Computação Quântica
Problems on the Quantum Fourier Transform (from the slides)

Question 1

Compute $\mathsf{QFT}_K(|00\cdots0\rangle)$.

Solution

For $K = 2^n$,

$$\mathsf{QFT}_K(|00\cdots 0\rangle) \;=\; \frac{1}{\sqrt{K}}\sum_{y=0}^{K-1} e^{2\pi i(\frac{0}{K})y}|y\rangle \;=\; \frac{1}{\sqrt{K}}\sum_{y_1,y_2\cdots,y_n=0}^{1}|y_1y_2\cdots y_n\rangle$$

Clearly,

$$\mathsf{QFT}_4(|00\rangle) \;=\; \frac{1}{2}(|00\rangle+|01\rangle+|10\rangle+|11\rangle)$$

and $\mathsf{QFT}_2 = \mathsf{H}$.

Question 2

Verify the following equality

$$\mathsf{QFT}_K(|x_1\cdots x_n\rangle) \;=\; \left(\frac{|0\rangle+e^{2\pi i(0.x_n)}|1\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|0\rangle+e^{2\pi i(0.x_{n-1}x_n)}|1\rangle}{\sqrt{2}}\right)\cdots\otimes\cdots\left(\frac{|0\rangle+e^{2\pi i(0.x_1x_2\cdots x_n)}|1\rangle}{\sqrt{2}}\right)$$

Solution

$$\begin{aligned}
\mathsf{QFT}_K(|x\rangle) \;&=\; \frac{1}{\sqrt{2^n}}\sum_{y=0}^{K-1} e^{2\pi i x y 2^{-n}}|y\rangle \\
&=\; \frac{1}{\sqrt{2^n}}\sum_{y_1,\cdots,y_n=0}^{1} e^{2\pi i x(\sum_{p=1}^{n} y_p 2^{-p})}\,|y_1\cdots y_n\rangle \\
&=\; \frac{1}{\sqrt{2^n}}\sum_{y_1,y_2=0}^{1}\bigotimes_{p=1}^{n} e^{2\pi i x y_p 2^{-p}}|y_p\rangle \\
&=\; \frac{1}{\sqrt{2^n}}\bigotimes_{p=1}^{n}\left(\sum_{y_p=0}^{1} e^{2\pi i x y_p 2^{-p}}|y_p\rangle\right) \\
&=\; \frac{1}{\sqrt{2^n}}\bigotimes_{p=1}^{n}\left(|0\rangle+e^{2\pi i x 2^{-p}}|1\rangle\right) \\
&=\; \left(\frac{|0\rangle+e^{2\pi i(0.x_n)}|1\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|0\rangle+e^{2\pi i(0.x_{n-1}x_n)}|1\rangle}{\sqrt{2}}\right)\cdots\otimes\cdots\left(\frac{|0\rangle+e^{2\pi i(0.x_1x_2\cdots x_n)}|1\rangle}{\sqrt{2}}\right)
\end{aligned}$$

Note that this general case follows exactly the same argument used for the case of QFT_4 applied to $|x\rangle = |x_1x_2\rangle$, as discussed in the slides. Recalling,

$$\begin{aligned}
\mathsf{QFT}_4(|x\rangle) &= \frac{1}{2}\sum_{y=0}^{3} e^{2\pi i x y 2^{-2}}|y\rangle \\
&= \frac{1}{2}\sum_{y_1,y_2=0}^{1} e^{2\pi i x(y_1 2^{-1}+y_2 2^{-2})}\,|y_1y_2\rangle \\
&= \frac{1}{2}\sum_{y_1,y_2=0}^{1} (e^{2\pi i x y_1 2^{-1}}|y_1\rangle \otimes e^{2\pi i x y_2 2^{-2}}|y_2\rangle) \\
&= \frac{1}{2}\sum_{y_1=0}^{1} (e^{2\pi i x y_1 2^{-1}}|y_1\rangle \otimes \sum_{y_2=0}^{1} e^{2\pi i x y_2 2^{-2}}|y_2\rangle) \\
&= \frac{(|0\rangle + e^{2\pi i x 2^{-1}|1\rangle})}{\sqrt{2}} \otimes \frac{(|0\rangle + e^{2\pi i x 2^{-2}|1\rangle})}{\sqrt{2}} \\
&= \frac{(|0\rangle + e^{2\pi i(x_1.x_2)|1\rangle})}{\sqrt{2}} \otimes \frac{(|0\rangle + e^{2\pi i(0.x_1x_2)|1\rangle})}{\sqrt{2}} \\
&= \frac{(|0\rangle + e^{2\pi i(0.x_2)|1\rangle})}{\sqrt{2}} \otimes \frac{(|0\rangle + e^{2\pi i(0.x_1x_2)|1\rangle})}{\sqrt{2}}
\end{aligned}$$

The first reduction resorts to the following fact for $|y\rangle = |y_1y_2\rangle$

$$\frac{y}{2^n} = \sum_{j=1}^{n} y_j 2^{-j}$$

The last one to

$$e^{2\pi i(a.b)} = e^{2\pi i a}e^{2\pi i(0.b)} = e^{2\pi i(0.b)}$$
