Computação Quântica

Problems on the Quantum Fourier Transform — May 2021

This exercise aims at improving your understanding of the quantum Fourier transform, a most relevant component in several quantum algorithms.

Recall the definition of QFT on K basis states:

$$\mathsf{QFT}_K(|x\rangle) \;=\; \frac{1}{\sqrt{K}}\sum_{y=0}^{K-1} e^{2\pi i(\frac{x}{K})y}|y\rangle$$

- Compute $\mathsf{QFT}_K(|00\cdots 0\rangle)$.
- The following equality

  $$\mathsf{QFT}_K(|x_1\cdots x_n\rangle) \;=\; \left(\frac{|0\rangle + e^{2\pi i(0.x_n)}|1\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|0\rangle + e^{2\pi i(0.x_n x_{n-1})}|1\rangle}{\sqrt{2}}\right)\cdots\otimes\cdots\left(\frac{|0\rangle + e^{2\pi i(0.x_1 x_2\cdots x_n)}|1\rangle}{\sqrt{2}}\right)$$

  was used in the lecture slides without proof. Verify it holds indeed.
- One can show, as we did in the lectures, that QFT is a unitary gate by building a unitary quantum circuit for its computation. Give an alternative, direct proof that the linear transformation defined above is unitary.
- Reproduce the circuit for QFT_4 and QFT_8, and compute the corresponding matrices. Give your calculation in detail.

Notes

Question 1

Fix $K = 2^n$,

$$\mathrm{QFT}_K(|00\cdots0\rangle) \;=\; \frac{1}{\sqrt{K}}\sum_{y=0}^{K-1} e^{2\pi i(\frac{0}{K})y}|y\rangle \;=\; \frac{1}{\sqrt{K}}\sum_{y_1,y_2\cdots,y_n=0}^{1}|y_1y_2\cdots y_n\rangle$$

Clearly,

$$\mathrm{QFT}_4(|00\rangle) \;=\; \frac{1}{2}(|00\rangle+|01\rangle+|10\rangle+|11\rangle)$$

and $\mathrm{QFT}_2 = H$.

Question 2

Let us first consider the case of QFT_4 applied to $|x\rangle = |x_1x_2\rangle$.

$$\begin{aligned}
\mathrm{QFT}_4(|x\rangle) &= \frac{1}{2}\sum_{y=0}^{3} e^{2\pi i x y 2^{-2}}|y\rangle \\
&= \frac{1}{2}\sum_{y_1,y_2=0}^{1} e^{2\pi i x(y_1 2^{-1}+y_2 2^{-2})}\,|y_1y_2\rangle \\
&= \frac{1}{2}\sum_{y_1,y_2=0}^{1} (e^{2\pi i x y_1 2^{-1}}|y_1\rangle \otimes e^{2\pi i x y_2 2^{-2}}|y_2\rangle) \\
&= \frac{1}{2}\sum_{y_1=0}^{1} (e^{2\pi i x y_1 2^{-1}}|y_1\rangle \otimes \sum_{y_2=0}^{1} e^{2\pi i x y_2 2^{-2}}|y_2\rangle) \\
&= \frac{(|0\rangle + e^{2\pi i x 2^{-1}|1\rangle})}{\sqrt{2}} \otimes \frac{(|0\rangle + e^{2\pi i x 2^{-2}|2\rangle})}{\sqrt{2}} \\
&= \frac{(|0\rangle + e^{2\pi i (x_1.x_2)|1\rangle})}{\sqrt{2}} \otimes \frac{(|0\rangle + e^{2\pi i (0.x_1x_2)|2\rangle})}{\sqrt{2}} \\
&= \frac{(|0\rangle + e^{2\pi i (0.x_2)|1\rangle})}{\sqrt{2}} \otimes \frac{(|0\rangle + e^{2\pi i (0.x_1x_2)|2\rangle})}{\sqrt{2}}
\end{aligned}$$

The first reduction resorts to the following fact for $|y\rangle = |y_1y_2\rangle$

$$\frac{y}{2^n} \;=\; \sum_{j=1}^{n} y_j 2^{-j}$$

The last one to

$$e^{2\pi i(a.b)} \;=\; e^{2\pi i a}e^{2\pi i(0.b)} \;=\; e^{2\pi i(0.b)}$$

The general case follows exactly the same argument.

$$
\begin{aligned}
\mathsf{QFT}_K(|x\rangle) &= \frac{1}{\sqrt{2^n}}\sum_{y=0}^{K-1} e^{2\pi i x y 2^{-n}}|y\rangle \\
&= \frac{1}{\sqrt{2^n}}\sum_{y_1,\cdots,y_n=0}^{1} e^{2\pi i x(\sum_{p=1}^{n} y_p 2^{-p})}\,|y_1\cdots y_n\rangle \\
&= \frac{1}{\sqrt{2^n}}\sum_{y_1,y_2=0}^{1}\bigotimes_{p=1}^{n} e^{2\pi i x y_p 2^{-p}}|y_p\rangle \\
&= \frac{1}{\sqrt{2^n}}\bigotimes_{p=1}^{n}\left(\sum_{y_p=0}^{1} e^{2\pi i x y_p 2^{-p}}|y_p\rangle\right) \\
&= \frac{1}{\sqrt{2^n}}\bigotimes_{p=1}^{n}\left(|0\rangle + e^{2\pi i x 2^{-p}}|1\rangle\right) \\
&= \left(\frac{|0\rangle + e^{2\pi i(0.x_n)}|1\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|0\rangle + e^{2\pi i(0.x_n x_{n-1})}|1\rangle}{\sqrt{2}}\right)\cdots\otimes\cdots\left(\frac{|0\rangle + e^{2\pi i(0.x_1 x_2\cdots x_n)}|1\rangle}{\sqrt{2}}\right)
\end{aligned}
$$

Question 3

A somehow indirect, but easy way to show an operator is unitary, is to recall that unitarian operators preserve internal products:

$$
(U|v\rangle, U|u\rangle) \;=\; \langle v|U^\dagger U|u\rangle \;=\; \langle v, u\rangle \tag{1}
$$

Let

$$
U \;=\; \mathsf{QFT}_K(|x\rangle) \;=\; \frac{1}{\sqrt{K}}\sum_{y=0}^{K-1} e^{2\pi i(\frac{x}{K})y}|y\rangle
$$

and compute

$$
\begin{aligned}
& \langle v|U^\dagger U|u\rangle \\
= & \qquad \{\text{ definitions}\} \\
& \left(\frac{1}{\sqrt{K}}\sum_{y=0}^{K-1} e^{2\pi i(\frac{v}{K})y}|y\rangle),\frac{1}{\sqrt{K}}\sum_{y=0}^{K-1} e^{2\pi i(\frac{u}{K})y}|y\rangle\right) \\
= & \qquad \{\ (\alpha|x\rangle,\beta|y\rangle) = \overline{\alpha}\beta\langle x|y\rangle\} \\
& \frac{1}{K}\sum_{y=0}^{K-1} e^{2\pi i(\frac{(u-v)}{K})y}
\end{aligned}
$$

<u>Case 1</u>: $u = v$. Then,

$$
\frac{1}{K}\sum_{y=0}^{K-1} e^{2\pi i(\frac{(u-v)}{K})y} \;=\; \frac{K}{K} \;=\; 1
$$

<u>Case 2</u>: $u \neq v$. Then,

$$
\frac{1}{K}\sum_{y=0}^{K-1} e^{2\pi i(\frac{(u-v)}{K})y} \;=\; \frac{1}{K}\sum_{y=0}^{K-1} r^k \;\text{ where } r = e^{2\pi i(\frac{(u-v)}{K})y}|y\rangle
$$

which boils down to[1]

$$\frac{1}{K}\frac{1-r^n}{1-r} = \frac{1}{K}\frac{1-e^{2\pi i(u-v)}}{1-r} = 0 \text{ because } (u-v) \text{ is an integer.}$$

Thus, equality (1) holds, recalling the both $|u\rangle$ and $|v\rangle$ are vectors in an orthonormal basis.

Question 4

The circuits are direct instances of the general case for QKT_K discussed in the lectures. QKT_4 uses the rotation gate

$$R_2 = \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i 2^{-2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

and QKT_8 resorts both to R_2 and

$$R_3 = \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i 2^{-3}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi 2^{-2}} \end{bmatrix}$$

The corresponding matrices are computed along the circuit. For exemple, for the second case, one obtains

$$\frac{1}{\sqrt{2^3}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \rho & \rho^2 & \rho^3 & \rho^4 & \rho^5 & \rho^6 & \rho^7 \\ 1 & \rho^2 & \rho^4 & \rho^6 & 1 & \rho^2 & \rho^4 & \rho^6 \\ 1 & \rho^3 & \rho^6 & \rho & \rho^4 & \rho^7 & \rho^2 & \rho^5 \\ 1 & \rho^4 & 1 & \rho^4 & 1 & \rho^4 & 1 & \rho^4 \\ 1 & \rho^5 & \rho^2 & \rho^7 & \rho^4 & \rho & \rho^6 & \rho^3 \\ 1 & \rho^6 & \rho^4 & \rho^2 & 1 & \rho^6 & \rho^4 & \rho^2 \\ 1 & \rho^7 & \rho^6 & \rho^5 & \rho^4 & \rho^3 & \rho^2 & \rho \end{bmatrix}$$

where $\rho = \sqrt{i} = e^{\frac{2\pi i}{8}}$.

[1] cf the sum of the first n of a geometric progression: $\sum_{i=0}^{n-1} ar^i = a_0 + \frac{1-r^n}{1-r}$.