Lecture 4: Computational complexity

MIEFis - Quantum Computation, L. S. Barbosa, 2020-21

Summary.
(1) Introduction: The focus. Growth rate of a function.

(2) Case study: closure operations.

(3) Complexity classes.

(4) Problems, problems, problems ...

---

1 Introduction

1.1 The focus

The focus of this lecture is *computational efficiency*, i.e. the attempt to quantify the amount of computational resources (e.g. time, space) required to solve a given problem. Or, putting it in different way, to answer the question of knowing how do such resources scale with the size of the problem (for a suitable definition / measure of *size*).

Actually, the efficiency of an algorithm will be discussed by studying how its number of basic operations scales as the size of the input increases.

A fundamental message is that the efficiency of an algorithm is, up to a considerable extent, much more important than the technology used to execute it.

Example

Integer multiplication by the repeated addition algorithm (to compute $x \cdot y$, just add x to itself $y - 1$ times) or by the usual grade-school algorithm illustrates the point. For example, multiplying 577 by 423 using repeated addition requires 422 additions, whereas doing it with the grade-school algorithm takes 3 multiplications of a number by a single digit and 3 additions. Even for 11-digit numbers, a pocket calculator running the grade-school algorithm would beat the best current supercomputer running repeated addition.

To study computational complexity some *abstractions* are in order. First of all, an algorithm can be abstracted as a computational task accepting as input and returning as output a sequence of bits. This is done without loss of generality because, with a linear overhead in its length, any string from any other alphabet can be encoded as a bit string.

Moreover, one may restrict attention to *decision problems*, i.e. problems that have a Boolean output, thus encoded as a single bit. For example, *given two numbers are they relatively prime?*. A function h with a Boolean output (which is special case of a function from strings to strings) can be identified with a set

$$L_h \mathrel{\widehat{=}} \{x \mid h(x) = 1\}$$

which forms a particular *language*. Thus, computing h is equivalent to the problem of deciding language L_h, i.e. given a string s decide whether $s \in L_h$.

Example

Consider the following *dinner party problem*: Given a list of acquaintances and a list of all pairs among them who do not get along, find the largest set of acquaintances you can invite to a dinner party such that every two invitees get along with one another.

Let us represent the possible invitees through the vertices of a graph with an edge connecting any two people who don't get along. Thus, the problem becomes that of finding a maximum sized *independent set*, i.e. the bigger set of vertices without any common edges, in a given graph. The problem corresponds to the following language

$$L \mathrel{\widehat{=}} \{(G, n) \mid \exists_{M \subseteq \mathsf{vertices}(G)} . \ |M| \geq n \wedge \forall_{x,y \in M} . \ (x, y) \notin \mathsf{edges}(G)\}$$

whose words are pairs composed by a graph G and a number n for which there exists a conflict-free set of invitees, of size at least n.

Observe this a particularly difficult problem: the obvious algorithm — try all possible subsets until a subset that does not include any pair of guests who don't get along is found is highly inefficient (how long will it take for a group of 100 invitees?)

The study of computational complexity focus on how various kinds of computational resources grow as a function of the input size. This analysis can be carried on in any formalization of a computing device — e.g. Turing machines or circuits. For future reference, let us recall the definition of the circuit model:

- Input (of size n): sequence $\omega = b_1 \cdots b_n 00 \cdots 0$ of n bits and a number of extra bits as working space for the computation.
- Computational step: application of a given Boolean operation (gate) to designated bits, thus updating the total bit string. This gates are typically a *universal* subset of Boolean operations.
- Circuit: prescribed sequence of computational steps. Each circuit is generated in a uniform way as a function of n (thus precluding the encoding of some hard computational problem into the changing structure of a family of circuits for different inputs).
- Output: the value of some designated subset of bits after the execution of the final step.

The time *complexity* question is rephrased as: *how many steps (in the worst case) does the algorithm require for any input of size* n*?*, which coincides with the number of gates in the circuit.

This model can be extended (as Turing machines do) to include classical probabilistic choices (useful for comparison with outputs of quantum measurements, that are generally probabilistic).

- the input $b_1 \cdots b_n 00 \cdots 0$ is extended to

$$b_1 \cdots b_n 00 \cdots 0 z_1 \cdots z_k 00 \cdots 0$$

  where $z_1 \cdots z_k$ is sequence of bits each of which is set to 0 or 1 uniformly at random.

- If the computation is repeated with the same input the random bits will generally be different, the output becomes a sample from a probability distribution over all possible output strings, which is generated by the uniformly random choice of $z_1 \cdots z_k$. Thus any output probability must always have the form $\frac{x}{2^k}$ for some integer $x \leq 2^k$.

In this setting (that can be used e.g. to implement probabilistic choices of gates), the output is correct with *suitably high probability*, according to some specified criteria.

Note that, while computability is *independent* of the particular notion of computation considered (cf, the Church-Turing thesis, which states that every physically realizable *computation device* can be simulated by a Turing machine), this is probably not the case when discussing computational complexity.

The corresponding conjecture is known as the *strong* Church-Turing thesis which claims that every physically realizable *computation model* can be simulated by a Turing machine with polynomial overhead. This is a bit controversial because some problems are known to be solvable in polynomial time on a quantum computer, but not known to be solvable in polynomial time on a classical one. Moreover, it seems that quantum computations cannot be efficiently simulated in a classical Turing machine [3].

1.2 Growth rate of a function

Example: transitive closure of $R \subseteq A^2$

from 'above' R^* is the smallest relations containing R that is transitive and reflexive.

from 'below'

$$R^* = \{(a, b) \mid a, b \in A \text{ there exists a path from } a \text{ to } b \text{ in } R\}$$

which suggests an algorithm:

Algorithm 1: TC1.

```
R* := ∅;
for i := 1..n do
    for each i-tuple (b_1, ··· , b_i) ∈ A^i do
        if it is a path then
            R* := R* ∪ {b_1, b_i}
    end
end
```

The following notation is adopted to capture the growth rate of a function:

$$\mathcal{O}(f) = \{g \in \mathbb{N}^{\mathbb{N}} \mid \exists_{c,d \in \mathbb{N}^+}. \forall_n. g(n) \leq c.f(n) + d\}$$

Stating that

$$h \in \mathcal{O}(f)$$

means that h *is no faster than* f. An equivalence relation relating rates of growth is defined by

$$f \sim g \text{ iff } f \in \mathcal{O}(g) \wedge g \in \mathcal{O}(f)$$

Example: $p(n) = 31n^2 + 17n + 3$

Clearly $p(n) \leq 48n^2 + 3$, because $n^2 \geq n$. Thus $p \in \mathcal{O}(-^2)$ with constants 48 and 3.
However, $-^2 \in \mathcal{O}(p)$ with constants 1 and 0.

Theorem

For any polynomial $p(n) = c_k n^k + \cdots + c_1 n + c_0$, $p \in \mathcal{O}(-^k)$ with constants $\sum_{1 \leq i \leq k} c_i$ and c_0.

Theorem

Any two polynomials p and q with the same degree verify $p \sim q$.

Theorem: the exponential barrier

The growth rate of function 2^n is higher than the one of an arbitrary polynomial.

Proof.
We want to show that

$$n^i \in \mathcal{O}(2^n) \quad \text{i.e. } n^i \leq c2^n + d$$

Let $c = (2i)^i$ and $d = (i^2)^i$, and consider two cases:

- $n \leq i^2 \Rightarrow n^i \leq c2^n + d$, because $n^i \leq d$
- $n > i^2 \Rightarrow n^i \leq c2^n + d$, because we may prove that $n^i \leq c2^n$ as follows

First observe that $n^i \leq (iq+i)^i = i^i(q+1)^i$, for q the integer quotient of n by i (i.e. $iq \leq n \leq i(q+1)$). Now,

$$\begin{array}{ll} & i^i(q+1)^i \\ \leq & \quad \{ n \leq 2^n \} \\ & i^i(2^{q+1})^i \\ \leq & \quad \{ \text{definition of } c \} \\ & c2^{qi} \\ \leq & \quad \{ \text{definition of } q \} \\ & c2^n \end{array}$$

Observe now that if a polynomial had the same growth rate than $-^2$, then any polynomial of a higher degree would have the same rate (because we've just proved that no polynomial grows as fast as $-^2$). But this leads to a contradiction because, as shown above, polynomials of different degrees have different rates of growth.

□

Clearly, 2^n has a higher rate of grow than any polynomial. Other exponential functions — for example, 27^n, n^n, $n!$, 2^{n^2} or 2^{2^n} — have even higher rates of growth.

2 Case study: Closure algorithms

2.1 Computing R^*

Algorithm 2: TC2.

$R^* := \emptyset$;
for $i := 1..n$ **do**
  for *each* i-*tuple* $(b_1, \cdots, b_i) \in A^i$ **do**
    if *it is a path* **then**
      $R^* := R^* \cup \{b_1, b_i\}$
  end
end

The algorithm examines each sequence $(b_1, \cdots, b_i)$; if this is a path add to the solution. Thus, the total number of basic operations (test and add) is

$$n(1 + n + n^2 + \cdots + n^n)$$

i.e. in each of the n iterations look for paths of length up to n. Therefore, TC1 $\in \mathcal{O}(n^{n+1})$.

Algorithm 3: TC3.

$R^* := R \cup \{(a, a) \mid a \in A\}$;
while $\exists_{a_i, a_j, a_k \in A}.\ (a_i, a_j), (a_j, a_k) \in R^*,\ (a_i, a_k) \notin R^*$ **do**
  $R^* := R^* \cup \{(a_i, a_k)\}$
end

- In each iteration one pair (if any) is added. Thus, the maximum number of additions corresponds to the maximum number of pairs available, i.e. n^2.
- In each iteration the algorithm searches for n^3 triples.

Therefore, TC3 $\in \mathcal{O}(n^2 \times n^3) = \mathcal{O}(n^5)$.

Exercise 1

The algorithm TC3 repeatedly searches for violations of the transitivity property. However, each triple must be checked again and again since the introduction of a new pair may entail new violations in triples that have already been checked. A better algorithm of $\mathcal{O}(n^2 \times n) = \mathcal{O}(n^3)$ can be obtained by imposing an order to the triples so that a new pair added does not violate the transitivity condition established for triples already considered. In the algorithm below, TC4, triples are ordered by the middle index (in increasing order). Explain why the algorithm works and its growth rate.

Algorithm 4: TC4.

$R^* := R \cup \{(a, a) \mid a \in A\}$;
for $j = 1, 2, \cdots, n$ **do**
  for $i = 1, 2, \cdots, n$ *and* $k = 1, 2, \cdots, n$ **do**
    if $(a_i, a_j), (a_j, a_k) \in R^*$ *but* $(a_i, a_k) \notin R^*$ **then**
      $R^* := R^* \cup \{(a_i, a_k)\}$
  end
end

2.2 Closure problems

A subset $C \subseteq A$ is *closed* for a relation $R \subseteq A^{n+1}$ if

$$b_{n+1} \in C \quad \Leftarrow \quad b_1, \cdots, b_n \in C \ \wedge \ (b_1, \cdots, b_n, b_{n+1}) \in R$$

e.g.

- $\mathbb{N}$ is closed for $+$
- the set of ancestors is closed for the relation *parent-of*
- any set is closed for $\subseteq$

Closure property: *The set* C *is closed under relations* $R_1, \cdots, R_m$

cf, the usual construction *the smallest set that contains* A *and has property* ϕ. But note that not all properties guarante the existence of a smallest set satisfying ϕ. However,

Theorem

If ϕ is a closure property defined by relations $R_1, \cdots, R_m$ on a set A and $B \subseteq A$, then there exists the smallest set C st $B \subseteq C$ and C has property ϕ.

Proof.
Let ϕ be defined by a relations $R_1, \cdots, R_m$ and S denote the set of subsets of A containing B and closed for each R_i. Clearly $S \neq \emptyset$ (why?). Then, define $C = \bigcap S$ (which is well defined because S is non empty). Then,

- $B \subseteq C$, by construction.
- C is closed under all relations $R_1, \cdots, R_m$. To see this, suppose $a_1, \cdots, a_{n-1} \in C$ and $(a_1, \cdots, a_{n-1}, a_n) \in R$. All sets in S contain $a_1, \cdots, a_{n-1}$ and because all of them are closed, all have a_n. Thus, $a_n \in C$.
- C is minimal: no strict subset C' of C exists (otherwise $C' \in S$ and $C \subseteq C'$).

□

Exercise 2

Set C in the theorem above is the *closure* of B under relations $R_1, \cdots, R_m$. Determine the closure of the singleton set containing yourself under the relation *parent of*. Similarly, determine the closure of set $\{0, 1\}$ under addition and the closure of the set of natural numbers under subtraction. Note that R^*, for a relation R is the closure of R under transitivity and reflexivity.

Theorem

Any closure property over a finite set can be computed in polynomial time.

Proof.

Algorithm 5: Computing a generic closure.

$C^\circ := C$;
while $\exists_{1 \leq i \leq k}$ *and* r_i *elements* $a_{j1} \cdots a_{j_{r_{i-1}}} \in C^\circ$ *and* $a_{j_{r_i}} \in D \setminus C^\circ$. $(a_{j1} \cdots a_{j_{r_i}}) \in R_i$ **do**
  $C^\circ := C^\circ \cup \{a_{j_{r_i}}\}$
end

Thus, the algorithm is $\mathcal{O}(n^{r+1})$ where n is the cardinal of D and r is the greatest arity of all relations considered.

□

Theorem

Any algorithm in polynomial time can be rendered as the computation of a closure over a set for a set of relations.

3 Complexity classes

3.1 Polynomial complexity

Consider the famous *Travelling Salesperson* problem: Given a map with n cities and distances among them, produce an itinerary that minimizes the total distance travelled.

Clearly, the problem can be solved (e.g. systematic examination of all itineraries). But, on the other hand, it remains unsolvable in any practical sense by current computers: too many itineraries — $(n-1)!$ — to be explored. Notice that a $(n-1)!$ algorithm goes faster than 2^n. For 40 cities the number of itineraries is enormous: 39!. Even if 10^{15} of them could be inspected per second (a value our of reach of current supercomputers) the required time for completing the calculation would be several billion lifetimes of the universe.

What is a *practically feasible algorithm*?

The general answer is: it should run for a number of steps bounded by a *polynomial* in the length of the input, i.e. have a polynomial rate of growth.

Thus, an algorithm whose running time can be upper-bounded by any polynomial function is considered *efficient*. On the other hand, it is regarded as *inefficient* if it can be lower-bounded by any exponential function. There are, of course, many growth rates that fall between polynomial and exponential — for example $n \log n$ — but the polynomial / exponential separation seems to be a basic frontier for all practical purposes. One may wonder why such a *quantitative gap* seems so important and full of deep, foundational implications. The following quote by a well-known expert in quantum computation, Scott Aaronson, is in order:

> *One might think that, once we know something is computable, whether it takes* 10 *seconds or 20 seconds to compute is obviously the concern of engineers rather than philosophers. But that conclusion would not be so obvious, if the question were one of* 10 *seconds versus* $10^{10^{10}}$ *seconds! And indeed, in complexity theory, the quantitative gaps we care about are usually so vast that one has to consider them qualitative gaps as well. Think, for example, of the difference between reading a* 400*-page book and reading every possible such book, or between writing down a thousand-digit number and counting to that number.*
>
> Scott Aaronson [1].

A language is *polynomially decidable* if there is a polynomially bounded Turing machine that decides it, i.e. a Turing machine which always halts after at most $p(n)$ steps, where $p(n)$ is a polynomial and n is the length of the input.

The class P of such languages is the quantitative analog of the class of recursive languages. As the latter, it is closed under complement, union, intersection, concatenations and Kleene star. But, on the other hand, not all recursive languages are polynomially decidable.

Theorem

$S \notin \mathsf{P}$, where

$$S = \{\text{"}M\text{"}\,\text{"}w\text{"} \mid M \text{ accepts input } w \text{ after at most } 2^{|w|} \text{ steps}\}$$

Proof.

If $S \in \mathsf{P}$, language

$$S' = \{\text{"}M\text{"} \mid M \text{ accepts input "}M\text{" after at most } 2^{|\text{"}M\text{"}|} \text{ steps}\}$$

and its complement are also in P. This means that there exists a polynomially bounded Turing machine B which accepts all descriptions of Turing machines that fail to accept their own description in 2^n steps, where n is the length of the description, and halt in $p(n)$ steps for a polynomial $p(n)$.

Does B accept its own description "B"?

- If YES then B fails to accept "B" within $2^{|\text{"B"}|}$ steps. However, B halts in |"B"| steps (because, by assumption, the complement of S' is in P). This means that B halts much before $2^{|\text{"B"}|}$. Thus it should reject "B", which leads to a contradiction. Note that there is always an integer n_0 such that $p(n) \leq 2^n$ for all $n \geq n_0$, and we may safely assume $|\text{"B"}| \geq n_0$.
- If NO a similar argument also leads to contradiction.

Summing up:

The class P (from *polynomial time*)

Is the class of all languages for which the membership problem has a classical algorithm that runs in polynomial time and gives the correct answer with certainty. As mentioned above, polynomial computations are regarded as *tractable* or *computable in practice*. On the other hand, non-polynomial computations are regarded as *intractable* as a small variation in the input size may require resources exceeding reasonable limits (e.g. the running time may exceed the number of atoms in the universe).

3.2 The classes BPP and PSPACE

BPP (from *bounded error probabilistic polynomial time*)

One objection raised against the definition of P is the fact that it sweeps behind the carpet the possibility of resorting to *randomness* as a computational resource. If such a thing exists in the Universe, one may conceive algorithms involving some sort of random choices such as initializing a variable with a value chosen at random from some range. Such algorithms, classified as *randomized* ou *probabilisitc*, are basically implementations of a random number generator[1].

[1] It turns out that generating random bits, i.e. tossing fair coins, is enough.

They can be implemented in *probabilisitc* Turing machines. These are Turing machines with two, rather than one, transition relations in their finite control. At each step one may choose, with equal probability, which of them to apply, a decision which is independent of all previous choices made.

The BPP class plays for randomized algorithms a role similar to that of P for the determinisitc case: it captures efficient probabilistic computation. Formally, it is the class of all languages whose membership problem has a classical randomised algorithm that runs in polynomial time and gives the correct answer with probability at least $\frac{2}{3}$ for every input. The class corresponds to the formalisation of decision problems that are feasible on a classical computer.

The choice of $\frac{2}{3}$ above is a bit arbitrary and can be replaced by any other number $\frac{1}{2} + \delta$, with $0 < \delta < \frac{1}{2}$. This is proved as follows:

- Consider an algorithm for a decision problem that works correctly with probability $\frac{1}{2} + \delta$, and repeat its execution k times.
- Take the majority vote of all k answers as the output.
- According to the so-called *Chernoff bound* or *amplification* lemma (see [5] for a proof), this answer is correct with a probability at least $1 - e^{-2k\delta^2}$ approaching 1 exponentially fast. Thus there will be value k such that this probability will exceed $1 - \epsilon$ for any $\epsilon > 0$.
- If the original algorithm had polynomial running time $\tau(n)$, this one will have $k\tau(n)$, which is still polynomial in n.

Clearly $P \subseteq BPP$, but strict inclusion remains an open questions: most probably, randomized computation may be no more powerful than deterministic computation.

PSPACE (from *polynomial space complexity*)

Is the class of of all decision problems that can be solved within a polynomially bounded amount of space as a function of the input size. Clearly

$$P \subseteq BPP \subseteq PSPACE$$

because any polynomial time computation occurs in polynomial space since polynomial many 1- and 2-bit gates can act on at most polynomial many bits in total. Similarly in any randomised polynomial time computation, for each fixed choice of the random bits, we can perform the associated computation in polynomial space. Then doing this sequentially in turn (re-using the same polynomial space allocation) for each of the exponentially many choices of the random bits, we can keep a running total of accept and reject answers, and thus get $BPP \subseteq PSPACE$.

It is not known whether any of these inclusions are strict.

3.3 The class NP

Most interesting problems mentioned above for which no polynomial algorithm exists — *Traveling Salesperson, Satisfiability, Independent Set, Integer Partition*, etc., can be solved by polyno-

mially bounded *nondeterministic* Turing machines. As in the probabilistic case, such machines have two transition functions which are arbitrarily chosen before being applied. Given an input, the machine decides if there is a sequence of (non-determinisitic) choices leading to an acceptance state.

This defines the class NP (of *nondeterministic* polynomial languages), i.e. of languages that can be decided by a polynomially bounded nondeterministic Turing machine. Note the meaning of *decision* in this context. For a nondeterministic Turing machine to decide a language means that all the computations of the machine must reject an input not in the language, whereas an input from the language must be accepted by at least one computation. Actually, the qualifier NP does not refer to non-polynomial, but to non-deterministic.

Although determinism and nondeterminism in the definition of Turing machines do not interfere on their expressiveness in what concerns decidability, separating determinism from nondeterminism at the polynomial level, remains unsolved. It is the famous $P \neq NP$ conjecture, which addresses the question whether or not the two classes are the same.

Another way to think about the complexity class NP is as the class that captures the set of problems whose solutions can be efficiently *verified*, i.e. verified in polynomial time with respect to the length of the input (as a Turing machine only reads one bit in each step). By contrast, the class P contains decision problems that can be efficiently *solved*.

Formally, a language $L \subseteq \{0,1\}^*$ is in NP if there is a polynomial $p : \mathbb{N} \longrightarrow \mathbb{N}$ and a polynomial-time Turing machine M such that, for every input $v \in \{0,1\}^*$,

$$v \in L \;\Leftrightarrow\; \exists_{t \in \{0,1\}^{p(|v|)}}.\; M(v,t) = 1$$

Turing machine M is called a *certifier* for L, and string t a *certificate*, or a *witness* for input v.

Example

The *dinner party problem* discussed above in NP. Its language contains all pairs (G, n) such that graph G has a subgraph of at least n vertices with no edges between them. A polynomial-time verification algorithm proceeds as follows:

- given a pair (G, n) and a string $t \in \{0,1\}^*$, output 1 iff t encodes a list of n vertices of G such that there is no edge between any two members of the list.
- Clearly, (G, n) is in the language iff there exists a t such that $M(G, n, t) = 1$. Therefore, the language is in NP. The sequence t of v vertices is the certificate that (G, n) is in the language.

Note that for G with N vertices, the certifier string t can be encoded with at most $\mathcal{O}(n \log N)$ bits. This is polynomial in the size of the representation of graph G.

Clearly $P \subseteq NP$, by making $p(|v|) = 0$ and thus reducing the certifier t to the empty string. The reverse inclusion is far less obvious. As Scott Aaronson puts it [1],

If $\mathsf{P} = \mathsf{NP}$ *then the ability to check the solutions to puzzles efficiently would imply the ability to find solutions efficiently. An analogy would be if anyone able to appreciate a great symphony could also compose one themselves!*

Or, formulated in another way,

If $\mathsf{P} = \mathsf{NP}$ *then whenever a theorem had a proof of reasonable length, we could find that proof in a reasonable amount of time. In such a situation, we might say that "for all practical purposes", Hilbert's dream of mechanizing mathematics had prevailed, despite the undecidability results of Gödel, Church, and Turing.*

The *dinner party problem* has an important completeness property: All problems (languages) in NP can be reduced to it in polynomial time (just as all recursively enumerable languages reduce to the halting problem). Formally, a language L *reduces* to another language H if there is a polynomial-time computable function $\mathsf{h} : \{0,1\}^* \longrightarrow \{0,1\}^*$ such that

$$\forall_{v \in \{0,1\}^*}.\ v \in \mathsf{L} \Leftrightarrow \mathsf{f}(v) \in \mathsf{H}$$

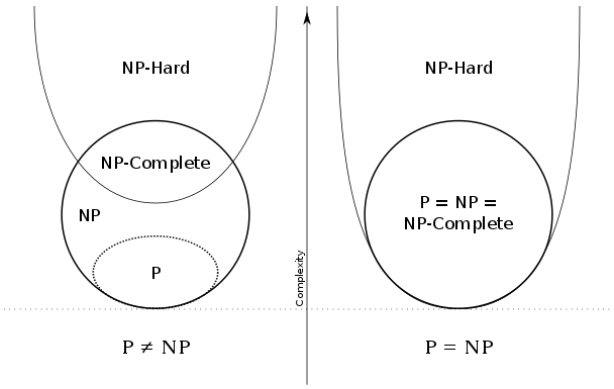
Such NP problems are called NP-complete, and they are the hardest of all NP problems.

The number of real-life problems that are known to be NP-complete is enormous. Examples include

- Sudoku and jigsaw puzzles;
- the *Traveling Salesperson* problem;
- the satisfability problem for propositional formulas;
- more generally, the problem of finding whether a mathematical statement has a proof with a determined number of symbols or fewer, in some formal system.

among many others. Note that each of them has a polynomial algorithm iff $\mathsf{P} = \mathsf{NP}$.

NP-hard problems live even beyond NP, i.e. they are at least as hard as the hardest problems in NP. A problem is NP-hard if every problem in NP can be reduced to it in polynomial time. There are problem which are NP-hard but not NP-complete. For example the *halting problem* which fails to be decidable. The following diagram, borrowed from the WWW, sums up our discussion:



4 Problems, problems, problems ...

Reachability. Given two nodes of a finite graph decide if there is a path connecting them.

Is a variant of the reflexive-transitive closure problem. Can be solved by computing this closure in time $\mathcal{O}(n^3)$ and inspect the result.

A *problem* is a set of inputs, typically infinite, with a Boolean question to be asked of each input. As mentioned below, a problem needs to be encoded as a language problem so that its complexity can be analysed in a common setting. For example, the *Reachability* problem can be reduced to a decision problem for the language

$$R = \{K(G)s(i)s(j) \mid \text{there is a path in } G \text{ connecting nodes } n_i \text{ to } n_j\}$$

where K and s are suitable binary encoding functions for graphs and integers.

Euler Cycle. Given a graph is there a closed path in it that uses each edge exactly once?

Note that the path can go many times through the same node (or even not at all if there are isolated nodes). It can be proved that the necessary condition on a graph to have such a path is that i) all nodes have equal numbers of incoming and outgoing edges, and ii) for each pair of nodes, neither of which isolated, there is a path connecting them. So, clearly the corresponding language

$$G = \{K(G) \mid G \text{ has an Euler cycle}\}$$

where $K(G)$ is some encoding of graphs as strings, is in P.

Hamilton Cycle. Given a graph is there a cycle that passes through each node exactly once?

No polynomial algorithm is known. Of course the trivial one (generate all paths and choose) is not polynomial.

Equivalence of Finite Automata. Given two deterministic automata, determine whether they recognise the same language?

The problem is polynomial, as it is the variant in which only regular expressions are considered. However, one cannot conclude about the latter just by reducing to the former: actually, the generation of a finite automaton from a regular expression may increase exponentially the number of states.

Integer Partition. Given a set of n nonnegative integers represented in binary, is there a subset S of the original set such that $\sum_{i \in S} a_i = \sum_{i \notin S} a_i$?

The algorithm is $\mathcal{O}(nV)$ where V is the sum of all numbers in the original set divided by 2. In spite of its polynomial appearance, the problem is not polynomial in the length of the input. The reason is that the integers are encoded in binary: if all integers are about 2^n, then S is close to $2^n \times \frac{n}{2}$.

Satisfiability. Is a Boolean formula in conjunctive normal form satisfiable?

No polynomial algorithm is known. However, if reduced to formulas with a maximum of two literals, it becomes polynomial.

Optimisation problems require us to find the best among many possible solutions, according to some cost function. The trick to transform optimisation into language problems is to fix each input with a *bound* on the cost function. For example, the *Traveling Salesperson* problem can be rephrased as *given an integer* $n \geq 2$*, a* $n \times n$ *distance matrix, and an integer* $b \geq 0$*, find a permutation of* n *such that its cost is less or equal to* b (which, to build up intuition, may be regarded as a budget).

Independent Set. Given an undirected graph and an integer $k \geq 2$ is there a subset s of the set of vertices with $|s| \geq k$ such that for any two vertices in s there is no edge connecting them? This is exactly the *dinner party* problem.

Clique. Given an undirected graph and an integer $k \geq 2$ is there a subset s of the set of vertices with $|s| \geq k$ such that for all vertices in s there is an edge connecting each pair?

Node Cover. Given an undirected graph and an integer $k \geq 2$ is there a subset s of the set of vertices with $|s| \leq k$ such that s covers all edges of the graph, e.g. to minimise the number of guards in a museum?

Note that a set of nodes covers an edge if it contains at least one endpoint of the edge.

No polynomial algorithms are known for these problems.

Further study.

My preference on complexity theory is Papadimitriou's wonderful book [5]; reference [4] provides an interesting alternative. S. Arora and B. Barak book [2] is a more recent textbook covering recent achievements in complexity theory (including challenges from quantum computation) and putting them in the context of the classical results.

References

[1] S. Aaronson. Why philosophers should care about computational complexity. In B. J. Copeland, C. Posy, and O. Shagrir, editors, *Computability: Turing, Gödel, Church, and Beyond*, pages 261–328. MIT Press, 2013.

[2] S. Arora and B. Barak. *Computational Complexity: A Modern Approach*. Cambridge University Press, 2009.

[3] D. Deutsch. Quantum theory, the Church-Turing principle and the universal quantum computer. *Proceedings of the Royal Society of London A*, 400:97–117, 1985.

[4] D. Z. Du and K. I. Ko. *Theory of Computational Complexity*. Addison-Wesley, 2000.

[5] C. H. Papadimitriou. *Computational Complexity*. Addison-Wesley, 1994.