Setting an Exponential Separation between Quantum and Classical Computation

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Previously...

The Problem

Take a function $f : \{0, 1\} \to \{0, 1\}$

Either $f(0) = f(1)$ or $f(0) \neq f(1)$

Tell us whether the first or second case hold

Classically, need to run f twice. Quantumly, once is enough

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Can we have more impressive differences in complexity?

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Global Phase Factor

Definition

Let $v, u \in \mathbb{C}^{2^n}$ be vectors. If $u = e^{i\theta} v$ we say that it is equal to v up to global phase factor $e^{i\theta}$

Theorem

$e^{i\theta} v$ and v are indistinguishable in the world of quantum mechanics

Proof sketch

Show that equality up to global phase is preserved by operators and normalisation + show that probability outcomes associated with v and $e^{i\theta} v$ are the same

Relative Phase Factor

Definition

We say that vectors $\sum_{x \in 2^n} \alpha_x |x\rangle$ and $\sum_{x \in 2^n} \beta_x |x\rangle$ differ by a relative phase factor if for all $x \in 2^n$

$$\alpha_x = e^{i\theta_x} \beta_x \qquad \text{(for some angle } \theta_x\text{)}$$

Example

Vectors $|0\rangle + |1\rangle$ and $|0\rangle - |1\rangle$ differ by a relative phase factor

Relative Phase Factor

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Example

Vectors $|0\rangle + |1\rangle$ and $|0\rangle - |1\rangle$ differ by a relative phase factor

Vectors that differ by a relative phase factor are distinguishable

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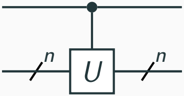
The Phase Kickback Effect pt. I

Recall that every quantum operation 

gives rise to a controlled quantum operation, which is depicted below



Let v be an eigenvector of U (*i.e.* $Uv = e^{i\theta} v$) and calculate

$$\begin{aligned}
& cU\Big((\alpha\,|0\rangle + \beta\,|1\rangle) \otimes v\Big) \\
&= cU(\alpha\,|0\rangle \otimes v + \beta\,|1\rangle \otimes v) \\
&= \alpha\,|0\rangle \otimes v + \beta\,|1\rangle \otimes e^{i\theta} v \\
&= (\alpha\,|0\rangle + e^{i\theta}\beta\,|1\rangle) \otimes v
\end{aligned}$$

The Phase Kickback Effect pt. II

What just happened?

The Phase Kickback Effect pt. II

What just happened?

- Global phase $e^{i\theta}$ (introduced to v) was 'kickedback' as a relative phase in the control qubit

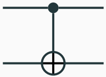
The Phase Kickback Effect pt. II

What just happened?

- Global phase $e^{i\theta}$ (introduced to v) was 'kickedback' as a relative phase in the control qubit
- Some information of U is now encoded in the control qubit

In general kickingback such phases causes interference patterns that give away information about U

Consider the controlled-not operation

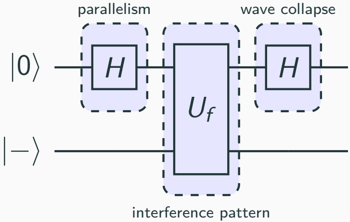


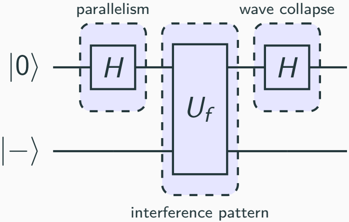
X has $|-\rangle$ as eigenvector with associated eigenstate -1. It thus yields the equation

$$cX\,|b\rangle\,|-\rangle = (-1)^b\,|b\rangle\,|-\rangle$$

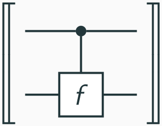
with $|b\rangle$ an element of the computational basis

Back to Deutsch's Problem





U_f can be seen as a generalised controlled not-operation



$$= |x\rangle\,|y\rangle \mapsto \begin{cases} |x\rangle\,|y\rangle & \text{if } f(x) = 0 \\ |x\rangle \neg |y\rangle & \text{if } f(x) = 1 \end{cases}$$

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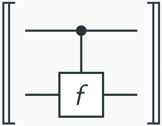
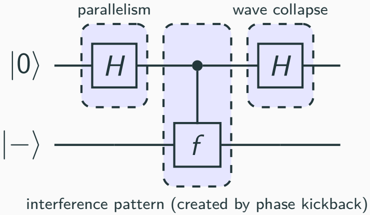


$$= |x\rangle\,|y\rangle \mapsto \begin{cases} |x\rangle\,|y\rangle & \text{if } f(x) = 0 \\ |y\rangle \neg\,|y\rangle & \text{if } f(x) = 1 \end{cases}$$

Recall that $|-\rangle$ is an eigenvector of X with eigenstate -1. Thus analogously to before we deduce

$$U_f\,|x\rangle\,|-\rangle = (-1)^{f(x)}\,|x\rangle\,|-\rangle$$

Back to Deutsch's Problem pt. III



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Going Beyond the Current Separation

Albeit looking almost magical how we handled Deutsch's problem, the corresponding complexity difference between quantum and classical is unimpressive

Can we come up with a more impressive separation?

Setting the Stage

Lemma

For $a, b \in \{0, 1\}$ *the equation* $(-1)^a(-1)^b = (-1)^{a \oplus b}$ *holds*

Prook sketch

Build a truth table for each case and compare the corresponding contents

Definition

Given two bit-strings $x, y \in \{0, 1\}^n$ we define their product $x \cdot y \in \{0, 1\}$ as $x \cdot y = (x_1 \wedge y_1) \oplus \cdots \oplus (x_n \wedge y_n)$

Setting the Stage

Lemma

For any three binary strings $x, a, b \in \{0,1\}^n$ *the equation* $(x \cdot a) \oplus (x \cdot b) = x \cdot (a \oplus b)$ *holds*

Proof sketch

Follows from the fact that for any three bits $a, b, c \in \{0,1\}$ the equation $(a \wedge b) \oplus (a \wedge c) = a \wedge (b \oplus c)$ holds

Setting the Stage

Lemma

For any element $|b\rangle$ *in the computational basis of* $\mathbb{C}^2$ *we have* $H|b\rangle = \frac{1}{\sqrt{2}} \sum_{z \in 2} (-1)^{b \wedge z} |z\rangle$

Proof sketch

Build a truth table and compare the corresponding contents

Theorem

For any element $|b\rangle$ *in the computational basis of* $\mathbb{C}^{2^n}$ *we have* $H^{\otimes n}|b\rangle = \frac{1}{\sqrt{2^n}} \sum_{z \in 2^n} (-1)^{b \cdot z} |z\rangle$

Proof sketch

Follows from induction on the size of n

Bernstein-Vazirani

The Problem

Take a function $f : \{0,1\}^n \to \{0,1\}$

You are promised that $f(x) = s \cdot x$ for some fixed bit-string s

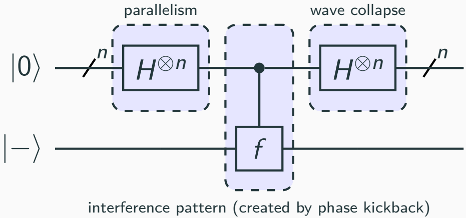
Find s

Classically, we run f n-times by computing

$$f(1\dots 0) = (s_1 \wedge 1) \oplus \cdots \oplus (s_n \wedge 0) = s_1$$
$$\vdots$$
$$f(0\dots 1) = (s_1 \wedge 0) \oplus \cdots \oplus (s_n \wedge 1) = s_n$$

Quantumly, we discover s by running f only once

The Circuit


parallelism
wave collapse
$|0\rangle$
n
$H^{\otimes n}$
$H^{\otimes n}$
n
$|-\rangle$
f
interference pattern (created by phase kickback)

The Computation

N.B. In order to not overburden notation we omit $|-\rangle$

$$H^{\otimes n}|0\rangle$$
$$= \frac{1}{\sqrt{2^n}} \sum_{z \in 2^n} |z\rangle \qquad \{\text{Theorem slide 18}\}$$
$$\overset{U_f}{\mapsto} \frac{1}{\sqrt{2^n}} \sum_{z \in 2^n} (-1)^{f(z)} |z\rangle \qquad \{\text{Definition slide 12}\}$$
$$\overset{H^{\otimes n}}{\mapsto} \frac{1}{2^n} \sum_{z \in 2^n} (-1)^{f(z)} \Big(\sum_{z' \in 2^n} (-1)^{z \cdot z'} |z'\rangle \Big) \qquad \{\text{Theorem slide 18}\}$$
$$= \frac{1}{2^n} \sum_{z \in 2^n} \sum_{z' \in 2^n} (-1)^{(z \cdot s) \oplus (z \cdot z')} |z'\rangle \qquad \{\text{Lemma slide 16}\}$$
$$= \frac{1}{2^n} \sum_{z \in 2^n} \sum_{z' \in 2^n} (-1)^{z \cdot (s \oplus z')} |z'\rangle \qquad \{\text{Lemma slide 17}\}$$

The Computation pt. II

Probability of measuring s at the end given by

$$\left| \frac{1}{2^n} \sum_{z \in 2^n} (-1)^{z \cdot (s \oplus s)} |s\rangle \right|^2$$
$$= \left| \frac{1}{2^n} \sum_{z \in 2^n} (-1)^{z \cdot 0} |s\rangle \right|^2$$
$$= \left| \frac{1}{2^n} \sum_{z \in 2^n} 1 \, |s\rangle \right|^2$$
$$= \left| \frac{2^n}{2^n} \right|^2$$
$$= 1$$

This means that somehow all values yielding wrong answers were completely cancelled

T.P.C. Show exactly how all the wrong answers were cancelled

Going Even Further Beyond

We went from running f n times to running just once

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Still not very impressive (at least for the Computer Scientist :-))

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Can we do even better?

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Deutsch-Josza

The Problem

Take a function $f : \{0,1\}^n \rightarrow \{0,1\}$

You are promised that f is either constant or balanced

Find out which case holds

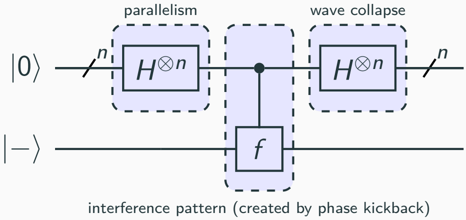
Classically, we evaluate half of the inputs ($\frac{2^n}{2} = 2^{n-1}$), evaluate one more and run the decision procedure,

- output always the same $\implies$ constant
- otherwise $\implies$ balanced

which requires running f $2^{n-1} + 1$ times

Quantumly, we know the answer by running f only once

The Circuit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We then proceed by case distinction. Assume that f is constant

$$\frac{1}{2^n} \sum_{z \in 2^n} (-1)^{f(z)} \Big(\sum_{z' \in 2^n} (-1)^{z \cdot z'} |z'\rangle \Big)$$
$$= \frac{1}{2^n} (\pm 1) \sum_{z \in 2^n} \Big(\sum_{z' \in 2^n} (-1)^{z \cdot z'} |z'\rangle \Big)$$

The Computation pt. II

Probability of measuring $|0\rangle$ at the end given by

$$\left|\frac{1}{2^n}(\pm 1)\sum_{z\in 2^n}(-1)^{z\cdot 0}\,|0\rangle\,\right|^2$$
$$= \left|\frac{1}{2^n}(\pm 1)\sum_{z\in 2^n} 1\,|0\rangle\,\right|^2$$
$$= \left|\frac{2^n}{2^n}\right|^2$$
$$= 1$$

So if f is constant we measure $|0\rangle$ with probability 1. Now if f is balanced...

The Computation pt. III

$$\frac{1}{2^n} \sum_{z \in 2^n} (-1)^{f(z)} \Big(\sum_{z' \in 2^n} (-1)^{z \cdot z'} |z'\rangle \Big)$$

$$= \frac{1}{2^n} \Bigg(\sum_{z \in 2^n, f(z)=0} (-1)^{f(z)} \Big(\sum_{z' \in 2^n} (-1)^{z \cdot z'} |z'\rangle \Big)$$

$$+ \sum_{z \in 2^n, f(z)=1} (-1)^{f(z)} \Big(\sum_{z' \in 2^n} (-1)^{z \cdot z'} |z'\rangle \Big) \Bigg)$$

$$= \frac{1}{2^n} \Bigg(\sum_{z \in 2^n, f(z)=0} \Big(\sum_{z' \in 2^n} (-1)^{z \cdot z'} |z'\rangle \Big)$$

$$+ \sum_{z \in 2^n, f(z)=1} (-1) \Big(\sum_{z' \in 2^n} (-1)^{z \cdot z'} |z'\rangle \Big) \Bigg)$$

The Computation pt. IV

Probability of measuring $|0\rangle$ at the end given by

$$\left|\tfrac{1}{2^n}\Big(\textstyle\sum_{z\in 2^n, f(z)=0}(-1)^{z\cdot 0}\,|0\rangle + \sum_{z\in 2^n, f(z)=1}(-1)(-1)^{z\cdot 0}\,|0\rangle\Big)\right|^2$$
$$= \left|\tfrac{1}{2^n}\Big(\textstyle\sum_{z\in 2^n, f(z)=0}|0\rangle + \sum_{z\in 2^n, f(z)=1}(-1)\,|0\rangle\Big)\right|^2$$
$$= \left|\tfrac{1}{2^n}\Big(\textstyle\sum_{z\in 2^n, f(z)=0}|0\rangle - \sum_{z\in 2^n, f(z)=1}|0\rangle\Big)\right|^2$$
$$= 0$$

So if f is balanced we measure $|0\rangle$ with probability 0

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Revisiting Deutsch-Josza

The Problem

Take a function $f : \{0,1\}^n \to \{0,1\}$. The latter either constant or balanced

Find out which case holds

Classically, evaluate half of the inputs ($\frac{2^n}{2} = 2^{n-1}$), evaluate one more and run the decision procedure,

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Quantumly, we know the answer by running f only once

However ...

Tackling Deutsch-Josza with Probabilities

To solve the problem with some margin of error evaluate two arbitrary inputs x and y,

- $f(x) = f(y) \implies$ constant
- $f(x) \neq f(y) \implies$ balanced

Tackling Deutsch-Josza with Probabilities

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Probability of giving the right answer?

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Probability of giving the right answer?

- f is constant $\implies$ right answer with probability 1
- f is balanced $\implies$ right answer with probability $\frac{2^{n-1}}{2^n} = \frac{1}{2}$

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- f is balanced $\implies$ right answer with probability $\frac{2^{n-1}}{2^n} = \frac{1}{2}$

Can we do better?

Tackling Deutsch-Josza with Probabilities pt. II

To solve the problem with some margin of error evaluate k arbitrary inputs $x_1, \ldots, x_k$,

- output always the same $\implies$ constant
- otherwise $\implies$ balanced

Tackling Deutsch-Josza with Probabilities pt. II

To solve the problem with some margin of error evaluate k arbitrary inputs $x_1, \ldots, x_k$,

- output always the same $\Longrightarrow$ constant
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Probability of giving the right answer?

Tackling Deutsch-Josza with Probabilities pt. II

To solve the problem with some margin of error evaluate k arbitrary inputs $x_1, \dots, x_k$,

- output always the same $\Longrightarrow$ constant
- otherwise $\Longrightarrow$ balanced

Probability of giving the right answer?

- f is constant $\Longrightarrow$ right answer with probability 1
- f is balanced $\Longrightarrow$ right answer with probability ...

$$1 - \left(\frac{2^{n-1}}{2^n}\right)^k = 1 - \frac{1}{2^k}$$

Probability of observing the same output in k tries

The Problem

Take a 2-1 function $f : \{0,1\}^n \rightarrow \{0,1\}^n$

There exists a string $s \in \{0,1\}^n$ s.t. $f(x) = f(y) \Rightarrow y = x \oplus s$

Find out s

Classically, evaluate inputs until collision is detected, *i.e.* $f(x) = f(y)$ for some x, y. Then compute $x \oplus y = x \oplus (x \oplus s) = s$

Since f is 2-1, after collecting 2^{n-1} evaluations with no collisions, next evaluation must cause a collision

So in the worst case we need $2^{n-1} + 1$ evaluations

Tackling Simon with Probabilities

How many evaluations do we need to have a collision with probability p?

Tackling Simon with Probabilities

How many evaluations do we need to have a collision with probability p?

To have a collision with probability $p = \frac{1}{k} \leq \frac{1}{2}$ we need

$$\approx \sqrt{(2 \cdot 2^n) \cdot p} = \sqrt{\frac{2}{k} \cdot 2^n} = \sqrt{\frac{2}{k}} \cdot 2^{\frac{n}{2}} \quad \text{evaluations}$$

See the Birthday's problem

Tackling Simon with Probabilities

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See the Birthday's problem

But quantumly, we solve the problem in polynomial time with probability $\approx \frac{1}{4}$

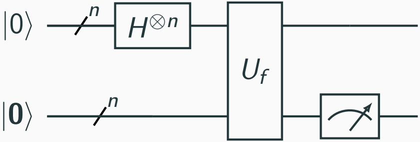
Simon's Algorithm: The Key Steps

1. Prepare superposition $\frac{1}{\sqrt{2}}(|x\rangle + |x \oplus s\rangle)$ for some string x
2. Use interference to extract a string y s.t. $y \cdot s = 0$
3. Repeat previous steps $n - 1$ times to obtain system of equations s.t. $y_k \cdot s = 0$
4. Solve the system for s using Gaussian elimination

↓

Complexity n^3

Simon's Algorithm: Preparing the Superposition



N.B. $U_f\,|x\rangle\,|y\rangle = |x\rangle\,|y \oplus f(x)\rangle$

$$U_f(H^{\otimes n} \otimes I)\,|0\rangle\,|\mathbf{0}\rangle$$
$$= U_f\Big(\tfrac{1}{\sqrt{2^n}}\textstyle\sum_{x\in 2^n}|x\rangle\,|\mathbf{0}\rangle\Big)$$
$$= \tfrac{1}{\sqrt{2^n}}\textstyle\sum_{x\in 2^n}|x\rangle\,|f(x)\rangle$$

We then measure the n-bottom qubits to obtain a superposition

$$\tfrac{1}{\sqrt{2}}(|x\rangle + |x \oplus s\rangle)$$

Simon's Algorithm: Extracting the String

Simon's Algorithm: Extracting the String

$$
\begin{aligned}
& H^{\otimes n} \tfrac{1}{\sqrt{2}}(|x\rangle + |x \oplus s\rangle) \\
&= \tfrac{1}{\sqrt{2^{n+1}}} \textstyle\sum_{y \in 2^n} (-1)^{x \cdot y} |y\rangle + (-1)^{(x \oplus s)\cdot y} |y\rangle && \{\text{Theorem slide 18}\} \\
&= \tfrac{1}{\sqrt{2^{n+1}}} \textstyle\sum_{y \in 2^n} (-1)^{x \cdot y} |y\rangle + (-1)^{x \cdot y \oplus s \cdot y} |y\rangle && \{\text{Lemma slide 17}\} \\
&= \tfrac{1}{\sqrt{2^{n+1}}} \textstyle\sum_{y \in 2^n} (-1)^{x \cdot y} |y\rangle + (-1)^{x \cdot y}(-1)^{s \cdot y} |y\rangle && \{\text{Lemma slide 16}\} \\
&= \tfrac{1}{\sqrt{2^{n+1}}} \textstyle\sum_{y \in 2^n} (-1)^{x \cdot y}(1 + (-1)^{s \cdot y}) |y\rangle
\end{aligned}
$$

Simon's Algorithm: Extracting the String

$$
\begin{aligned}
& H^{\otimes n}\frac{1}{\sqrt{2}}(|x\rangle + |x \oplus s\rangle) \\
&= \frac{1}{\sqrt{2^{n+1}}} \textstyle\sum_{y \in 2^n} (-1)^{x \cdot y}\,|y\rangle + (-1)^{(x \oplus s)\cdot y}\,|y\rangle && \{\text{Theorem slide 18}\} \\
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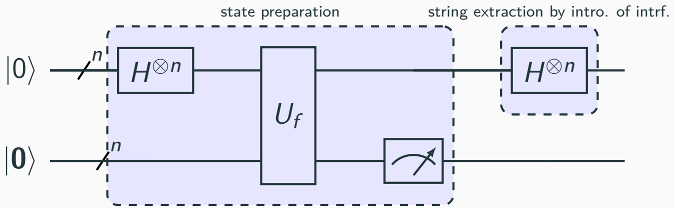
&= \frac{1}{\sqrt{2^{n+1}}} \textstyle\sum_{y \in 2^n} (-1)^{x \cdot y}(1 + (-1)^{s \cdot y})\,|y\rangle
\end{aligned}
$$

Destructive interference when $s \cdot y = 1$. We only observe $|y\rangle$ s.t. $s \cdot y = 0$

The Circuit


state preparation
string extraction by intro. of intrf.
$|0\rangle$
$|\mathbf{0}\rangle$
n
n
$H^{\otimes n}$
U_f
$H^{\otimes n}$

Simon's Algorithm: Solving the System to Extract s

A system of $n-1$ linearly independent equations,

$$\begin{cases} y_1 \cdot s = 0 \\ \dots \\ y_{n-1} \cdot s = 0 \end{cases}$$

has two solutions. One is $s = 0$ but it violates the 2-1 promise. So only the other solution is of interest

Simon's Algorithm: Solving the System to Extract s

A system of $n-1$ linearly independent equations,

$$\begin{cases} y_1 \cdot s = 0 \\ \dots \\ y_{n-1} \cdot s = 0 \end{cases}$$

has two solutions. One is $s = 0$ but it violates the 2-1 promise. So only the other solution is of interest

Probability of obtaining such a system of equations by running the circuit $n-1$ times?

Simon's Algorithm: Probability of Success

Homework

If $s \neq 0$ then for half of the inputs y we have $y \cdot s = 0$ and for the other half $y \cdot s = 1$

#	Possibilities of failure at each step	Probability of failure
1	$\{0\}$	$\frac{2^0}{2^{n-1}}$
2	$\{0, y_1\}$	$\frac{2^1}{2^{n-1}}$
3	$\{0, y_1, y_2, y_1 \oplus y_2\}$	$\frac{2^2}{2^{n-1}}$
...	...	...
$n-1$	$\{0, y_1, y_2, y_3 \dots\}$	$\frac{2^{n-2}}{2^{n-1}}$

Simon's Algorithm: Probability of Success

#	Possibilities of failure at each step	Probability of failure
1	$\{0\}$	$\frac{2^0}{2^{n-1}}$
2	$\{0, y_1\}$	$\frac{2^1}{2^{n-1}}$
3	$\{0, y_1, y_2, y_1 \oplus y_2\}$	$\frac{2^2}{2^{n-1}}$
...	...	...
$n-1$	$\{0, y_1, y_2, y_3 \dots\}$	$\frac{2^{n-2}}{2^{n-1}}$

Table yields the sequence of probabilities of failure,

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots, \frac{1}{2^{n-1}} \qquad \text{(from bottom to top)}$$

Probability of failing in the first $n-2$ steps is thus

$$\frac{1}{4} + \frac{1}{8} + \dots = \frac{1}{4}\left(1 + \frac{1}{2} + \dots\right) \leq \frac{1}{4} \cdot \left(\sum_{i \in \mathbb{N}} \frac{1}{2^i}\right) = \frac{1}{2}$$

Geometric series whose sum is equal to two

Simon's Algorithm: Probability of Success

Probability of succeeding in the first $n-2$ steps at least $\frac{1}{2}$

Probability of succeeding in the $(n-1)$-th step is $\frac{1}{2}$

Thus probability of succeeding in all $n-1$ steps at least $\frac{1}{4}$

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More advanced maths tell that the probability is slightly higher (around $0.28878\ldots$)

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Problems solved were somewhat contrived. In the next lectures we will analyse problems with broader applications