Quantum Computation
Estimating eigenvalues: An application of QFT

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The problem: Eigenvalue estimation

Several algorithms previously discussed (Simon, Deutsch-Joza, etc) resort to the following technique:

- Take a controlled version of an operator U and prepare the target qubit with an eigenvector,
- win order to push up (or kick back) the associated eigenvalue to the state of the control qubit as in

$$cU\,(a_0|0\rangle + a_1|1\rangle)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = \left((-1)^{f(0)}a_0|0\rangle + (-1)^{f(1)}a_1|1\rangle\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

The problem: Eigenvalue estimation

The question

Can this technique be generalised to estimate the eigenvalues of an arbitrary, *n*-qubit unitary operator U?

The eigenvalue estimation problem

Let $(|\psi\rangle, e^{i2\pi\phi})$, with $0 \leq \phi < 1$, be an eigenvector, eigenvalue pair for a unitary U. Determine ϕ.

Note that eigenvalues of unitary operators are always of this form.

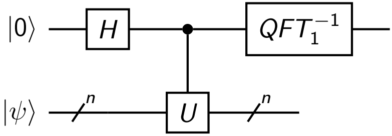
The strategy

Use a controled version of U to prepare a state from which ϕ can be found through the inverse of the QFT
(recall the phase estimation problem already discussed)

A Simple Example

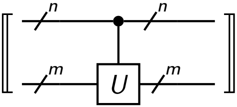
Take a unitary U with an eigenvector $|\psi\rangle$ whose eigenvalue is $e^{i2\pi\phi}$ ϕ is equal to one of the values $\{0 \cdot \frac{1}{2}, 1 \cdot \frac{1}{2}\}$. Find out ϕ.

This is obtained via the circuit



The general case

In less trivial cases, a multi-controlled version of U is reguired:



$= |x\rangle |y\rangle \mapsto |x\rangle U^x |y\rangle$

Intuitively it applies U to $|y\rangle$ a number of times equal to x

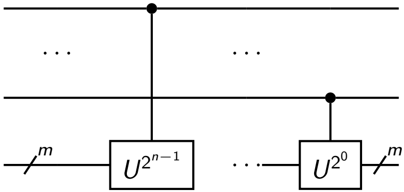
Examples

$|10\rangle |y\rangle \mapsto |10\rangle (UU |y\rangle)$ and $|00\rangle |y\rangle \mapsto |00\rangle |y\rangle$

Note that $|\psi\rangle$ is also an eigenvector of U^x, with eigenvalue $e^{i2\pi x\phi}$, for any integer x.

Multi-controlled operations

Recall that a binary number $x_1 \ldots x_n$ corresponds to the natural number $2^{n-1}x_1 + \cdots + 2^0 x_n$
We use this to build the previous multi-controlled operation in terms of simpler operations



Note that the multi-controlled operation uses n ‘simply’-controlled rotations U^{2^i}

Another Example

Take a unitary U with an eigenvector $|\psi\rangle$ whose eigenvalue is $e^{i2\pi\phi}$
ϕ is equal to one of the following values $\left\{0 \cdot \frac{1}{4}, 1 \cdot \frac{1}{4}, 2 \cdot \frac{1}{4}, 3 \cdot \frac{1}{4}\right\}$

The following circuit discovers ϕ

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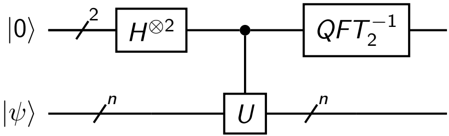
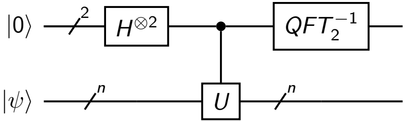


Another Example



$$|0\rangle\,|0\rangle$$

$$\overset{H^{\otimes 2}}{\longmapsto}\ \frac{1}{\sqrt{2^2}}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

$$\overset{\text{ctrl. } U}{\longmapsto}\ \frac{1}{\sqrt{2^2}}(|00\rangle + e^{i2\pi\phi}\,|01\rangle + e^{i2\pi\phi\cdot 2}\,|10\rangle + e^{i2\pi\phi\cdot 3}\,|11\rangle)$$

$$= \frac{1}{\sqrt{2^2}}(|00\rangle + e^{i2\pi x\cdot\frac{1}{4}}\,|01\rangle + e^{i2\pi x\cdot\frac{1}{4}\cdot 2}\,|10\rangle + e^{i2\pi x\cdot\frac{1}{4}\cdot 3}\,|11\rangle)$$

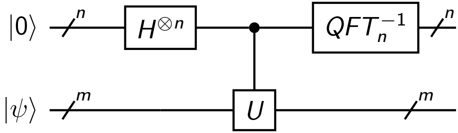
$$= \frac{1}{\sqrt{2^2}}(|00\rangle + \omega_2^{x}\,|01\rangle + \omega_2^{x\cdot 2}\,|10\rangle + \omega_2^{x\cdot 3}\,|11\rangle)$$

$$\overset{QFT_2^{-1}}{\longmapsto}\ |x\rangle$$

Yet Another Example

Take a unitary U with eigenvector $|\psi\rangle$ whose eigenvalue is $e^{i2\pi\phi}$ st $\phi \in \left\{0 \cdot \frac{1}{2^n}, \ldots, 2^n - 1 \cdot \frac{1}{2^n}\right\}$

The following circuit returns x such that $\phi = x \cdot \frac{1}{2^n}$



Exercise

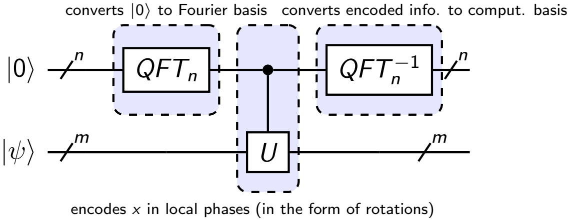
Prove that indeed the circuit returns x such that $\phi = x \cdot \frac{1}{2^n}$

Yet Another Example

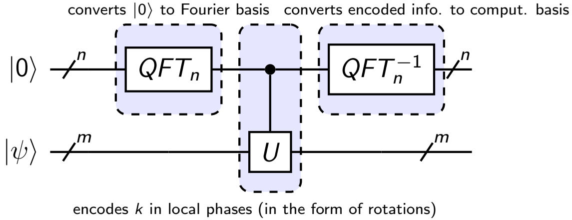
Exercise

Show that $QFT_n |0\rangle = H^{\otimes n} |0\rangle$.

Note that this allows to rewrite the previous circuit in the one below



Complexity of quantum eigenvalue estimation



How many gates does the circuit above use?

n 'Hadamards' $+$ n 'simply'-controlled gates $+$ n^2 gates for QFT_n^{-1}

... but precision is Limited

We assumed $0 \leq \phi < 1$ takes a value from $\left\{0 \cdot \frac{1}{2^n}, \ldots, 2^n - 1 \cdot \frac{1}{2^n}\right\}$
... an assumption that arose from having only n qubits to estimate ...

But what to do if ϕ takes none of these values?

Return the n-bit number k with $k \cdot \frac{1}{2^n}$ the value above closest to ϕ

Is the circuit above up to this task?

Setting the stage

Let $\omega_n = e^{i2\pi\cdot\frac{1}{2^n}}$ $\underbrace{\text{(division of the unit circle in } 2^n \text{ slices)}}_{\text{a.k.a. the n roots of unity}}$

To answer the previous question, we will use the following explicit defn. of QFT^{-1}

$$QFT_n^{-1}|x\rangle = \frac{1}{\sqrt{2^n}}\sum_{k=0}^{2^n-1}\omega_n^{-x\cdot k}|k\rangle$$

Exercise

Prove that QFT_n^{-1} is indeed the inverse of QFT_n

Setting the stage

Let $k \cdot \frac{1}{2^n}$ be the value in $\left\{0 \cdot \frac{1}{2^n}, \ldots, 2^n - 1 \cdot \frac{1}{2^n}\right\}$ closest to ϕ, i.e.

$$\exists_\epsilon \cdot 0 \leq |\epsilon| \leq \frac{1}{2^n} \quad \text{and} \quad k \cdot \frac{1}{2^n} + \epsilon = \phi$$

Note that the difference ϵ decreases when the number of qubits increases.

Recall the circuit

Setting the stage

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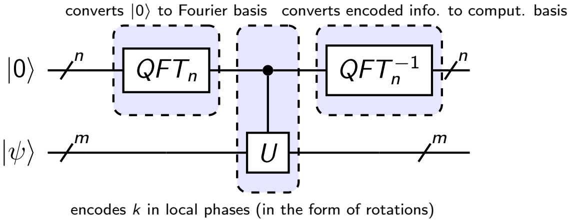


Computing the output again

$$|0\rangle$$

$$\overset{H^{\otimes n}}{\mapsto} \frac{1}{\sqrt{2^n}}(|0\rangle + |1\rangle + \cdots + |2^n - 1\rangle)$$

$$\overset{\text{ctrl. } U}{\mapsto} \frac{1}{\sqrt{2^n}}\left(|0\rangle + e^{i2\pi\phi\cdot 1} |1\rangle + \cdots + e^{i2\pi\phi\cdot 2^{n-1}} |2^n - 1\rangle \right)$$

$$= \frac{1}{\sqrt{2^n}}\left(|0\rangle + e^{i2\pi(k\cdot\frac{1}{2^n}+\epsilon)\cdot 1} |1\rangle + \cdots + e^{i2\pi(k\cdot\frac{1}{2^n}+\epsilon)\cdot 2^{n-1}} |2^n - 1\rangle \right)$$

$$= \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} e^{i2\pi(k\cdot\frac{1}{2^n}+\epsilon)\cdot j} |j\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} e^{i2\pi k\cdot\frac{1}{2^n}\cdot j} e^{i2\pi\epsilon\cdot j} |j\rangle$$

$$\overset{QFT^{-1}}{\mapsto} \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} e^{i2\pi k\cdot\frac{1}{2^n}\cdot j} e^{i2\pi\epsilon\cdot j} \left(\frac{1}{\sqrt{2^n}} \sum_{l=0}^{2^n-1} e^{-i2\pi j\cdot\frac{1}{2^n}\cdot l} |l\rangle \right)$$

$$= \frac{1}{2^n} \sum_{j=0}^{2^n-1} e^{i2\pi k\cdot\frac{1}{2^n}\cdot j} e^{i2\pi\epsilon\cdot j} \left(\sum_{l=0}^{2^n-1} e^{-i2\pi j\cdot\frac{1}{2^n}\cdot l} |l\rangle \right)$$

$$= \frac{1}{2^n} \sum_{j=0}^{2^n-1} \sum_{l=0}^{2^n-1} e^{i2\pi\epsilon\cdot j} e^{i2\pi j\cdot\frac{1}{2^n}\cdot(k-l)} |l\rangle$$

Looking into the final state

The amplitude of $|k\rangle$ is $\frac{1}{2^n}\sum_{j=0}^{2^n-1} e^{i2\pi\epsilon\cdot j}$
which is a finite geometric series.
Therefore,

$$\frac{1}{2^n}\sum_{j=0}^{2^n-1} e^{i2\pi\epsilon j} = \begin{cases} 1 & \text{if } \epsilon = 0 \\ \frac{1}{2^n}\frac{1-e^{i2\pi\epsilon 2^n}}{1-e^{i2\pi\epsilon}} & \text{if } \epsilon \neq 0 \end{cases}$$

Let us proceed under the assumption $\epsilon \neq 0$.

A geometric detour

$|1 - e^{i\theta}|$ for some angle θ is the Euclidean distance between 1 and $e^{i\theta}$ (length of the straight line segment between both points)
Consider also arc length θ between 1 and $e^{i\theta}$ (distance between the two points by running along the unit circle)

Theorem

Let d^E and d^a be respectively the Euclidean distance and arc length between 1 and $e^{i\theta}$. Then,

a. $d^E \leq d^a$

b. *if* $0 \leq \theta \leq \pi$ *we have* $\frac{d^a}{d^E} \leq \frac{\pi}{2}$

Finally!

Recall $\left| \frac{1}{2^n} \frac{1-e^{i2\pi\epsilon 2^n}}{1-e^{i2\pi\epsilon}} \right|^2$ is the probability of measuring $|k\rangle$

$$\begin{aligned}
\left| \frac{1}{2^n} \frac{1-e^{i2\pi\epsilon 2^n}}{1-e^{i2\pi\epsilon}} \right|^2 &= \left(\frac{1}{2^n}\right)^2 \frac{\left|1-e^{i2\pi\epsilon 2^n}\right|^2}{\left|1-e^{i2\pi\epsilon}\right|^2} \\
&\geq \left(\frac{1}{2^n}\right)^2 \frac{\left|1-e^{i2\pi\epsilon 2^n}\right|^2}{(2\pi\epsilon)^2} && \{\text{Thm. [a.]}\} \\
&\geq \left(\frac{1}{2^n}\right)^2 \frac{\left(\frac{2}{\pi}\cdot 2\pi\epsilon 2^n\right)^2}{(2\pi\epsilon)^2} && \{\text{Thm. [b.]}\} \\
&= \left(\frac{1}{2^n}\right)^2 \frac{(4\epsilon 2^n)^2}{(2\pi\epsilon)^2} \\
&= \left(\frac{1}{2^n}\right)^2 \frac{(2\cdot 2^n)^2}{\pi^2} = \frac{2^2}{\pi^2} = \frac{4}{\pi^2}
\end{aligned}$$

Working with a superposition of eigenvectors

The algorithm requires an eigenvector as input,
but sometimes is highly difficult to build such a vector.

Often it is easier to feed instead a superposition of eigenvectors.

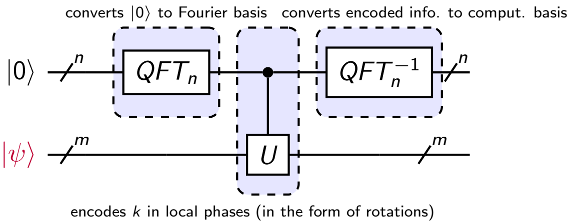
Indeed, by the spectral theorem one knows that the eigenvectors $\{|\psi_1\rangle, \dots, |\psi_N\rangle\}$ of U (with associated eigenvalues $e^{i2\pi\phi_1}, \dots, e^{i2\pi\phi_N}$) form a basis for the $N(= 2^n)$-dimensional vector space on which U acts.

Thus, one may define

$$|\psi\rangle = \frac{1}{\sqrt{N}}(|\psi_1\rangle + \cdots + |\psi_N\rangle)$$

to feed the circuit

Working with a superposition of eigenvectors



Exercise

Show that if $\forall_{i \leq N} \cdot \phi_i \in \left\{0 \cdot \frac{1}{2^n}, \ldots, 2^n - 1 \cdot \frac{1}{2^n}\right\}$ then the circuit's output is

$$\frac{1}{\sqrt{N}}\Big(|k_1\rangle\, |\psi_1\rangle + \cdots + |k_N\rangle\, |\psi_N\rangle \Big) \qquad \left(\phi_i = k_i \cdot \frac{1}{2^n}\right)$$

Working with a superposition of eigenvectors

Similarly, it can be shown that in general the circuit's output is

$$\frac{1}{\sqrt{N}}\left(\left|\tilde{\phi_1}\right\rangle|\psi_1\rangle+\cdots+\left|\tilde{\phi_N}\right\rangle|\psi_N\rangle\right)$$

where each $\tilde{\phi_i}$ is the best n-bit approximation of ϕ_i with probability $\geq \frac{4}{\pi^2}$