Quantum Computation
Revisiting the quantum Fourier transform

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Recap

The previous lecture discussed an algorithm to extract the phase factor $w \in [0, 1[$ from a generic n-qubit quantum state. Writing w as $\frac{x}{2^n}$, for x an integer representable in n qubits, the estimation process was described by

$$\frac{1}{\sqrt{2^n}} \sum_{y \in 2^n} e^{2\pi i (\frac{x}{2^n}) y} |y\rangle \quad \rightsquigarrow \quad |x\rangle$$

Its inverse is QFT, the quantum Fourier transform, a most useful routine in Quantum Computation.

Let us revisit its construction in a systematic way.

The quantum Fourier transform

Essentially, the QFT performs a change-of-basis operation which encodes information of computational basis states in local phases.

For 1 qubit state this is exactly what the Hadamard gate accomplishes:

$$H|0\rangle = \frac{1}{\sqrt{2}}\big(\,|0\rangle + 1\,|1\rangle\,\big) \qquad H|1\rangle = \frac{1}{\sqrt{2}}\big(\,|0\rangle + (-1)\,|1\rangle\,\big)$$

QFT: 1 qubit

Thus, $QFT_1 = H$:

$$QFT_1 |0\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + 1 |1\rangle \big) \qquad QFT_1 |1\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + (-1) |1\rangle \big)$$

Operation H^{-1} allows to extract information encoded in local phases

↓ $= H$

Exercise

Let $\omega_1 = e^{i2\pi\frac{1}{2}}$. Show that $QFT_1 |x\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + \omega_1^{1\cdot x} |1\rangle \Big)$

↓ angle of π radians

QFT: 1 qubit

Note that ω_1 represents a rotation of π radians, diving the unit circle into two slices..

Actually, the two 2^{th}-roots of the identity are

$$\omega_1^0 = 1 \quad \text{and} \quad \omega_1^1 = e^{\frac{i2\pi}{2}} = e^{i\pi} = -1$$

Also note that

$$\omega_1^{1.x} \; = \; e^{\frac{i2\pi x}{2}} \; = \; e^{i2\pi \frac{x}{2}} \; = \; e^{i2\pi(0.x)}$$

as used in the previous lecture.

QFT: 2 qubits

Let $\omega_2 = e^{i2\pi\frac{1}{4}}$

$$QFT_2\,|00\rangle = \frac{1}{\sqrt{2}}\big(\,|0\rangle + \omega_2^{2\cdot 0}\,|1\rangle\,\big) \otimes \frac{1}{\sqrt{2}}\big(\,|0\rangle + \omega_2^{1\cdot 0}\,|1\rangle\,\big)$$
$$QFT_2\,|01\rangle = \frac{1}{\sqrt{2}}\big(\,|0\rangle + \omega_2^{2\cdot 1}\,|1\rangle\,\big) \otimes \frac{1}{\sqrt{2}}\big(\,|0\rangle + \omega_2^{1\cdot 1}\,|1\rangle\,\big)$$
$$QFT_2\,|10\rangle = \frac{1}{\sqrt{2}}\big(\,|0\rangle + \omega_2^{2\cdot 2}\,|1\rangle\,\big) \otimes \frac{1}{\sqrt{2}}\big(\,|0\rangle + \omega_2^{1\cdot 2}\,|1\rangle\,\big)$$
$$QFT_2\,|11\rangle = \frac{1}{\sqrt{2}}\big(\,|0\rangle + \omega_2^{2\cdot 3}\,|1\rangle\,\big) \otimes \frac{1}{\sqrt{2}}\big(\,|0\rangle + \omega_2^{1\cdot 3}\,|1\rangle\,\big)$$

In general

$$QFT_2\,|x\rangle = \frac{1}{\sqrt{2}}\big(\,|0\rangle + \omega_2^{2\cdot x}\,|1\rangle\,\big) \otimes \frac{1}{\sqrt{2}}\big(\,|0\rangle + \omega_2^{1\cdot x}\,|1\rangle\,\big)$$

Exercise

Show that, for $x = |x_1x_2\rangle$, $QFT_2\,|x\rangle$ can be written as

$$QFT_2\,|x\rangle = \frac{1}{\sqrt{2}}\big(\,|0\rangle + e^{i2\pi(0.x1)}\,|1\rangle\,\big) \otimes \frac{1}{\sqrt{2}}\big(\,|0\rangle + e^{i2\pi(0.x_1x_2)}\,|1\rangle\,\big)$$

QFT: 2 qubits

Exercise

Compute the phase coeficients in the expressions above and use Bloch sphere to study $QFT_2 \left|x\right\rangle$.

Hint

$$\begin{array}{lll} \omega_2^{2.0} & = & 1 \\ \omega_2^{2.1} & = & -1 \\ \omega_2^{2.2} & = & 1 \\ \omega_2^{2.3} & = & -1 \end{array} \qquad \begin{array}{lll} \omega_2^{1.0} & = & 1 \\ \omega_2^{1.1} & = & e^{i\frac{\pi}{2}} \\ \omega_2^{1.2} & = & -1 \\ \omega_2^{1.3} & = & e^{i\frac{3}{2}\pi} \end{array}$$

Note that

- for every ω_2-rotation on the second qubit there are *two* such rotations on the first qubit
- $\omega_2^2 = \omega_1$, or, in general, $\omega_n^2 = \omega_{n-1}$

QFT: 2 qubits

In order to derive a circuit for QFT_2, compute

$$
\begin{aligned}
QFT_2 \left|x\right\rangle &= \tfrac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{2\cdot x} \left|1\right\rangle \big) \otimes \tfrac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{1\cdot x} \left|1\right\rangle \big) \\
&= \tfrac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{2(2x_1+x_2)} \left|1\right\rangle \big) \otimes \tfrac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{2x_1+x_2} \left|1\right\rangle \big) \\
&= \tfrac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{4x_1+2x_2} \left|1\right\rangle \big) \otimes \tfrac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{2x_1+x_2} \left|1\right\rangle \big) \\
&= \tfrac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{4x_1}\omega_2^{2x_2} \left|1\right\rangle \big) \otimes \tfrac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{2x_1}\omega_2^{x_2} \left|1\right\rangle \big) \\
&= \tfrac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{2x_2} \left|1\right\rangle \big) \otimes \tfrac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{2x_1}\omega_2^{x_2} \left|1\right\rangle \big) \\
&= \underbrace{\tfrac{1}{\sqrt{2}}\big(\left|0\right\rangle + (-1)^{x_2} \left|1\right\rangle \big)}_{H|x2\rangle} \otimes \underbrace{\tfrac{1}{\sqrt{2}}\big(\left|0\right\rangle + (-1)^{x_1}\omega_2^{x_2} \left|1\right\rangle \big)}_{\text{some controlled rot. on } H|x1\rangle}
\end{aligned}
$$

QFT: 2 qubits

Define

$$R_2 |0\rangle = |0\rangle \quad \text{and} \quad R_2 |1\rangle = \omega_2 |1\rangle$$

which rotates a vector in the xz-plane $\frac{\pi}{2}$ radians

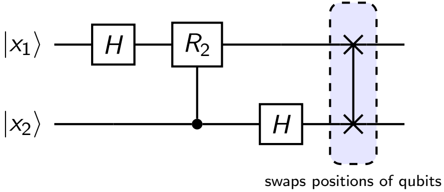
It yields a controlled-R_2 operation

$$|x\rangle |0\rangle \mapsto |x\rangle |0\rangle \qquad |x\rangle |1\rangle \mapsto R_2 |x\rangle |1\rangle$$

or, equivalently,

$$|0\rangle |y\rangle \mapsto |0\rangle |y\rangle \qquad |1\rangle |y\rangle \mapsto \omega_2^y |1\rangle |y\rangle$$

Putting all pieces together to derive the QFT circuit for 2 qubits:



QFT: 3 qubits

$$QFT_3 |x\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + \omega_3^{4\cdot x} |1\rangle \big) \otimes \big(|0\rangle + \omega_3^{2\cdot x} |1\rangle \big) \otimes \big(|0\rangle + \omega_3^{1\cdot x} |1\rangle \big)$$

for $\omega_n = e^{i2\pi \cdot \frac{1}{2^n}}$.

N.B.
In the sequel the normalisation factor $\frac{1}{\sqrt{2}}$ will be dropped in each state to increase readability

QFT: 3 qubits

Recalling that a binary number $x_1 \ldots x_n$ represents the natural number

$$2^{n-1} \cdot x_1 + \cdots + 2^0 \cdot x_n$$

and that

$$\omega_n^2 = \omega_{n-1} \text{ and thus } \omega_n^{2^{n-1}} = e^{i\pi} = -1$$

define QFT_3 as follows:

QFT: 3 Qubits

$$
\begin{aligned}
&QFT_3 |x\rangle \\
&= \big(\,|0\rangle + \omega_3^{4\cdot x}|1\rangle\,\big) \otimes \big(\,|0\rangle + \omega_3^{2\cdot x}|1\rangle\,\big) \otimes \big(\,|0\rangle + \omega_3^{1\cdot x}|1\rangle\,\big) \\
&= \big(\,|0\rangle + (-1)^{x}|1\rangle\,\big) \otimes \big(\,|0\rangle + \omega_3^{2\cdot x}|1\rangle\,\big) \otimes \big(\,|0\rangle + \omega_3^{1\cdot x}|1\rangle\,\big) \\
&= \big(\,|0\rangle + (-1)^{x_3}|1\rangle\,\big) \otimes \big(\,|0\rangle + \omega_3^{2\cdot x}|1\rangle\,\big) \otimes \big(\,|0\rangle + \omega_3^{1\cdot x}|1\rangle\,\big) \\
&= H|x_3\rangle \otimes \big(\,|0\rangle + \omega_3^{2\cdot(4x_1+2x_2+x_3)}|1\rangle\,\big) \otimes \big(\,|0\rangle + \omega_3^{1\cdot x}|1\rangle\,\big) \\
&= H|x_3\rangle \otimes \big(\,|0\rangle + \omega_3^{2\cdot(4x_1+2x_2)}\omega_3^{2\cdot x_3}|1\rangle\,\big) \otimes \big(\,|0\rangle + \omega_3^{1\cdot x}|1\rangle\,\big) \\
&= H|x_3\rangle \otimes \big(\,|0\rangle + \omega_2^{2\cdot(2x_1+x_2)}\omega_2^{x_3}|1\rangle\,\big) \otimes \big(\,|0\rangle + \omega_3^{4x_1+2x_2+x_3}|1\rangle\,\big) \\
&= H|x_3\rangle \otimes \big(\,|0\rangle + \omega_2^{2\cdot(2x_1+x_2)}\omega_2^{x_3}|1\rangle\,\big) \otimes \big(\,|0\rangle + \omega_3^{4x_1+2x_2}\omega_3^{x_3}|1\rangle\,\big) \\
&= H|x_3\rangle \otimes \big(\,|0\rangle + \omega_2^{2\cdot(2x_1+x_2)}\omega_2^{x_3}|1\rangle\,\big) \otimes \big(\,|0\rangle + \omega_3^{2\cdot(2x_1+x_2)}\omega_3^{x_3}|1\rangle\,\big) \\
&= H|x_3\rangle \otimes \underbrace{\big(\,|0\rangle + \omega_2^{2\cdot(2x_1+x_2)}\omega_2^{x_3}|1\rangle\,\big) \otimes \big(\,|0\rangle + \omega_2^{2x_1+x_2}\omega_3^{x_3}|1\rangle\,\big)}_{\text{some controlled-rotations on } QFT_2|x_1x_2\rangle}
\end{aligned}
$$

QFT: 3 qubits

Take $R_3 |0\rangle = |0\rangle$ and $R_3 |1\rangle = \omega_3 |1\rangle$. Intuitively, R_3 rotates a vector in the xz-plane 'one 2^3-th of the unit circle'.
It yields a controlled-R_3 operation defined by

$$|x\rangle |0\rangle \mapsto |x\rangle |0\rangle \text{ and } |x\rangle |1\rangle \mapsto R_3 |x\rangle |1\rangle$$

Equivalently

$$|0\rangle |y\rangle \mapsto |0\rangle |y\rangle \text{ and } |1\rangle |y\rangle \mapsto \omega_3^y |1\rangle |y\rangle$$

Putting all pieces together we derive the QFT circuit for 3 qubits

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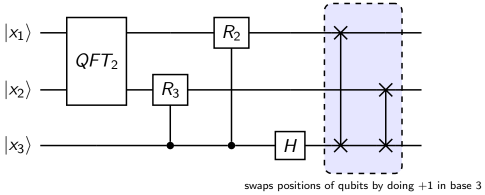


QFT: n qubits

Calculation easily extends to QFT_n (*in lieu* of QFT_3) :

Let $\omega_n = e^{i2\pi \cdot \frac{1}{2^n}}$ (division of the unit circle in 2^n slices)

$$QFT_n |x\rangle = \frac{1}{\sqrt{2}} \big(|0\rangle + \omega_n^{2^{n-1} \cdot x} |1\rangle \big) \otimes \cdots \otimes \big(|0\rangle + \omega_n^{2^0 \cdot x} |1\rangle \big)$$

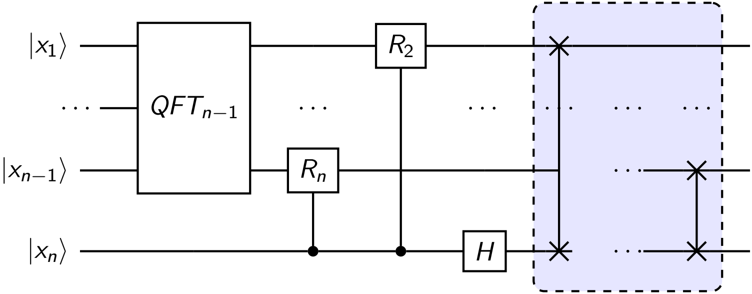
Take $R_n |0\rangle = |0\rangle$ and $R_n |1\rangle = \omega_n |1\rangle$. Intuitively, R_n rotates a vector in the xz-plane 'one 2^n-th of the unit circle'

It yields a controlled-R_n operation defined by $|x\rangle |0\rangle \mapsto |x\rangle |0\rangle$ and $|x\rangle |1\rangle \mapsto R_n |x\rangle |1\rangle$. Equivalently

$$|0\rangle |y\rangle \mapsto |0\rangle |y\rangle \text{ and } |1\rangle |y\rangle \mapsto \omega_n^y |1\rangle |y\rangle$$

QFT: n qubits

This suggests a recursive definition for the general *QFT* circuit:



swaps positions of qubits by doing $+1$ in base n

An equivalent formulation of QFT

Although we have been working with

$$QFT_n |x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + \omega_n^{2^{n-1}\cdot x} |1\rangle) \otimes \cdots \otimes \frac{1}{\sqrt{2}}(|0\rangle + \omega_n^{1\cdot x} |1\rangle)$$

we are already familiar with an equivalent, useful definition

$$QFT_n |x\rangle = \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} \omega_n^{k\cdot x} |k\rangle$$

Examples with $n = 1$ and $n = 2$

$$QFT_1 |x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + \omega_1^{x} |1\rangle)$$
$$QFT_2 |x\rangle = \frac{1}{\sqrt{2^2}}(|00\rangle + \omega_2^{x} |01\rangle + \omega_2^{2\cdot x} |10\rangle + \omega_2^{3\cdot x} |11\rangle)$$