# Quantum phase Estimation

Estimate eigenvalues of unitary operators
(without even diagonalizing the unitary)

Provable exponential / polynomial speedups

→ Factoring - Shor's Algorithm
→ solving linear systems of eq. - HHL
→ Eigenvalues of molecular Hamiltonians

Estimating eigenvalues

why is it a phase?

Let $M$ be a unitary/Hermitian matrix $\left(\begin{array}{c}\text{Hermitian} \\ M = M^\dagger\end{array}\right)$

$$\begin{cases} MM^\dagger = I = M^\dagger M & \text{(unitary)} \\ MM^\dagger = M^2 = M^\dagger M & \text{(Hermitian)} \end{cases}$$

They are normal matrices $MM^\dagger = M^\dagger M$.

Then spectral decomposition states that $\exists$ orthonormal basis $\{|\psi_k\rangle\}$ s.t.:

$$M = \sum_k \lambda_k |\psi_k\rangle\langle\psi_k|$$

$$M = \sum_k \lambda_k |\varphi_k \rangle\langle \varphi_k|$$

$$M |\varphi_j\rangle = \sum_k \lambda_k |\varphi_k\rangle\langle\varphi_k | |\varphi_j\rangle$$

$$= \lambda_j |\varphi_j\rangle$$

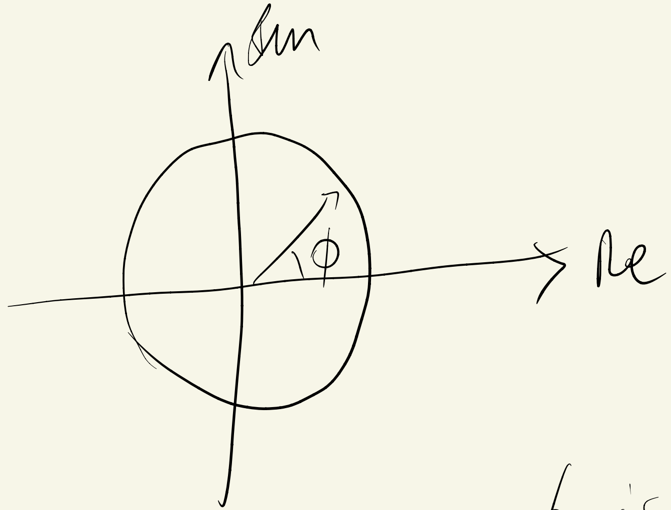
$$\boxed{M|\varphi_j\rangle = \lambda_j |\varphi_j\rangle}$$

Moreover, unitary operators <u>must</u> preserve norm of vectors:

$$\| U|\varphi_j\rangle \| = \| \lambda_j |\varphi_j\rangle \| = |\lambda| \, \| |\varphi_j\rangle \|$$

must $|\lambda| = 1$

lie on the unit circle




$$\lambda = \cos(\phi) + i\sin\phi$$
$$= e^{i\phi}$$

$\phi$ is $2\pi$ periodic $\quad \lambda = e^{2\pi i\theta}$

$$\boxed{U|\varphi_j\rangle = e^{2\pi i\theta}|\varphi_j\rangle}$$

QPE estimates $\theta$ in binary floating point representation.

Let $U_{2^n \times 2^n}$ be unitary and $K$ be the # bits for the approximation:

Opt :

1) Initialize $|0\rangle^n |0\rangle^K$

2) Encode eigenstate $|\psi_j\rangle |0\rangle^K$

3) Apply operator $\mathcal{O}$ s.t.:

$$\mathcal{O}\,|\psi_j\rangle|0\rangle^{k} \longmapsto |\psi_j\rangle \frac{1}{\sqrt{2^n}} \sum_{i=0} e^{2\pi i \theta_j i} |i\rangle$$

3) Apply operator $O$ s.t.:

$$O\,|\psi_j\rangle|0\rangle^{k} \longmapsto |\psi_j\rangle \frac{1}{\sqrt{2^n}} \sum_{i=0} e^{2\pi i \theta_j i} |i\rangle$$

Can we prepare it having only access to $U$?

$$|\theta\rangle \xmapsto{QFT} \frac{1}{\sqrt{2^n}} \sum_{j=0} e^{2\pi i \theta \cdot j} |j\rangle$$

$$|\theta\rangle \xmapsto{QFT} \frac{1}{\sqrt{2^n}} \sum_{j=0} e^{2\pi i \theta \cdot j} |j\rangle$$

$$\xleftarrow{QFT^{-1}}$$

Apply inverse gives us the k-bit approximation $|\tilde{\theta}\rangle$

4) Apply inverse QFT

# Quantum phase estimation:

Given unitary $U_{2^n \times 2^n}$, eigenstate $|\psi_j\rangle$, $K$ bits.

→ prepare $|0\rangle^n |0\rangle^k$

→ Encode eigenstate $|\psi_j\rangle |0\rangle^k$

→ prepare operator $O$ s.t

$$O\,|\psi_j\rangle|0\rangle^K \longmapsto |\psi_j\rangle \frac{1}{\sqrt{2^n}} \sum_{j=0} e^{2\pi i \theta_j} |j\rangle$$

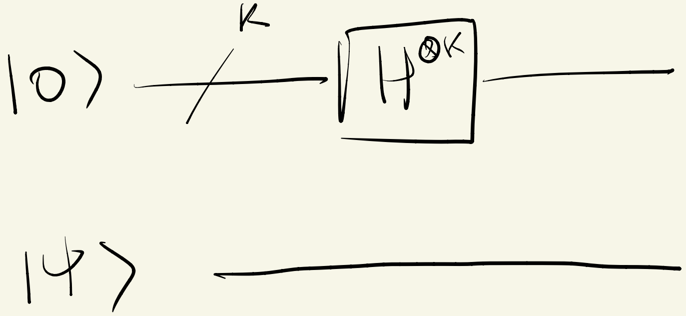
→ Apply inverse QFT to $k$ bit register

$$QFT^{-1} \frac{1}{\sqrt{2^n}} \sum_{j=0} e^{2\pi i \theta_j} |j\rangle \longmapsto |\tilde{\theta}\rangle$$

Encode the operator

0

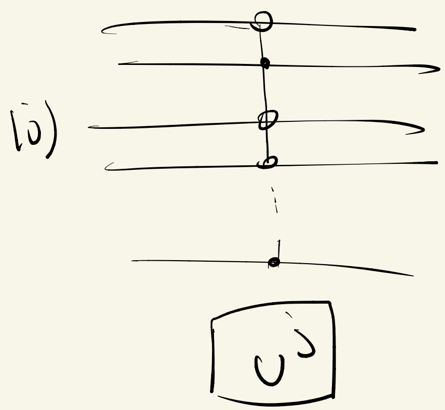
K-bits for approximation




$$\frac{1}{\sqrt{2^K}} \sum_j |j\rangle$$

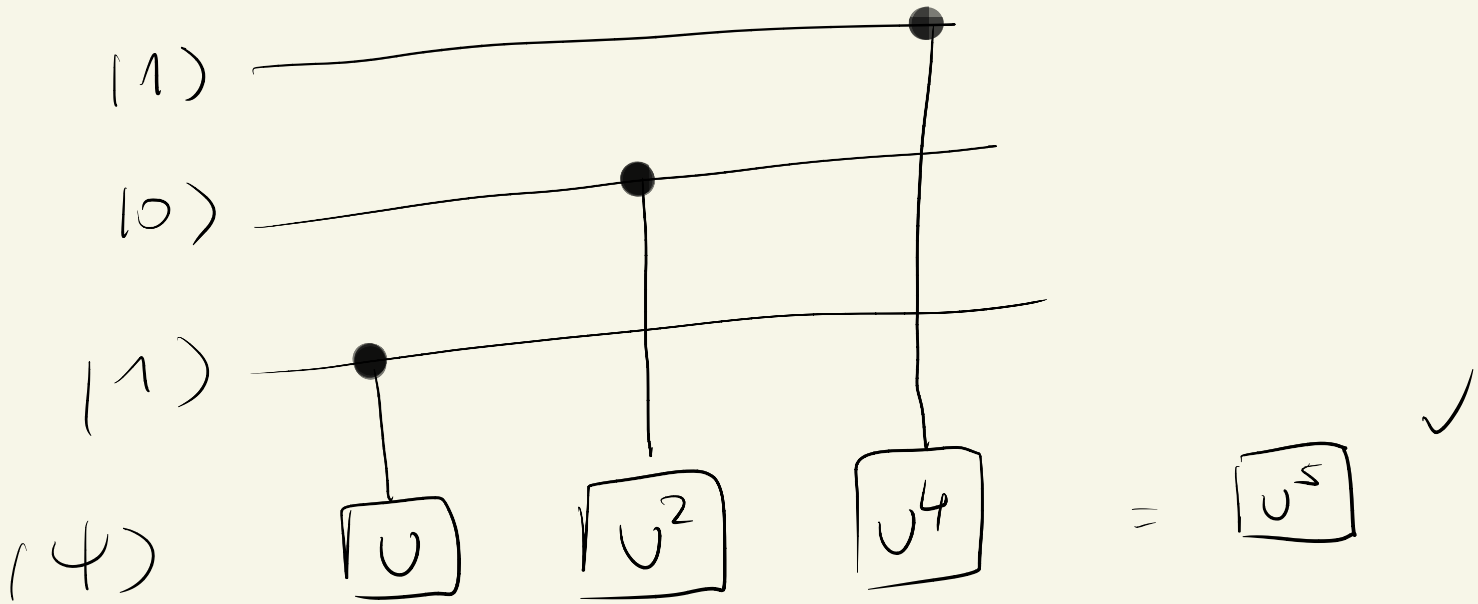
$$U^j |\psi\rangle |j\rangle \longmapsto e^{2\pi i \theta j} |\psi\rangle |j\rangle$$

for each $|j\rangle \rightarrow$




Problem: $O(2^K)$ multicontrolled operations.

let's imagine that $|j\rangle = |101\rangle = |5\rangle$




Generalization: For qubit $q$ in range $\{0, \ldots, K-1\}$

Apply $C\text{-}U^{2^q}$

Quantum Fourier Transform

has a product state

decomposition!

check Nielsen and Chuang Pag 218

$$
\begin{aligned}
|j\rangle &\to \frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} e^{2\pi i jk/2^n} |k\rangle && (5.5)\\
&= \frac{1}{2^{n/2}} \sum_{k_1=0}^{1} \dots \sum_{k_n=0}^{1} e^{2\pi i j \left(\sum_{l=1}^{n} k_l 2^{-l}\right)} |k_1 \dots k_n\rangle && (5.6)\\
&= \frac{1}{2^{n/2}} \sum_{k_1=0}^{1} \dots \sum_{k_n=0}^{1} \bigotimes_{l=1}^{n} e^{2\pi i j k_l 2^{-l}} |k_l\rangle && (5.7)\\
&= \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[ \sum_{k_l=0}^{1} e^{2\pi i j k_l 2^{-l}} |k_l\rangle \right] && (5.8)\\
&= \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[ |0\rangle + e^{2\pi i j 2^{-l}} |1\rangle \right] && (5.9)\\
&= \frac{\left(|0\rangle + e^{2\pi i 0.j_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i 0.j_{n-1}j_n}|1\rangle\right)\cdots\left(|0\rangle + e^{2\pi i 0.j_1 j_2 \cdots j_n}|1\rangle\right)}{2^{n/2}}. && (5.10)
\end{aligned}
$$

The product representation (5.4) makes it easy to derive an efficient circuit for the quantum Fourier transform. Such a circuit is shown in Figure 5.1. The gate $R_k$ denotes the unitary transformation

$$
R_k \equiv \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{bmatrix}. \tag{5.11}
$$

$$|j\rangle \rightarrow \frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} e^{2\pi i j k/2^n} |k\rangle \tag{5.5}$$

$$= \frac{1}{2^{n/2}} \sum_{k_1=0}^{1} \ldots \sum_{k_n=0}^{1} e^{2\pi i j \left(\sum_{l=1}^{n} k_l 2^{-l}\right)} |k_1 \ldots k_n\rangle \tag{5.6}$$

$$= \frac{1}{2^{n/2}} \sum_{k_1=0}^{1} \ldots \sum_{k_n=0}^{1} \bigotimes_{l=1}^{n} e^{2\pi i j k_l 2^{-l}} |k_l\rangle \tag{5.7}$$

H + c-$R_K$

For every control bit

$j$

$$= \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[ \sum_{k_l=0}^{1} e^{2\pi i j k_l 2^{-l}} |k_l\rangle \right] \tag{5.8}$$

$$= \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[ |0\rangle + e^{2\pi i j 2^{-l}} |1\rangle \right] \tag{5.9}$$

$$= \frac{\left(|0\rangle + e^{2\pi i 0.j_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i 0.j_{n-1}j_n}|1\rangle\right) \cdots \left(|0\rangle + e^{2\pi i 0.j_1 j_2 \cdots j_n}|1\rangle\right)}{2^{n/2}}. \tag{5.10}$$

The product representation (5.4) makes it easy to derive an efficient circuit for the quantum Fourier transform. Such a circuit is shown in Figure 5.1. The gate $R_k$ denotes the unitary transformation

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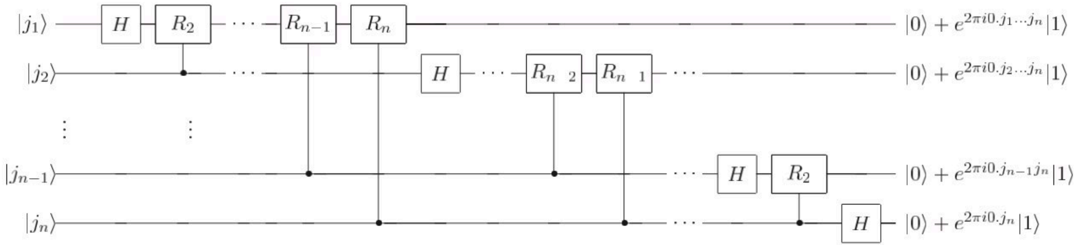
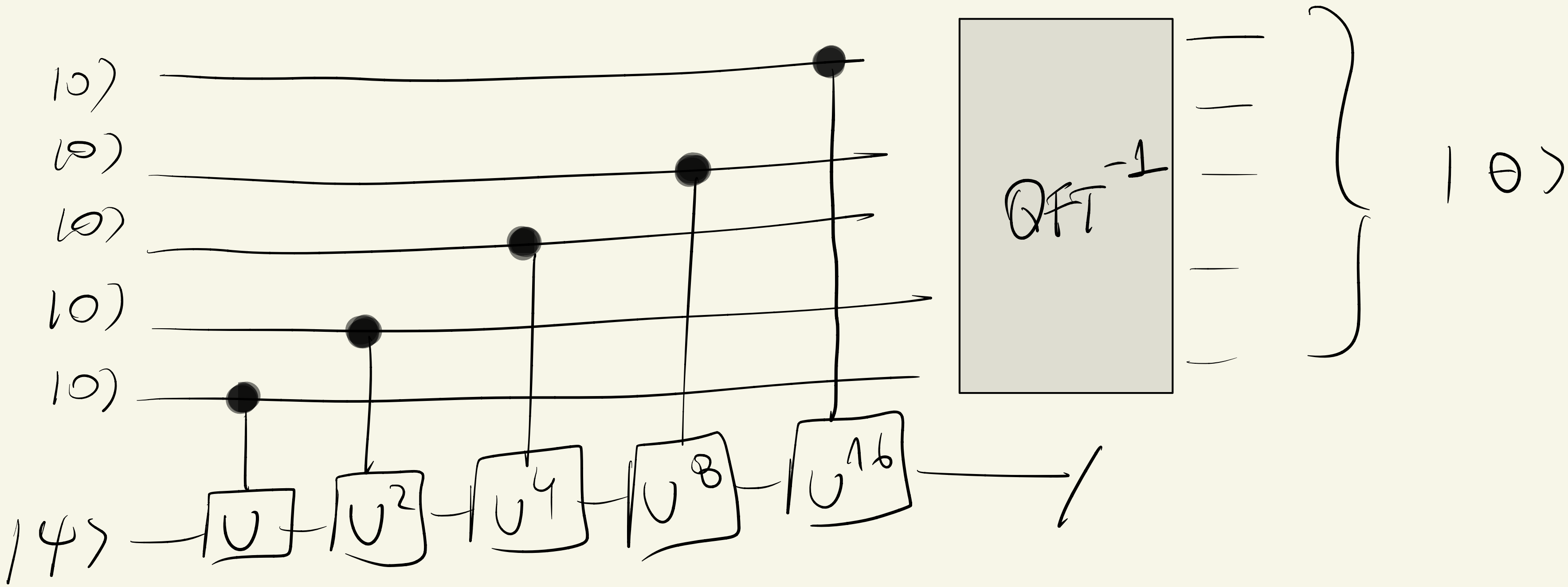



Figure 5.1. Efficient circuit for the quantum Fourier transform. This circuit is easily derived from the product representation (5.4) for the quantum Fourier transform. Not shown are swap gates at the end of the circuit which reverse the order of the qubits, or normalization factors of $1/\sqrt{2}$ in the output.

Use pennylane's

qml.QFT (wires = estimation_wires)

Phase estimation full circuit:




* For a given K,

$$P(\tilde{\theta}) \geq 4/\pi^2 > 40\%$$

* Accuracy $\varepsilon$ and Prob $> 1-\delta$

$$K > \log\left(\frac{1}{\varepsilon}\left(\frac{1}{2\delta}+2\right)\right)$$

## CAVEATS

How to implement both

- $U$ and $CU^{2^j}$

- eigenstate $|\psi\rangle$

**Exercise**: consider $U = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/5} \end{pmatrix}$

angle we want to estimate
$\theta = 1/5 = 0.2$

- qml.PhaseShift(angle, wire)

- eigenstate ?