code

- Is this the smallest code capable of correcting all single-qubit errors?

$$2^{\#_{sx}} \geq \#_{Ex} \qquad \#_{sx} = \#_{sz} = \#_s/2 = (N-1)/2$$

$$2^{\#_{sz}} \geq \#_{Ez} \qquad \#_{Ez} = \#_{Ex} = N+1$$

$$2^{\frac{N-1}{2}} \geq N+1$$

N must be at least 7

Short codes

- Is this the smallest code capable of correcting all single-qubit errors?

$$2^{\#_s} \geq \#_E$$

$$\#_s = N - 1$$

$$\#_E = \#_{Ex} + \#_{Ey} + \#_{Ez} + 1 = 3N + 1 \qquad \sigma_y = -i\sigma_x\sigma_z$$

$$2^{N-1} \geq 3N + 1 \longrightarrow N = 5$$

The 5-qubit code

K^1	X	Z	Z	X	I
K^2	I	X	Z	Z	X
K^3	X	I	X	Z	Z
K^4	Z	X	I	X	Z

You can check that for every X, Y, and Z affecting only one qubit, there is one and only one set of syndromes associated

The 5-qubit code

K^1	X	Z	Z	X	I
K^2	I	X	Z	Z	X
K^3	X	I	X	Z	Z
K^4	Z	X	I	X	Z

There is a constraint do in code creation with mixed or not mixed stabilizers

All stabilizers must commute [$[K_i, K_j] = 0$ for all i, j

CSS codes

- The Steane code is the smallest CSS code that correct one error
  - CSS -> Calderbank, Shor, and Steane
- This class of codes correct X and Z errors independently

$$
\begin{array}{ccccccccc}
K^1 & I & I & I & X & X & X & X \\
K^2 & X & I & X & I & X & I & X \\
K^3 & I & X & X & I & I & X & X \\
\hdashline
K^4 & I & I & I & Z & Z & Z & Z \\
K^5 & Z & I & Z & I & Z & I & Z \\
K^6 & I & Z & Z & I & Z & Z & I
\end{array}
$$

Parity check matrices

$$
H_x = \begin{matrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{matrix}
$$

$$
H_z = \begin{matrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{matrix}
$$

CSS codes

- Most codes today are CSS
  - For reasons that I am not entirely convinced by
- Let me use this moment to introduce the parity check matrices
  - We can consider an error vector $\vec{e} = \{\vec{e}_z, \vec{e}_x\}$
  - And a parity check matrix $\mathbf{H}$ that measures the parities defined by the stabilizers
- The syndrome measurement of an error is given by

$$\vec{S} = \mathbf{H}\vec{\mathbf{e}}$$

CSS codes

$$H = \begin{pmatrix} H_X & 0 \\ 0 & H_Z \end{pmatrix}$$

K^1	I	I	I	X	X	X	X
K^2	X	I	X	I	X	I	X
K^3	I	X	X	I	I	X	X
K^4	I	I	I	Z	Z	Z	Z
K^5	Z	I	Z	I	Z	I	Z
K^6	I	Z	Z	I	Z	Z	I

Parity check matrices

$$H_x = \begin{matrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{matrix}$$

$$H_z = \begin{matrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{matrix}$$

CSS codes

- The stabilizer commutation condition for CSS codes is just

$$H_x H_z^T = 0 \mod 2$$

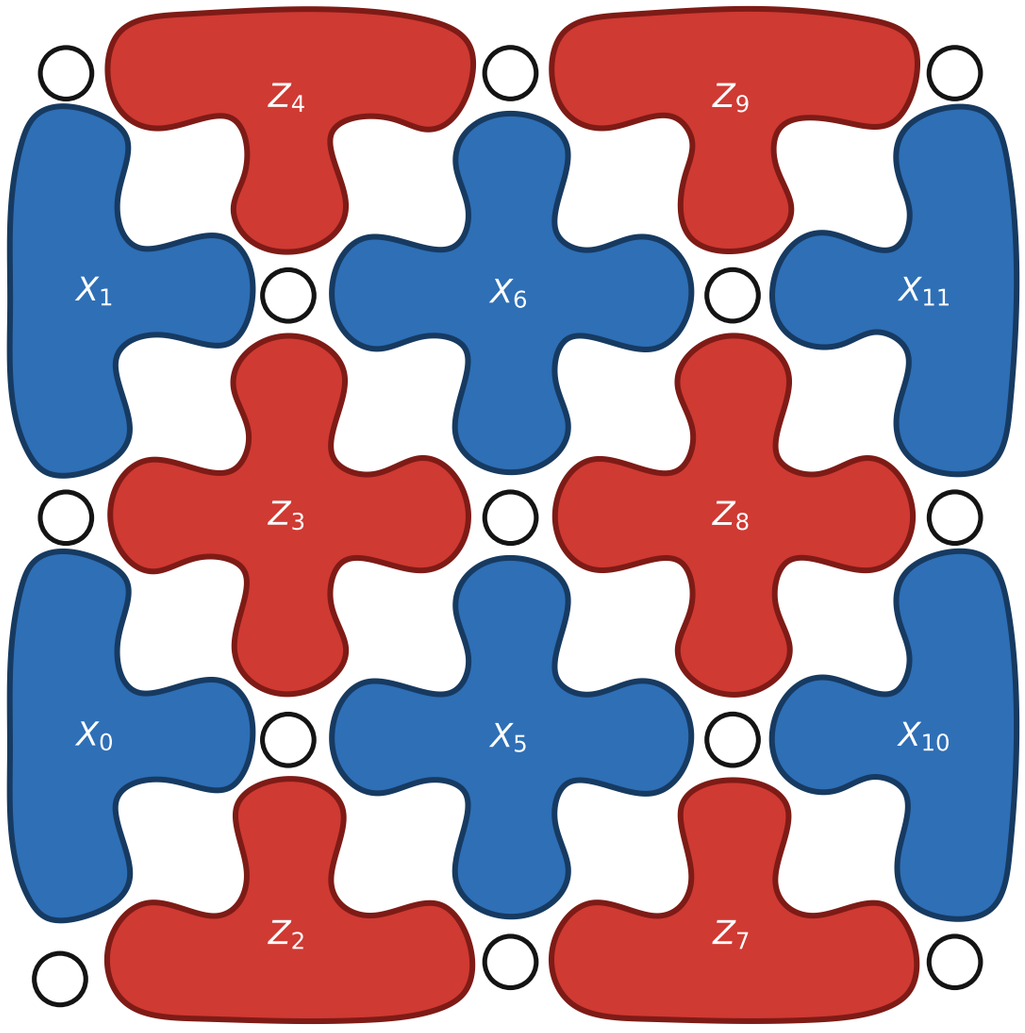
$$H_x = \begin{matrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{matrix} \qquad H_z = \begin{matrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{matrix}$$

Surface codes

- Is an example of a CSS code construction that respects the commutation constraint
  - I am not a big fan of surface codes
  - They are overused in research — we have much better codes
- For quantum computers they are a bit more convenient than the alternatives, but with very low logical-to-physical encoding rates
- They are, however, very good for explaining things

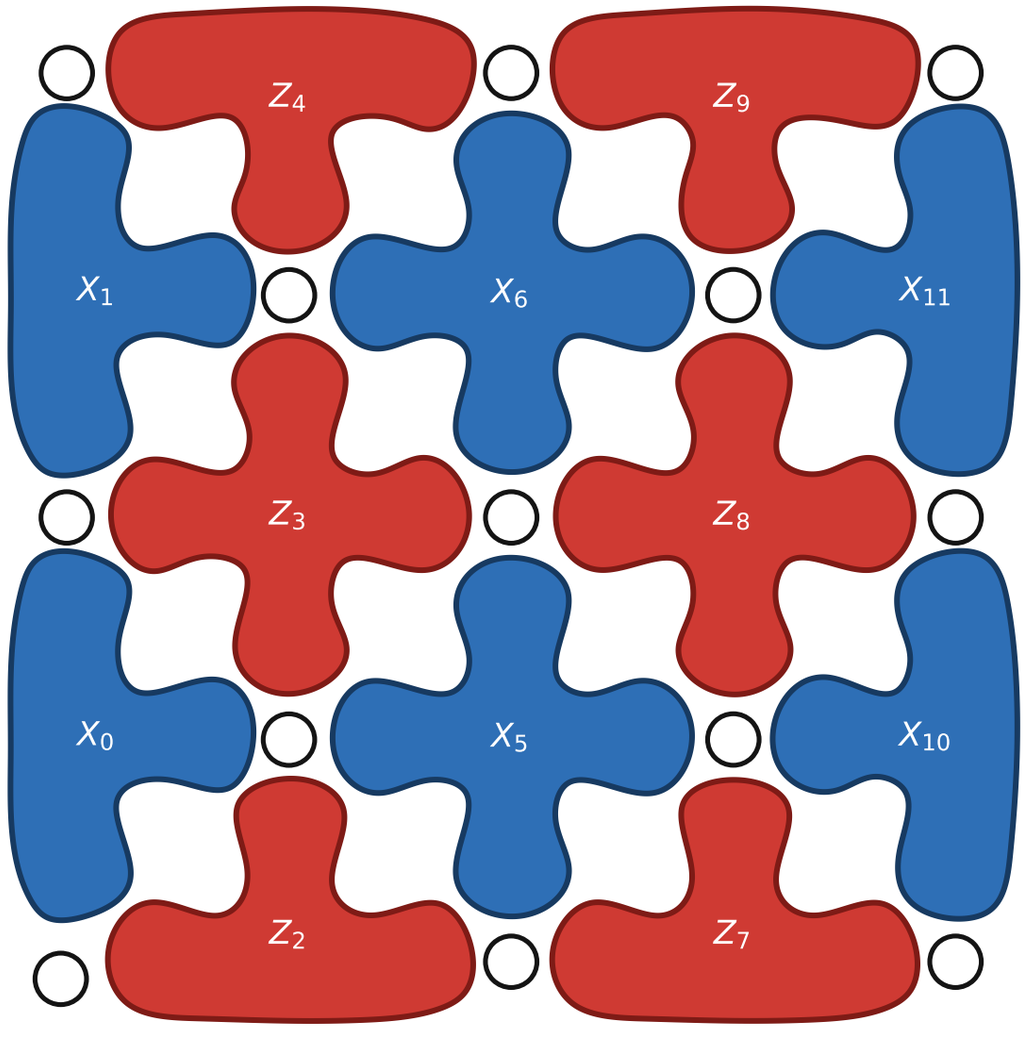
Surface codes

- What is a surface code?
- A surface code is a quantum error-correcting code laid out on a grid, with Z and X stabilizers each connected only to neighboring qubits.
- k is always 1
- n=13
- d=3 ($d \sim \sqrt{n}$)
- [13,1,3]



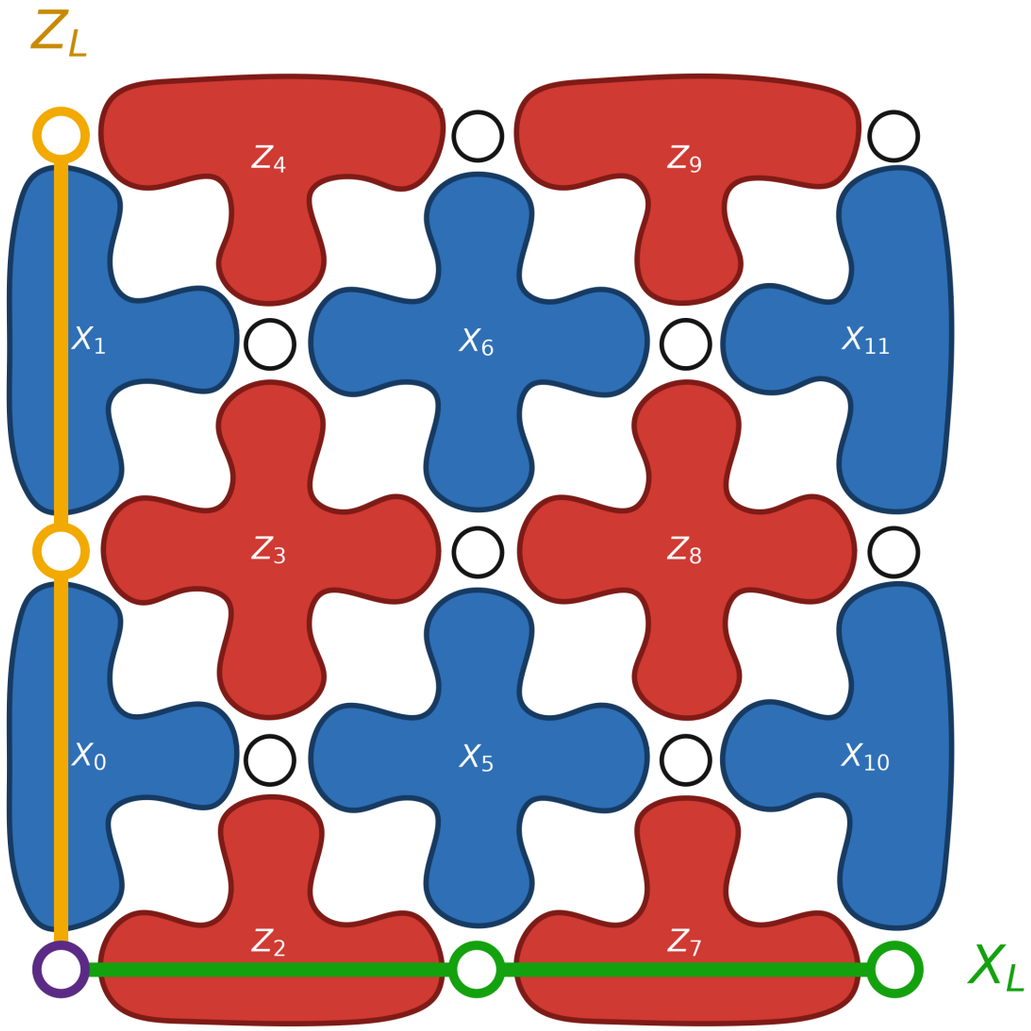
Surface codes

- Things to notice
  - All Z stabilizers and X stabilizers share two qubits
    - So $[H_x, H_z] = 0$
  - All connections are local



Surface codes

- I lied two times in this class,
  - **First lie**, I said a logical states is
  - $|\phi_L> = ..$
- Logical states are not defined like this in modern quantum error correction
- This kind of assumes we have access to a state, or at least that the state is well defined
  - This is not always true



Logical measurements as operators

- The Shor code again

$$|0\rangle \rightarrow |0_L\rangle \equiv \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}}$$

$$|1\rangle \rightarrow |1_L\rangle \equiv \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}$$

- For the logical Z $X^{\mathrm{L}} = X_1 X_2 X_3,$ For the logical X $Z^{\mathrm{L}} = Z_1 Z_4 Z_7$

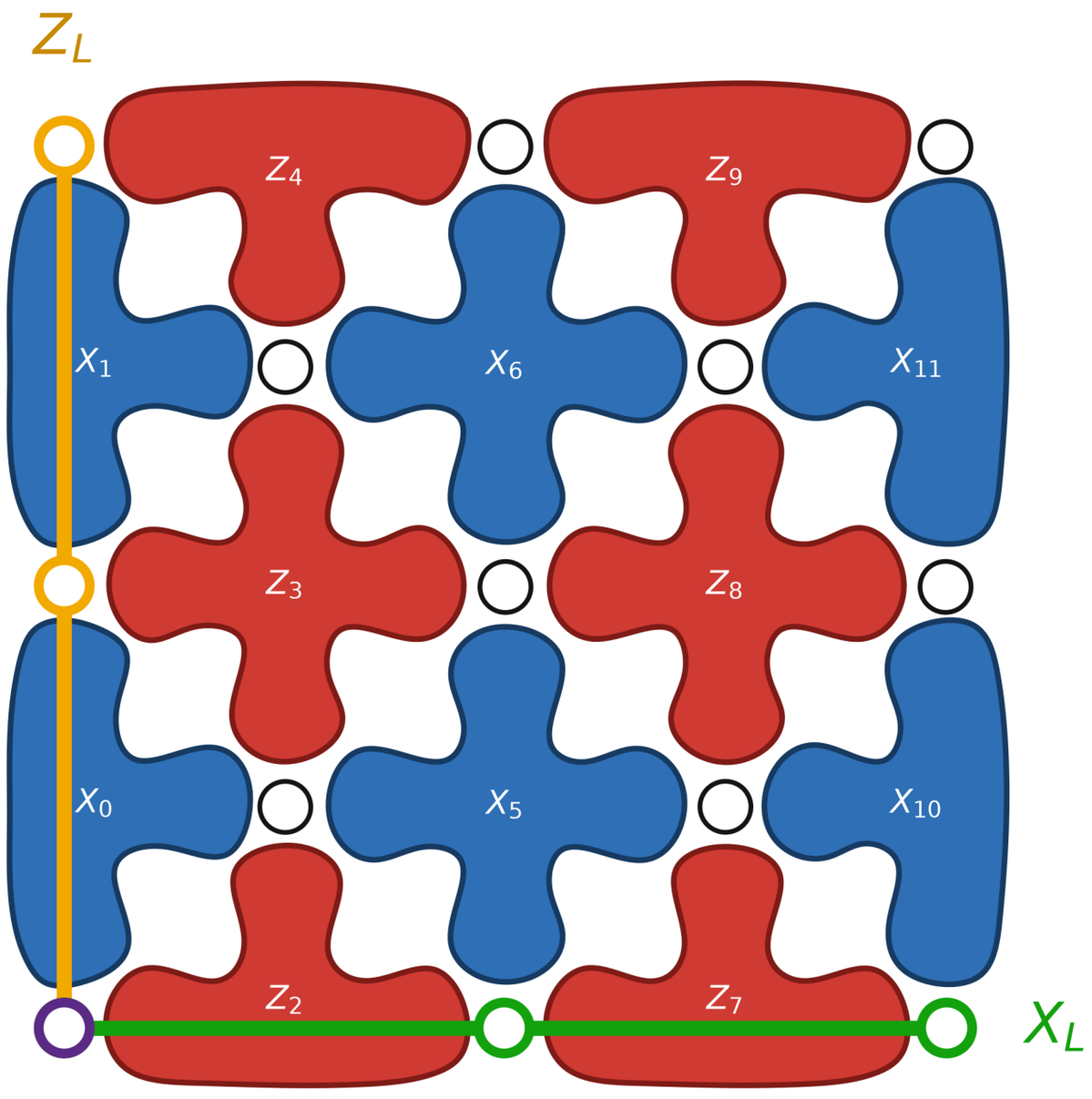
- There is a second set of states that give the same logical information and pass all the stabilizer tests

$$|0_L\rangle \propto (|000\rangle + |011\rangle + |101\rangle + |110\rangle)(|000\rangle + |011\rangle + |101\rangle + |110\rangle)(|000\rangle + |011\rangle + |101\rangle + |110\rangle)$$

$$|1_{\mathrm{L}}\rangle \propto (|001\rangle + |010\rangle + |100\rangle + |111\rangle)(|001\rangle + |010\rangle + |100\rangle + |111\rangle)(|001\rangle + |010\rangle + |100\rangle + |111\rangle)$$

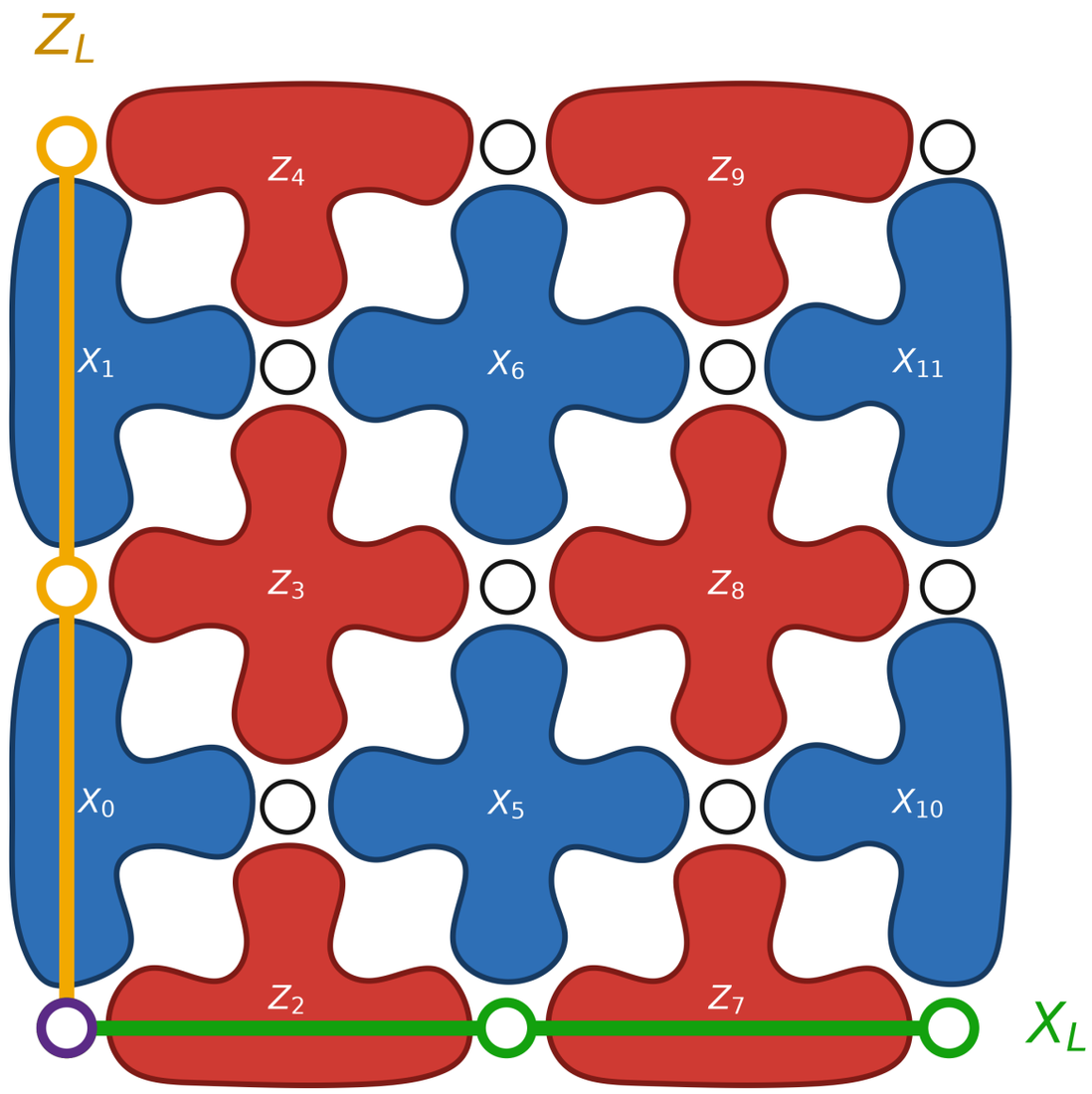
Surface codes

- Logical operators cross the entire codes
- Some errors do not change the logical state



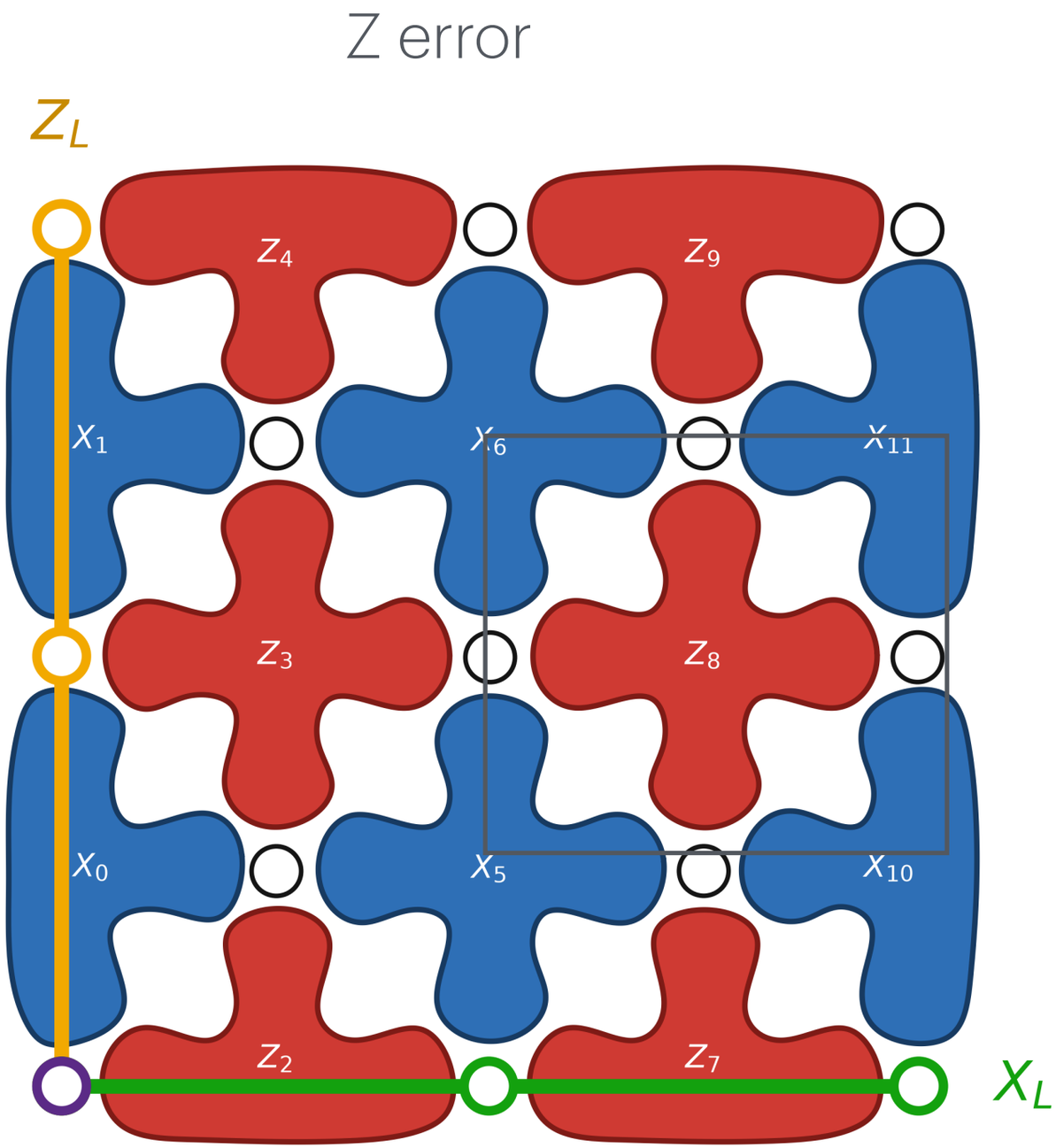
Surface codes

- There are tree types of error on surface codes
  - Degenerate errors
  - Correctable errors
  - And actual logical errors



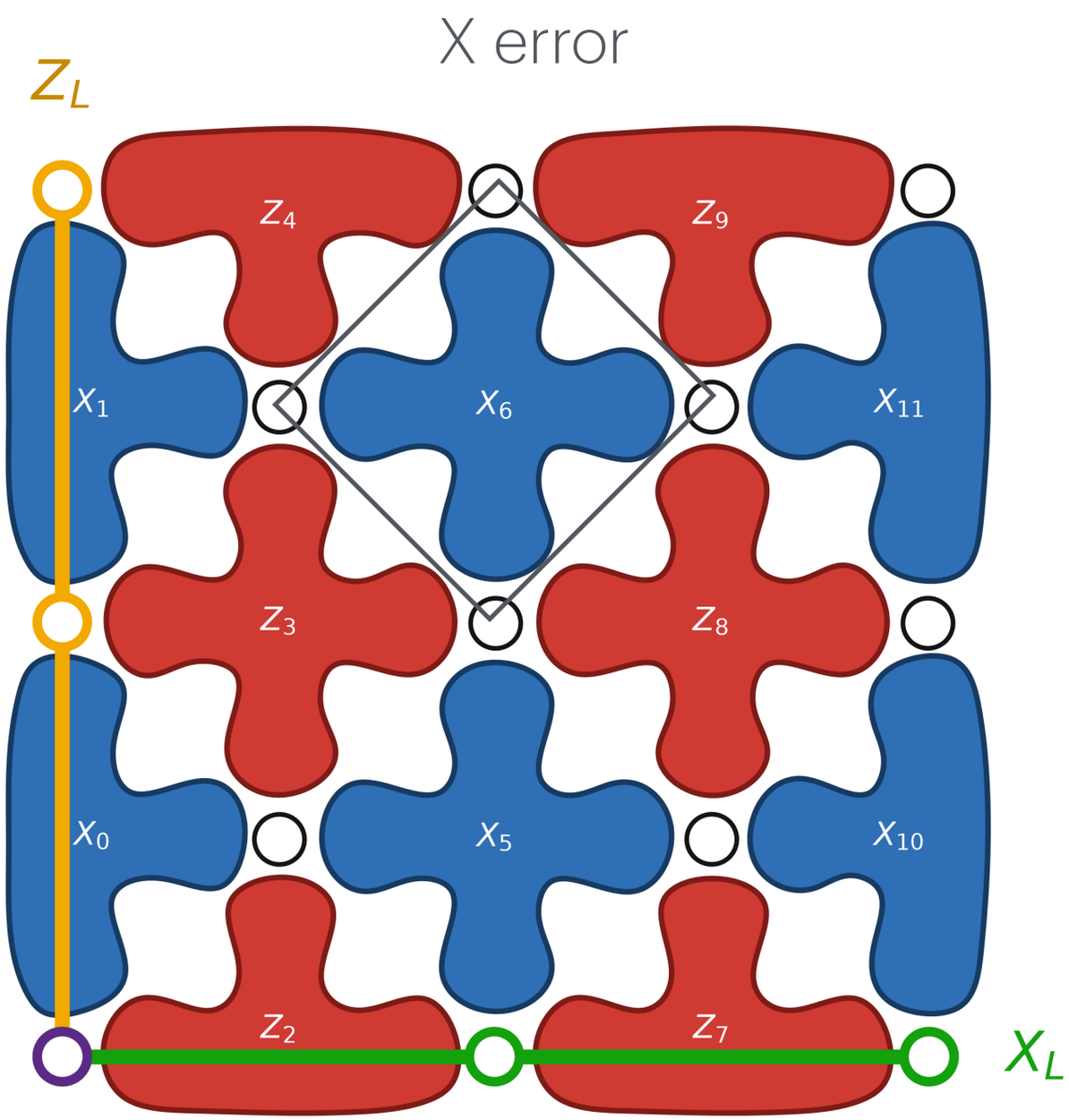
Surface codes

- There are tree types of error on surface codes
  - **Degenerate errors**
  - Correctable errors
  - And actual logical errors



Surface codes

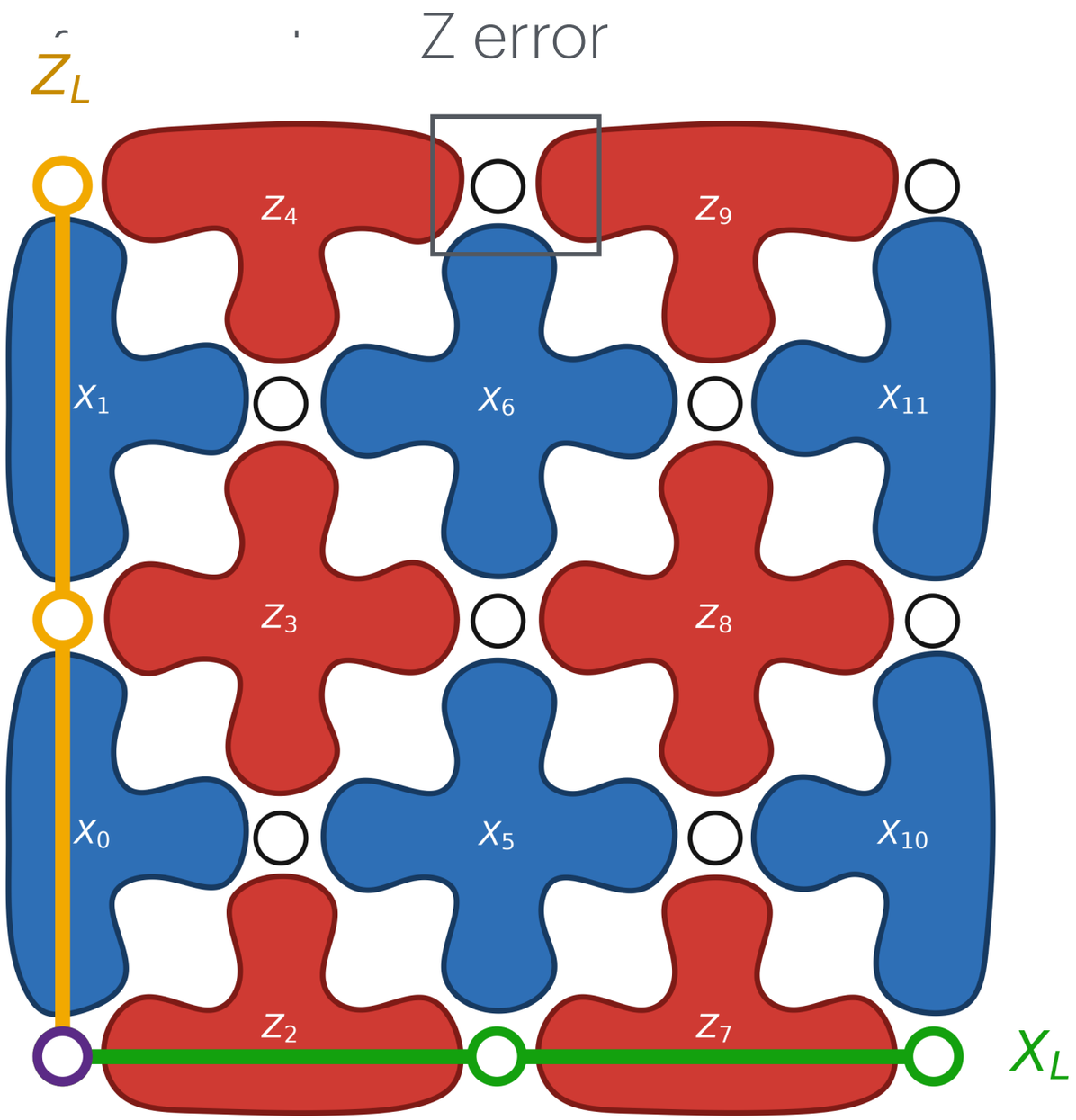
- There are tree types of error on surface codes
  - **Degenerate errors**
  - Correctable errors
  - An actual logical errors



Surface codes

- There are tree types of error on

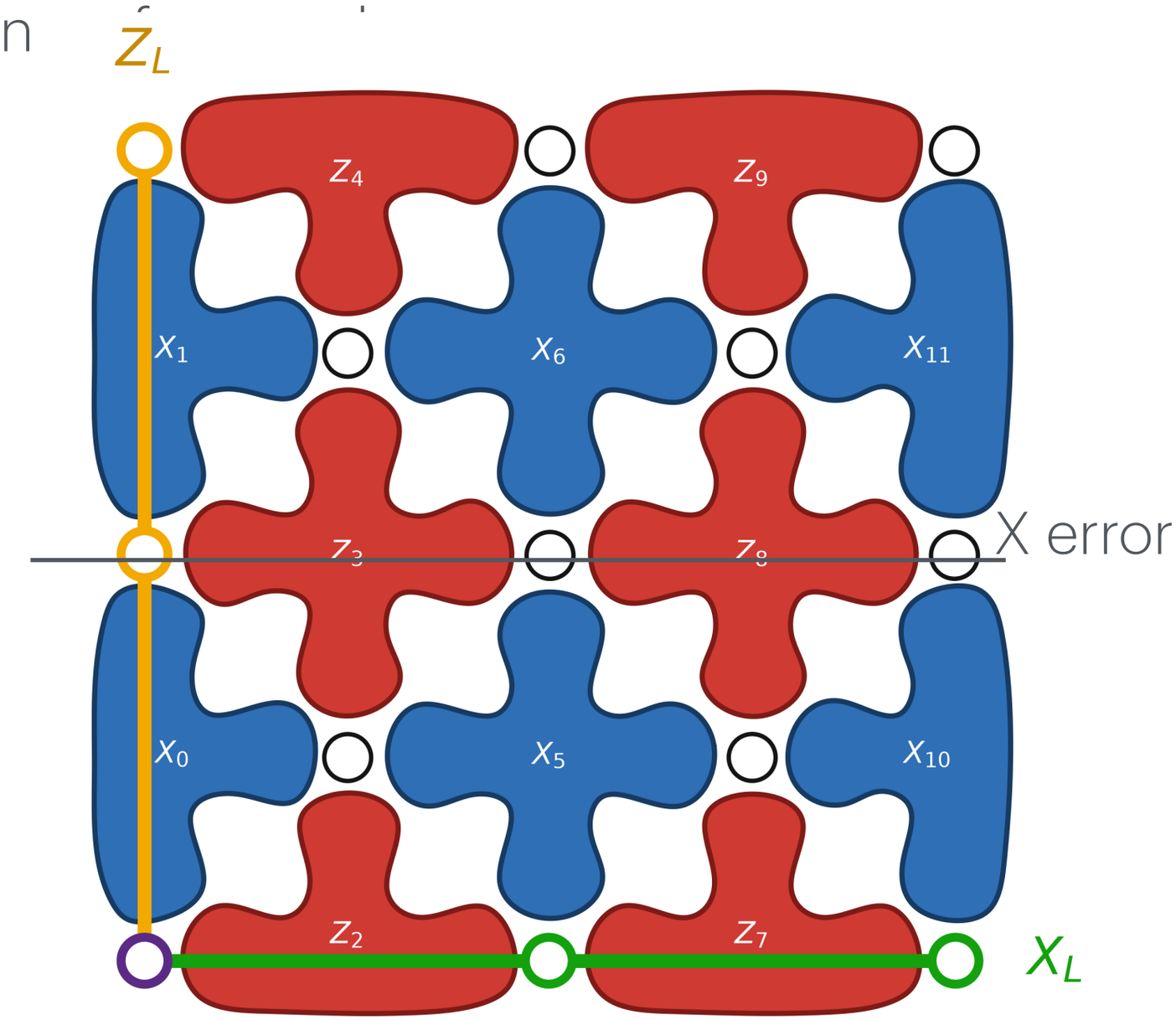
- Degenerate errors
- **Correctable errors**
- An actual logical errors



Surface codes

- There are tree types of error on

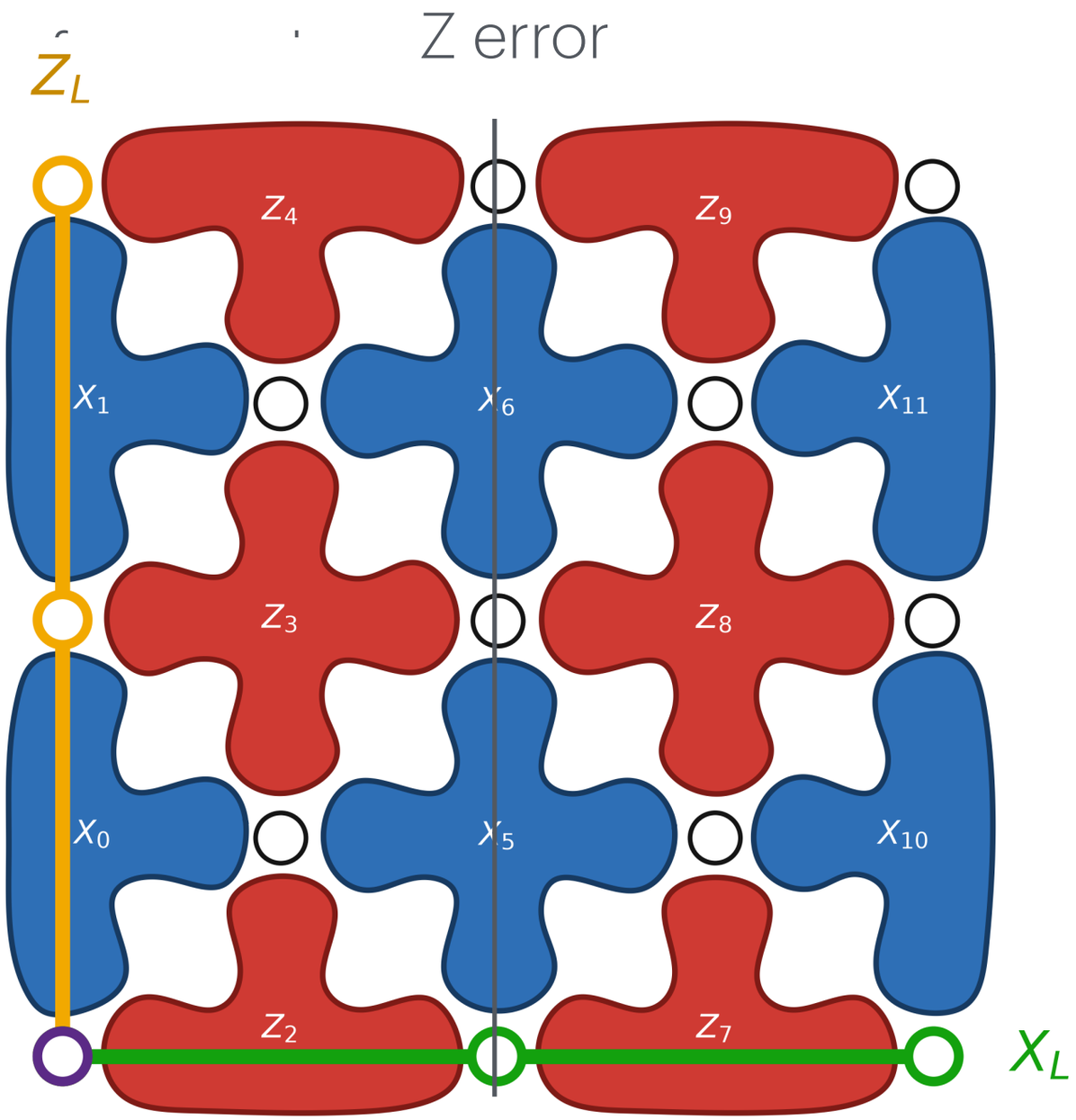
- Degenerate errors
- Correctable error
- **An actual logical errors**



Surface codes

- There are tree types of error on

- Degenerate errors
- Correctable error
- **And actual logical errors**



Surface codes

- Logical errors,
  - Let us suppose the real error is $e^x_{\text{real}} = [0001000000000]$
  - Our decoder, using the syndrome information predicted an error
    - $e^x_{\text{dec}} = [0010000000000]$
- The error is logical if and only if $d := e^x_{\text{real}} \ominus e^x_{\text{dec}} = e^x_{\text{real}} \oplus e^x_{\text{dec}}$
  - Is in $d \in \text{kernel}\mathrm{H_z} \backslash \text{span} H_x$

Decoders

- I did not talk about **decoders**
- A decoder is algorithm that takes into account the **syndrome** S and predict the **most likely** (or close to it) error that occurred e_{pred}
  - **Not always** the error that **actually** that occurred e_{real}
  - **Two error might** produce the **same syndrome** but have different **logical effects,**
    - and the decoder might predict the wrong ones
- **Decoders will be left for your projects**

Decoders

- Your project will be to **choose** a **code**:
  - Classical codes, surface codes, toric code, LDPC
- Simulate an error channel with **Stim**
  - bit flips, phase flips, etc.
  - Circuit-level noise for the brave ones (not recommended)
- Test a **decoder**
- Tesseract, belief propagation, BP+OSD, relay BP, matching
- Compute the **LER curves**

Decoders

- The project will be evaluated based on:
  - Understanding of the code used, rate, distance, and limitations
  - Understanding of the error model used
  - **Most importantly, understanding of the decoder used — especially its limitations**
- Better understanding of the concepts will be valued over choosing hard codes and decoders