Lecture 9:
Estimating eigenvalues:
An application of QFT

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The problem: Eigenvalue estimation

The eigenvalue estimation problem

Let $(|\psi\rangle, e^{i2\pi\phi})$, with $0 \leq \phi < 1$, be an eigenvector, eigenvalue pair for a unitary U. Determine ϕ.

Note that eigenvalues of unitary operators are always of this form. Why?

The strategy

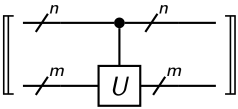
- Use a controlled version of U to prepare a state from which ϕ can be found.
- Then, resort to the inverse of the QFT to find it.
- The accuracy of the estimation increases with the number of qubits available for the recovery state

Thus, the problem reduces to the already discussed

phase estimation problem

The general case

A multi-controlled version of U is reguired:



$$= |x\rangle\,|y\rangle \mapsto |x\rangle\, U^{x}\,|y\rangle$$

decimal representation of x

Intuitively it applies U to $|y\rangle$ a number of times equal to x

Examples

$|10\rangle\,|y\rangle \mapsto |10\rangle\,(UU\,|y\rangle)$ and $|00\rangle\,|y\rangle \mapsto |00\rangle\,|y\rangle$

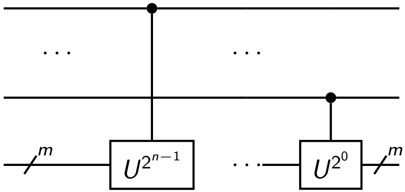
Note that $|\psi\rangle$ is also an eigenvector of U^{x}, with eigenvalue $e^{i2\pi x\phi}$, for any integer x.

Multi-controlled operations

Recall that a binary number $x_1 \dots x_n$ corresponds to the natural number

$$2^{n-1}x_1 + \cdots + 2^0 x_n$$

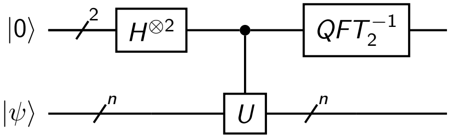
We use this to build the previous multi-controlled operation in terms of n 'simply'-controlled rotations U^{2^i}



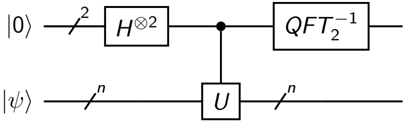
An Example

Take a unitary U with an eigenvector $|\psi\rangle$ whose eigenvalue is $e^{i2\pi\phi}$
ϕ is equal to one of the following values $\left\{0 \cdot \frac{1}{4}, 1 \cdot \frac{1}{4}, 2 \cdot \frac{1}{4}, 3 \cdot \frac{1}{4}\right\}$

The following circuit discovers ϕ



Another Example



$$|0\rangle\,|0\rangle$$

$$\overset{H^{\otimes 2}}{\mapsto}\ \frac{1}{\sqrt{2^2}}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

$$\overset{\text{ctrl. } U}{\mapsto}\ \frac{1}{\sqrt{2^2}}(|00\rangle + e^{i2\pi\phi}\,|01\rangle + e^{i2\pi\phi\cdot 2}\,|10\rangle + e^{i2\pi\phi\cdot 3}\,|11\rangle)$$

$$= \frac{1}{\sqrt{2^2}}(|00\rangle + e^{i2\pi x\cdot\frac{1}{4}}\,|01\rangle + e^{i2\pi x\cdot\frac{1}{4}\cdot 2}\,|10\rangle + e^{i2\pi x\cdot\frac{1}{4}\cdot 3}\,|11\rangle)$$

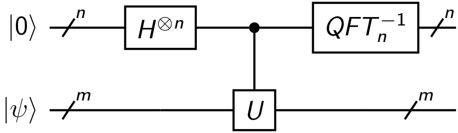
$$= \frac{1}{\sqrt{2^2}}(|00\rangle + \omega_2^{x}\,|01\rangle + \omega_2^{x\cdot 2}\,|10\rangle + \omega_2^{x\cdot 3}\,|11\rangle)$$

$$\overset{QFT_2^{-1}}{\mapsto}\ |x\rangle$$

Yet Another Example

Take a unitary U with eigenvector $|\psi\rangle$ whose eigenvalue is $e^{i2\pi\phi}$ st $\phi \in \left\{0 \cdot \frac{1}{2^n}, \ldots, 2^n - 1 \cdot \frac{1}{2^n}\right\}$

The following circuit returns x such that $\phi = x \cdot \frac{1}{2^n}$



Exercise

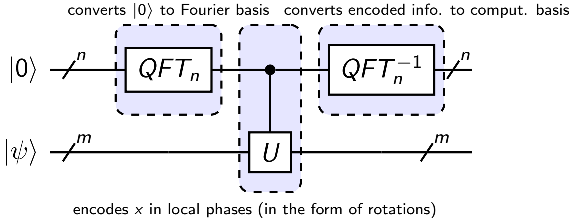
Prove that indeed the circuit returns x such that $\phi = x \cdot \frac{1}{2^n}$

Yet Another Example

Exercise

Show that $QFT_n |0\rangle = H^{\otimes n} |0\rangle$.

Note that this allows to rewrite the previous circuit in the one below



... but precision is Limited

We assumed $0 \leq \phi < 1$ takes a value from $\left\{0 \cdot \frac{1}{2^n}, \ldots, 2^n - 1 \cdot \frac{1}{2^n}\right\}$
... an assumption that arose from having only n qubits to estimate ...

But what to do if ϕ takes none of these values?

Return the n-bit number k with $k \cdot \frac{1}{2^n}$ the value above closest to ϕ

Is the circuit above up to this task?

Setting the stage

Let $\omega_n = e^{i2\pi \cdot \frac{1}{2^n}}$

and consider the following explicit definition. of QFT^{-1}

$$QFT_n^{-1} |x\rangle = \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} \omega_n^{-k \cdot x} |k\rangle$$

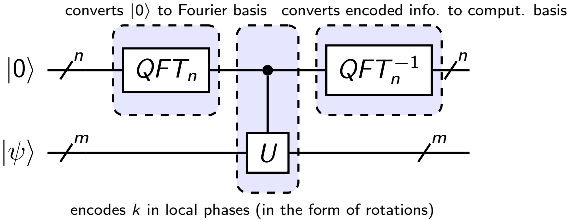
Setting the stage

Let $k \cdot \frac{1}{2^n}$ be the value in $\left\{0 \cdot \frac{1}{2^n}, \ldots, 2^n - 1 \cdot \frac{1}{2^n}\right\}$ closest to ϕ, i.e.

$$\exists_\epsilon \cdot 0 \leq |\epsilon| \leq \frac{1}{2^n} \quad \text{and} \quad k \cdot \frac{1}{2^n} + \epsilon = \phi$$

Note that the difference ϵ decreases when the number of qubits increases.

Recall the circuit



Computing the output again

$$|0\rangle$$

$$\overset{H^{\otimes n}}{\mapsto} \frac{1}{\sqrt{2^n}}(|0\rangle + |1\rangle + \cdots + |2^n - 1\rangle)$$

$$\overset{\text{ctrl.}\, U}{\mapsto} \frac{1}{\sqrt{2^n}}\left(|0\rangle + e^{i2\pi\phi\cdot 1} |1\rangle + \cdots + e^{i2\pi\phi\cdot 2^{n-1}} |2^n - 1\rangle \right)$$

$$= \frac{1}{\sqrt{2^n}}\left(|0\rangle + e^{i2\pi(k\cdot\frac{1}{2^n}+\epsilon)\cdot 1} |1\rangle + \cdots + e^{i2\pi(k\cdot\frac{1}{2^n}+\epsilon)\cdot 2^{n-1}} |2^n - 1\rangle \right)$$

$$= \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} e^{i2\pi(k\cdot\frac{1}{2^n}+\epsilon)\cdot j} |j\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} e^{i2\pi k\cdot\frac{1}{2^n}\cdot j} e^{i2\pi\epsilon\cdot j} |j\rangle$$

$$\overset{QFT^{-1}}{\mapsto} \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} e^{i2\pi k\cdot\frac{1}{2^n}\cdot j} e^{i2\pi\epsilon\cdot j} \left(\frac{1}{\sqrt{2^n}} \sum_{l=0}^{2^n-1} e^{-i2\pi j\cdot\frac{1}{2^n}\cdot l} |l\rangle \right)$$

$$= \frac{1}{2^n} \sum_{j=0}^{2^n-1} e^{i2\pi k\cdot\frac{1}{2^n}\cdot j} e^{i2\pi\epsilon\cdot j} \left(\sum_{l=0}^{2^n-1} e^{-i2\pi j\cdot\frac{1}{2^n}\cdot l} |l\rangle \right)$$

$$= \frac{1}{2^n} \sum_{j=0}^{2^n-1} \sum_{l=0}^{2^n-1} e^{i2\pi\epsilon\cdot j} e^{i2\pi j\cdot\frac{1}{2^n}\cdot(k-l)} |l\rangle$$

Looking into the final state

The amplitude of $|k\rangle$ is

$$\frac{1}{2^n}\sum_{j=0}^{2^n-1} e^{i2\pi\epsilon\cdot j}$$

which is a finite geometric series.

Therefore,

$$\frac{1}{2^n}\sum_{j=0}^{2^n-1} e^{i2\pi\epsilon j} = \begin{cases} 1 & \text{if } \epsilon = 0 \\ \frac{1}{2^n}\frac{1-e^{i2\pi\epsilon 2^n}}{1-e^{i2\pi\epsilon}} & \text{if } \epsilon \neq 0 \end{cases}$$

Let us proceed under the assumption $\epsilon \neq 0$.

A geometric detour

$|1 - e^{i\theta}|$ for some angle θ is the Euclidean distance between 1 and $e^{i\theta}$ (length of the straight line segment between both points)

Consider also arc length θ between 1 and $e^{i\theta}$ (distance between the two points by running along the unit circle)

Theorem

Let d^E and d^a be respectively the Euclidean distance and arc length between 1 and $e^{i\theta}$. Then,

a. $d^E \leq d^a$

b. *if* $0 \leq \theta \leq \pi$ *we have* $\frac{d^a}{d^E} \leq \frac{\pi}{2}$

Finally!

Recall $\left|\frac{1}{2^n}\frac{1-e^{i2\pi\epsilon 2^n}}{1-e^{i2\pi\epsilon}}\right|^2$ is the probability of measuring $|k\rangle$

$$
\begin{aligned}
\left|\frac{1}{2^n}\frac{1-e^{i2\pi\epsilon 2^n}}{1-e^{i2\pi\epsilon}}\right|^2 &= \left(\frac{1}{2^n}\right)^2 \frac{\left|1-e^{i2\pi\epsilon 2^n}\right|^2}{\left|1-e^{i2\pi\epsilon}\right|^2} \\
&\geq \left(\frac{1}{2^n}\right)^2 \frac{\left|1-e^{i2\pi\epsilon 2^n}\right|^2}{(2\pi\epsilon)^2} && \{\text{Thm a.}\} \\
&\geq \left(\frac{1}{2^n}\right)^2 \frac{\left(\frac{2}{\pi}\cdot 2\pi\epsilon 2^n\right)^2}{(2\pi\epsilon)^2} && \{\text{Thm b.}\} \\
&= \left(\frac{1}{2^n}\right)^2 \frac{(4\epsilon 2^n)^2}{(2\pi\epsilon)^2} \\
&= \left(\frac{1}{2^n}\right)^2 \frac{(2\cdot 2^n)^2}{\pi^2} = \frac{2^2}{\pi^2} = \frac{4}{\pi^2}
\end{aligned}
$$

Working with a superposition of eigenvectors

The algorithm requires an eigenvector as input,
but sometimes is highly difficult to build such a vector.

Often it is easier to feed instead a superposition of eigenvectors.

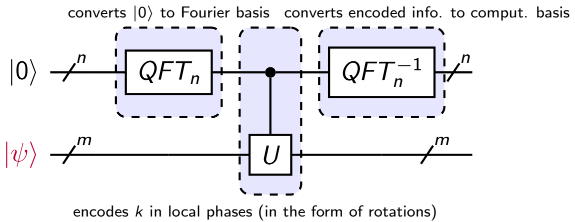
Indeed, by the spectral theorem one knows that the eigenvectors $\{|v_1\rangle, \ldots, |v_N\rangle\}$ of U (with associated eigenvalues $e^{i2\pi\phi_1}, \ldots, e^{i2\pi\phi_N}$) form a basis for the $N(= 2^n)$-dimensional vector space on which U acts.

Thus, one may define

$$|\psi\rangle = \frac{1}{\sqrt{N}}(|v_1\rangle + \cdots + |v_N\rangle)$$

to feed the circuit

Working with a superposition of eigenvectors



Exercise

Show that if $\forall_{i \leq N} \cdot \phi_i \in \left\{0 \cdot \frac{1}{2^n}, \ldots, 2^n - 1 \cdot \frac{1}{2^n}\right\}$ then the circuit's output is

$$\frac{1}{\sqrt{N}}\Big(|x_1\rangle\, |v_1\rangle + \cdots + |x_N\rangle\, |v_N\rangle \Big) \qquad \left(\phi_i = x_i \cdot \frac{1}{2^n}\right)$$