Entanglement and Teleportation

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The Problem

Two secure labs and in one of these a qubit

Terrain between the two labs full of entities that wish to access the qubit's state

How to transfer this quantum state from one lab to the other?

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Terrain between the two labs full of entities that wish to access the qubit's state

How to transfer this quantum state from one lab to the other?

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Quantumly, we can do better thanks to entanglement

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Mathematical Notion of Entanglement

Definition

A vector $u \in V \otimes W$ is entangled if it cannot be written as a tensor $v \otimes w$ such that $v \in V$ and $w \in W$

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Example

All four states below are entangled

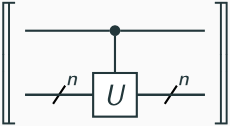
$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \qquad \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

They form a basis of $\mathbb{C}^4$, which is often called the Bell basis

An Important Ingredient for Building Bell States and Beyond

Every quantum operation  gives rise to a 'controlled' quantum operation



$$= \begin{cases} cU\,|0\rangle\,|b\rangle = |0\rangle\,|b\rangle \\ cU\,|1\rangle\,|b\rangle = |1\rangle\,U\,|b\rangle \end{cases}$$

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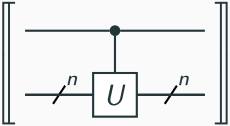
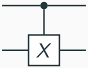
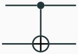
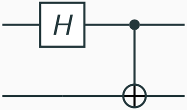


$$= \begin{cases} cU\left|0\right\rangle\left|b\right\rangle = \left|0\right\rangle\left|b\right\rangle \\ cU\left|1\right\rangle\left|b\right\rangle = \left|1\right\rangle U\left|b\right\rangle \end{cases}$$

N.B. The circuit 

is often denoted as 

Building Bell States



Every vector in the computational basis of $\mathbb{C}^4$ when fed to the circuit above yields a Bell state

Postulates of Measurement

Maps M_0 and M_1 of type $\mathbb{C}^2 \to \mathbb{C}^2$ for measuring a qubit

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad M_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

A map M_k, $k \in \{0, 1\}$ possibly tensored with identities $\mathrm{id} : \mathbb{C}^2 \to \mathbb{C}^2$ called a measurement

Postulates

For a state $v \in \mathbb{C}^{2^n}$ and measurement $M : \mathbb{C}^{2^n} \to \mathbb{C}^{2^n}$

- probability of outcome represented by M is $\langle Mv, Mv \rangle$
- state after the observed outcome is $\frac{1}{\|Mv\|} Mv$

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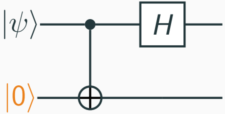
Quantum Teleportation Intra-Gate pt. I

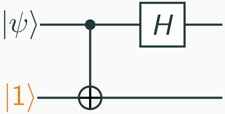
We transfer the top wire qubit's state to the bottom wire



$$
\begin{aligned}
&(H \otimes I)cX(\alpha |0\rangle + \beta |1\rangle) |0\rangle \\
&= (H \otimes I)cX(\alpha |00\rangle + \beta |10\rangle) \\
&= (H \otimes I)(\alpha |00\rangle + \beta |11\rangle) \\
&= |+\rangle \alpha |0\rangle + |-\rangle \beta |1\rangle \\
&= \frac{1}{\sqrt{2}}(|0\rangle \alpha |0\rangle + |1\rangle \alpha |0\rangle + |0\rangle \beta |1\rangle - |1\rangle \beta |1\rangle) \\
&= \frac{1}{\sqrt{2}}\Big(|0\rangle (\alpha |0\rangle + \beta |1\rangle) + |1\rangle (\alpha |0\rangle - \beta |1\rangle)\Big)
\end{aligned}
$$

Quantum Teleportation Intra-Gate pt. II

We transfer the top wire qubit's state to the bottom wire



$$(H \otimes I) cX (\alpha |0\rangle + \beta |1\rangle) |1\rangle$$
$$= \dots$$
$$= \frac{1}{\sqrt{2}} \Big(|0\rangle (\alpha |1\rangle + \beta |0\rangle) + |1\rangle (\alpha |1\rangle - \beta |0\rangle) \Big)$$

Quantum Teleportation Intra-Gate pt. III

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Quantum Teleportation Intra-Gate pt. III

Are we done?

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Fortunately we can do better. We use entanglement to establish a secure 'communication channel' and proceed in the following manner . . .

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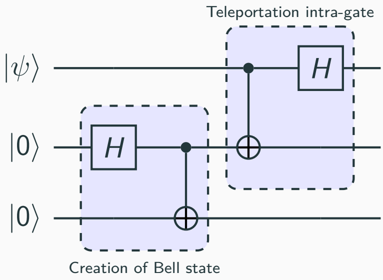


Bottom qubits become entangled and thus connected, even if they are far away from each other later on

Quantum Teleportation pt. II

$$((H \otimes I) \otimes I)(cX \otimes I)\Big((\alpha |0\rangle + \beta |1\rangle) \otimes \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\Big)$$

$$= \tfrac{1}{\sqrt{2}}((H \otimes I) \otimes I)(cX \otimes I)\Big(\alpha |000\rangle + \alpha |011\rangle + \beta |100\rangle + \beta |111\rangle\Big)$$

$$= \tfrac{1}{\sqrt{2}}((H \otimes I) \otimes I)\Big(\alpha |000\rangle + \alpha |011\rangle + \beta |110\rangle + \beta |101\rangle\Big)$$

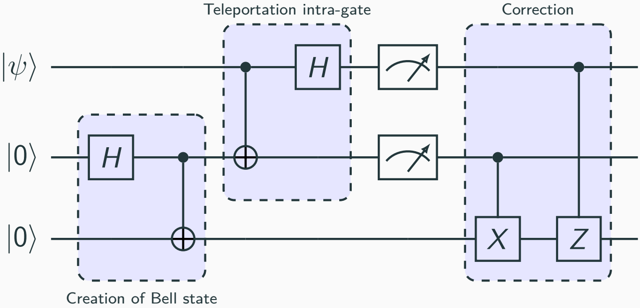
$$= \tfrac{1}{\sqrt{2}}((H \otimes I) \otimes I)\Big(|0\rangle (\alpha |00\rangle + \alpha |11\rangle) + |1\rangle (\beta |10\rangle + \beta |01\rangle)\Big)$$

$$= \tfrac{1}{2}\Big((|0\rangle + |1\rangle)(\alpha |00\rangle + \alpha |11\rangle) + (|0\rangle - |1\rangle)(\beta |10\rangle + \beta |01\rangle)\Big)$$

$$= \tfrac{1}{2}\Big(|00\rangle (\alpha |0\rangle + \beta |1\rangle) + |01\rangle (\alpha |1\rangle + \beta |0\rangle) + |10\rangle (\alpha |0\rangle - \beta |1\rangle) \ \ldots$$

$$\cdots + |11\rangle (\alpha |1\rangle - \beta |0\rangle)\Big)$$

Quantum Teleportation pt. III



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Did We Just Break Physics?

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No-cloning

Did not end up with two copies of $|\psi\rangle$, because the state of the top qubit was destroyed by the measurement

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Did not end up with two copies of $|\psi\rangle$, because the state of the top qubit was destroyed by the measurement

FTL communication

Did not communicate faster than light, because teleportation required us to send two classical bits

What's Next?

First glimpse of applications of quantum phenomena to algorithmics and communication. Namely

- superposition & interference
- entanglement

Next we will overview more sophisticated applications of these phenomena