Quantum Computation
(Lecture 7)

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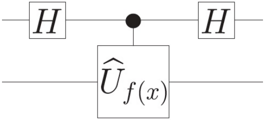
MSc Physics Engineering

Universidade do Minho, 2021-22

The problem: Eigenvalue estimation

Several algorithms previously discussed (Simon, Deutsch-Joza, etc) resort to the following technique:

- Take a controlled version of an operator U_f and prepare the target qubit with an eigenvector;
- with the effect of pushing up (or kicking back) the associated eigenvalue to the state of the control qubit as in



$$U_f\,(a_0|0\rangle + a_1|1\rangle)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = \left((-1)^{f(0)}a_0|0\rangle + (-1)^{f(1)}a_1|1\rangle\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

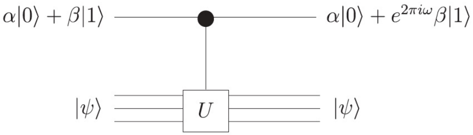
The problem

The question

Can this technique be generalised to estimate the eigenvalues of an arbitrary, *n*-qubit unitary operator U?

Let cU be a controlled version of a unitary operator U, and $(|\phi\rangle, e^{2\pi i w})$ an eigenvector, eigenvalue pair. Then,

$$\begin{aligned} cU|0\rangle|\phi\rangle &= |0\rangle|\phi\rangle \\ cU|1\rangle|\phi\rangle &= |1\rangle U|\phi\rangle = |1\rangle e^{2\pi i w}|\phi\rangle = e^{2\pi i w}|1\rangle|\phi\rangle \end{aligned}$$



The eigenvalue of U is encoded into the relative phase factor between the basis states of the control qubit of cU, thus becoming a measurable quantity.

The problem

The eigenvalue estimation problem

Given a circuit for an operator U, and an eigenvector, eigenvalue pair, $(|\phi\rangle, e^{2\pi i w})$, determine a good estimate for w.

The idea

Prepare a state

$$\frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{2\pi i w y}|y\rangle =$$

$$\left(\frac{|0\rangle + e^{2\pi i (2^{n-1} w)}|1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle + e^{2\pi i (2^{n-2} w)}|1\rangle}{\sqrt{2}}\right) \otimes \cdots \otimes \left(\frac{|0\rangle + e^{2\pi i w}|1\rangle}{\sqrt{2}}\right)$$

and resort to QFT^{-1} to obtain an estimate for w.

The strategy

To prepare this state note that

- $|\phi\rangle$ is also an eigenvector of U^2, with eigenvalue $(e^{2\pi i w})^2 = e^{4\pi i w}$.
- in general, this applies to U^q, with eigenvalue $e^{2q\pi i w}$, for any integer q.

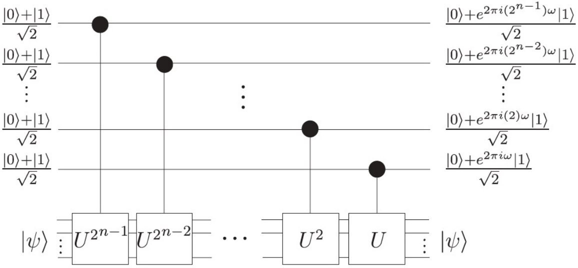
Thus, it is enough to build a controlled-U gate, set the target qubit to the eigenstate $|\phi\rangle$, and compute for the relevant j,

$$cU^{2^j}\left(\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|\phi\rangle\right) = \left(\frac{|0\rangle + e^{2\pi i(2^j w)}|1\rangle}{\sqrt{2}}\right)$$

The strategy

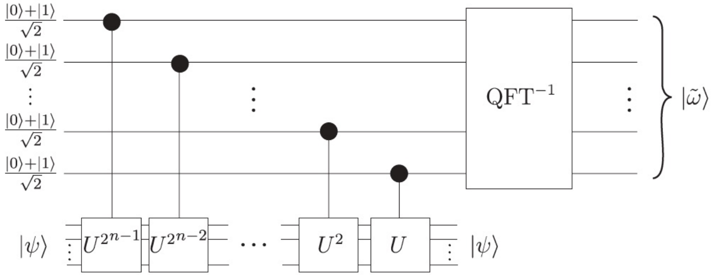
The envisaged circuit implements a sequence of controlled-U^{2^j} gates each controlled on the j-significant bit of

$$x = 2^{n-1}x_{n-1} + \cdots + 2x_1 + x_0$$



The strategy

... followed by QFT^{-1}



The strategy

Observe that

- Applying this sequence of controlled-U^{2^j} gates is equivalent to the successive application of U a total of x times, as captured by the following cU^x gate:

$$cU^x(|x\rangle|\phi\rangle) = (|x\rangle U^x|\phi\rangle)$$

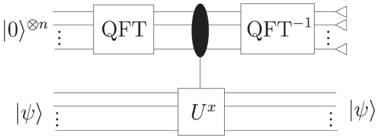
- On the other hand, the control qubits are prepared through $H^{\otimes n}|0\rangle^{\otimes n}$ as

$$\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\otimes\cdots\otimes\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)$$

  which can be accomplished by QFT again:

$$H^{\otimes n}|0\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x\rangle = QFT|0\rangle^{\otimes n}$$

The algorithm



1. Prepare a n-qubit register, identified as the control register, with $|0\rangle^{\otimes n}$ and apply QFT to it.
2. Apply cU^x to the eigenstate $|\phi\rangle$ controlled on the state of the control register.
3. Apply QFT^{-1} to the control register.
4. Measure the control register to obtain a string of bits encoding the integer x.
5. Output the value $\tilde{w} = \frac{x}{2^n}$ as an estimate for w.

Going generic

What if $|\phi\rangle$ is an arbitrary state?

By the spectral theorem one knows that the eigenvectors $\{|\phi_1\rangle, |\phi_2\rangle, \cdots\}$ (with eigenvalues $e^{2\pi i w_j}$, for $j = 1, 2, \cdots$) of U form a basis for the 2^n-dimensional vector space on which U acts. Thus, one may write

$$|\phi\rangle = \sum_{j=0}^{2^n-1} \alpha_j |\phi_j\rangle$$

The algorithm above maps, for each eigenvector of U,

$$|0\rangle^{\otimes} |\phi_j\rangle \mapsto |\tilde{w}_j\rangle |\phi_j\rangle$$

which, by linearity, entails

$$|\phi\rangle \mapsto \sum_{j=0}^{2^n-1} \alpha_j |\tilde{w}_j\rangle |\phi_j\rangle$$