# Algebra of Quantum Operations

Problem Set 2

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*The goal of the problem set 2 is to recall the algebra of quantum operation by applying quantum operations in Haskell.*

You can find the documentation page for the Data.Complex Haskell package helpful for operations regarding complex numbers.

**Let's recall some essential concepts**

**Qubit**

A single-qubit state $|\psi\rangle$ can be in a superposition of basis states $|0\rangle$ and $|1\rangle$:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{1}$$

where $\alpha$ and $\beta$ are complex coefficients with normalisation $|\alpha|^2+|\beta|^2 = 1$. The bra-ket notation simplifies the description of quantum states in a complex vector space. In other words, $|0\rangle$ and $|1\rangle$ are shorthand for the following column vectors:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

**Matrix in Haskell:**

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow \text{[[1,0],[0,-1]]}$$

**Vectors in Haskell:**

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \text{[[1],[0]]};\ |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \text{[[0],[1]]}$$

1. The Hadamard gate,

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

   can map the bases states $|0\rangle$ and $|1\rangle$ to the superposition basis $|+\rangle$ and $|-\rangle$, respectively.

   Write both states ($|+\rangle$ and $|-\rangle$) as a superposition of basis states $|0\rangle$ and $|1\rangle$, as presented in equation 1.

2. The joint state of a system of qubits is described by the tensor product $\otimes$. For two (separable) qubit states $|q_0\rangle$ and $|q_1\rangle$, the joint state of the system may be written in the bra-ket notation with an implicit tensor product:

$$|q_0\rangle \otimes |q_1\rangle = |q_0\rangle|q_1\rangle = |q_0 q_1\rangle$$

   Using the tensor product function from Problem Set 1:

   (a) Write the vector representation of states $|00\rangle$, and $|11\rangle$.

   (b) Write the vector representation of the state $|010\rangle$.

3. In a complex vector space $\mathbb{C}^n$, the norm of a vector is expressed as:

$$||v|| := \sqrt{|v_1|^2 + ... + |v_n|^2} = \sqrt{v_1 v_1 + ... + v_n v_n}$$

   (a) Implement a function to determine the norm of a vector $\mathbb{C}^n$.

   (b) Implement a function "normalise"that takes a column vector $v$ representing an n-qubit quantum state, and returns a state $v/||v||$ with normalised coefficients (i.e. such that the sum of the absolute squares over all coefficients is equal to 1).

4. Matrices representing quantum operations are always *unitary*. A general form of a unitary matrix representing single-qubit operations may be expressed as:

$$U(\theta, \phi, \lambda) = \begin{pmatrix} \cos(\theta/2) & -e^{i\lambda}\sin(\theta/2) \\ e^{i\phi}\sin(\theta/2) & e^{i\lambda+i\phi}cos(\theta/2) \end{pmatrix}$$

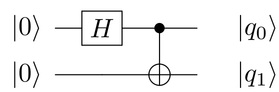
   with $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$ and $0 \leq \lambda < 2\pi$.

   (a) Implement a function u3, that takes $(\theta, \phi, \lambda)$.

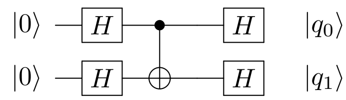
   (b) Apply u3(0,pi,0) and u3(pi/2,0,pi). Do you know these gates by other names?

5. In the quantum circuit model of quantum computation, qubits are represented as horizontal lines, with a sequence of boxes over n-lines representing a sequence of n-qubit quantum operations being performed in left-to-right order. Using the functions gate and tensor:

(a) Write the state of the two-qubit system $|q_0 q_1\rangle$:




(b) Write the state of the two-qubit system $|q_0 q_1\rangle$. Build a truth table (i.e. result over input states $|00\rangle$; $|01\rangle$; $|10\rangle$; $|11\rangle$) for the circuit below. Can you describe the operation being performed?




6. One cannot obtain the matrix representing the CNOT gate in a quantum circuit by applying the tensor products when the control and target qubits are in non-consecutive registers. In figure 1 are three example cases of CNOT matrices.

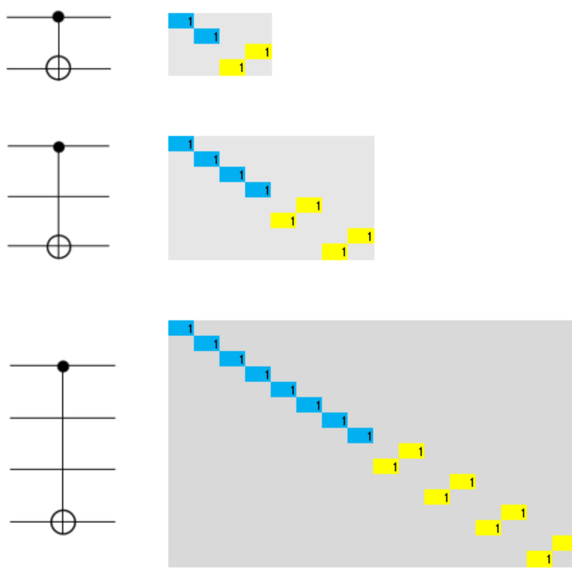


Figure 1: Different instances of CNOT gates in a quantum circuit. Entries sitting on the main diagonal are in blue, while entries outside of it are in yellow; grey space represents null entries.

Implement a function that takes an integer n representing the number of qubits over which the quantum gate is performed and returns a matrix in the form [[Complex Float]] characterising the operation.