Lecture 1: Automata & discrete mathematical structures

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Summary
(1) Automata as a classical computational model
(2) Sets, functions, relations. Isomorphism and cardinality.
(3) Ordered structures: preorders, partial orders, lattices. Complete lattices. Ideals and filters. Boolean algebras. Application: the theorem of Knaster-Tarski. Lattices as algebraic structures.
(4) Reversibility. Groups as a prototypical algebraic structure. Groups of permutations. Action of a group. Application: Cayley theorem.

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1 Automata as a classical computational model

$\boxed{\text{Automata}}$

An automaton over a set N of names is a tuple $\langle \mathsf{S}, s_0, \mathsf{N}, \downarrow, \longrightarrow \rangle$ where

- $\mathsf{S} = \{s_0, s_1, s_2, ...\}$ is a set of states, with a distinguished initial state s_0
- $\downarrow \subseteq \mathsf{S}$ is the set of terminating or final states

$$\downarrow s \;\equiv\; s \in \downarrow$$

- $\longrightarrow \subseteq \mathsf{S} \times \mathsf{N} \times \mathsf{S}$ is the transition relation, often given as an N-indexed family of binary relations

$$s \xrightarrow{a} s' \;\equiv\; \langle s', a, s \rangle \in \longrightarrow$$

$\boxed{\text{Variants}}$

- deterministic
- non deterministic
- finite
- image finite
- ...

Morphism

A morphism relating two automata over N, $\langle \mathsf{S}, \mathsf{N}, \mathsf{s}_0, \downarrow, \longrightarrow \rangle$ and $\langle \mathsf{T}, \mathsf{N}, \mathsf{t}_0, \downarrow', \longrightarrow' \rangle$, is a function $\mathsf{h} : \mathsf{S} \longrightarrow \mathsf{S}'$ st

$$\begin{aligned} s \xrightarrow{a} s' \quad &\Rightarrow \quad h\, s \overset{a}{\longrightarrow'} h\, s' \\ s \downarrow \quad &\Rightarrow \quad h\, s \downarrow' \\ t_0 \ = \ h(s_0) & \end{aligned}$$

Morphisms preserve transitions, initial state and termination

Reachability

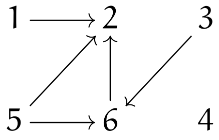
The reachability relation, $\longrightarrow^* \subseteq \mathsf{S} \times \mathsf{N}^* \times \mathsf{S}$, is defined inductively

- $s \xrightarrow{\epsilon}{}^{*}\ s$ for each $s \in \mathsf{S}$, where $\epsilon \in \mathsf{N}^*$ denotes the empty word;
- if $s \xrightarrow{a} s''$ and $s'' \xrightarrow{\sigma}{}^{*}\ s'$ then $s \xrightarrow{a\sigma}{}^{*}\ s'$, for $a \in \mathsf{N}, \sigma \in \mathsf{N}^*$

A state $t \in \mathsf{S}$ is reachable from $s \in \mathsf{S}$ iff there is a word $\sigma \in \mathsf{N}^*$ st $s \xrightarrow{\sigma}{}^{*}\ t$

Matricial representation

(with $\mathsf{N} = \emptyset$)



- The states of a system correspond to column vectors;
- The automata dynamics is encoded in Boolean matrices: $M[i, j] = 1$ if and only if there is an edge (path of length 1) from vertex j to vertex i;
- Multiplying the current state vector by matrix M yields progress from one state to another in one time step;
- Multiple step dynamics are obtained via matrix multiplication.

2 Sets, relations, cardinality

- Set, function, composition, isomorphism.
- Powerset ($\mathbf{2}^A$); partition.
- Binary relations; $\mathbf{2}^{A\times B} \cong \mathbf{2}^{A^B}$
- Equivalence relations. Quotient set as a partition.

$\boxed{\text{Finite and infinite sets.}}$

- injective *vs* sujective functions
- equicardinality *vs* isomorphism.
- finite *vs* infinite
- countable *vs* uncountable

$\boxed{\text{Ranking cardinality}}$

$$|A| \leq |B| \ \text{ iff } \ \text{there is an injection } f : A \longrightarrow B$$

Relation $\leq$ above is a total order. Note that proving antisymmetry (i.e. the Cantor-Bernstein-Schroeder theorem) and totality (which requires the axiom of choice) is extremely hard.

$\boxed{\text{Theorem}}$
$\mathbb{N}$ and $\mathbb{Z}$ have the same cardinal

Proof (hint).
Consider $h : \mathbb{N} \longrightarrow \mathbb{Z}$ defined as follows and show it is a bijection

$$h(x) \ = \ \begin{cases} 2x & \Longleftarrow x \geq 0 \\ -2x+1 & \Longleftarrow \text{otherwise} \end{cases}$$

$\boxed{\text{Theorem}}$
$\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ have the same cardinal

Proof (hint).
Consider $h : \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$ defined as follows and show it is a bijection

$h(x, y) \ = \$ The number of elements on all previous diagonals + The index of the current pair on its diagonal

i.e.

$$h(x, y) \ = \ \frac{(x+y)(x+y+1)}{2+x}$$

Theorem

The union of a finite number of countably infinite sets is countably infinite.

$$\begin{tikzcd}
X & x_0 \arrow[d] & x_1 \arrow[d] & x_3 \arrow[d] & \cdots \\
Y & y_0 \arrow[d] & y_1 \arrow[d] & y_3 \arrow[d] & \cdots \\
Z & z_0 \arrow[uur] & z_1 \arrow[uur] & z_3 \arrow[uur] & \cdots
\end{tikzcd}$$

Theorem

$|\mathbb{N}| < |\mathbb{R}|$

Proof (hint).

The difficult part is to prove that $\mathbb{N} \neq \mathbb{R}$. The proof resorts to *diagonalization*.

Theorem

$\mathbf{2}^{\mathbb{N}}$ is uncountable.

Proof.

If this is not the case, and $\mathbf{2}^{\mathbb{N}}$ is countably infinite, there is an enumeration of sets such that

$$\mathbf{2}^{\mathbb{N}} = \{R_1, R_2, \cdots\}$$

Let $D = \{n \in \mathbb{N} \mid n \notin R_n\}$. Set D is a set of natural numbers and thus should appear somewhere in the enumeration $R_1, R_2, \cdots$. Suppose $D = R_j$ for some value j. Does $j \in R_j$? If yes, by definition of D, $j \notin D$, which contradicts $D = R_j$. If, alternatively, $j \notin R_j$ then $j \in D$ which is again a contradiction.

Exercise (Cantor Theorem)

Generalise the previous proof to show that, for any set X, $|X| < |\mathbf{2}^X|$.

Note. This theorem sheds light on the limits of computability: *there are more problems that we might want to solve than there are programs to solve them, even though both are infinite*. To see this restrict your attention to one type of problem: deciding whether a string has some property (e.g. having even length, being palindromes, or a legal Haskell program). A property can be identified with the set of strings that happen to share it. Clearly, the number of possible programs is no bigger than the number of strings, while the number of sets of strings is strictly greater. This shows the existence of unsolvable problems, i.e. problems that can be formulated but not possibly solved.

Theorem: the pigeonhole principle

Let m objects be distributed into n bins. If $m > n$, then some bin contains at least two objects
Proof (hint).
By contrapositive (i.e. show that if every bin contains at most one object, then $m \leq n$.)

Applications: Given a large enough number of objects with a bounded number of properties, eventually at least two of them will share a property.

Ex. 1 - Theorem

Suppose that every point in the real plane is coloured either red or blue. Then for any distance $d > 0$, there are two points exactly distance d from one another that are the same color.

Ex. 2 - Theorem

For any natural number n, there is a nonzero multiple of n whose digits are all 0s and 1s.

3 Orders

- pre-order
- partial order
- lattice, bounded lattice and complete lattice

Exercise 1

In a poset $(P, \sqsubseteq)$ define the supremum of S, represented by $\sqcup S$, as the least upper bound (lub) of S. The *dual* notion of infimum, $\sqcap S$ is defined as the greatest lower bound (glb) of S.

Characterise lub and glb in $(\mathcal{P}(X), \subseteq)$ and $(\mathbb{N}, div)$, where div is integer division. Suppose P is a poset with a top and a bottom element $\top$ and $\bot$, respectively, i.e. $\sqcap P = \bot$ and $\sqcup P = \top$. Explain why $\sqcap\emptyset = \top$ and $\sqcup\emptyset = \bot$.

Exercise 2

A morphism between posets $(P, \sqsubseteq)$ and $(Q, \subseteq)$ is a function $f : P \longrightarrow Q$ such that

$$x \sqsubseteq y \ \Rightarrow \ f(x) \subseteq f(y)$$

i.e. a monotonic function. What extra structure must a morphism between lattices, bounded lattices or complete lattice preserve?

Exercise 3

A lattice is complete if infimum and supremum are defined for arbitrary subsets. Characterise as complete lattices i) the set of all sub-spaces of a vectorial space; ii) the set of sub-groups of a group; iii) any finite lattice.

The Knaster-Tarski theorem.

A most relevant result about complete lattices for the semantics of computation is the theorem of Knaster-Tarski [5] on the existence of fixed points of a monotonic function. Such special points (for which $x = f(x)$ give meaning to recursive functions.

$\boxed{\text{Theorem}}$

Let $(U, \sqsubseteq)$ be a complete lattice and $f : U \longrightarrow U$ a monotonic function. The least and the greatest fixed points of f are given by

$$\begin{aligned} m &= \bigsqcap\{x \in U \mid f(x) \sqsubseteq x\} \\ M &= \bigsqcup\{x \in U \mid x \sqsubseteq f(x)\} \end{aligned}$$

respectively.

Proof.

Let us show that m is the least fixed point of f. Let $X = \{x \in U \mid f(x) \sqsubseteq x\}$ and choose $x \in X$ arbitrarily. Clearly, $m \sqsubseteq x$ and, f being monotonic, $f(m) \sqsubseteq f(x)$. On the other hand, $f(x) \sqsubseteq x$, because $x \in X$. Thus, we may conclude that, for all $x \in X$, $f(m) \sqsubseteq x$. This means that m is the *least* pre-fixed point of f. In particular, $f(m) \sqsubseteq m$, which leads us to $f(f(m)) \sqsubseteq f(m)$. We conclude that $f(m) \in X$ and, thus, $m \sqsubseteq f(m)$. But then $f(m) = m$ as expected. The second part of the theorem comes from this one; if f is monotonic in $(U, \sqsubseteq)$, then it is also monotonic in the complete lattice formed by the inverse order $(U, \sqsupseteq)$. If M is the least fixed point of f in $(U, \sqsupseteq)$, it will be the *greatest* fixed point of the same function in $(U, \sqsubseteq)$.

$\boxed{\text{Lattices as algebraic structures}}$

Lattices can as well be seem as algebraic structures taking $\sqcup$ and $\sqcap$ as binary operations satisfying the axioms for commutativity, associativity, idempotence, and the following absorption laws:

$$\begin{aligned} a \sqcup (a \sqcap b) = a \\ a \sqcap (a \sqcup b) = a \end{aligned}$$

Clearly, $a \leq b \Leftrightarrow a \sqcup b = b \Leftrightarrow a \sqcap b = a$.

4 Groups

A group (G, θ, u) is a set G with a binary operation θ which is associative, and equipped with an identity element u and an inverse:

$$a^{-1}\theta a = u = a\theta a^{-1}$$

Note that in *monoid* θ lacks inverse, and a *semigroup* also drops the identity element.

$\boxed{\textbf{Exercise } 4}$

Show that $(\mathbb{R}^+, \times, 1)$ and $(\mathbb{R}^+, +, 0)$ are groups. Prove that a bijection between them is obtained by functions ln_e and e^-.

$\boxed{\textbf{Exercise } 5}$

Show that $S_n = (\{\sigma : n \longrightarrow n \mid \sigma \text{ is a permutation}\}, \cdot, id)$ is a group. This is usually called the *symmetry group of degree* n.

$\boxed{\textbf{Exercise } 6}$

Prove the following properties:

1. $a\theta b = a\theta c \Rightarrow b = c$ (dually, $b\theta a = c\theta a \Rightarrow b = c$).
2. $a^{-1^{-1}} = a$.
3. $(a\theta b)^{-1} = b^{-1}\theta a^{-1}$.
4. The equation $a\theta x = b$ has a unique solution $x = a^{-1}\theta b$.

Cayley Theorem.

The set of bijections $f : X \longrightarrow X$ over a set X with functional composition forms a group of *transformations* (which is the identity? And the inverse?). The following is a main result in the theory of groups:

$\boxed{\text{Theorem}}$
Every group is isomorphic to a group of transformations

Proof.
Let (G, θ, u) be a group. For each element a of G define a map $f_a : G \longrightarrow G$ such that $f_a(x) = a\theta x$.

Let us show that a new group T can be defined over the set of transformations above:

1. The (functional) composition of two elements of T is in T:

$$(f_a \cdot f_b)(x) = f_a(f_b(x)) = f_a(b\theta x) = a\theta(b\theta x) = (a\theta b)\theta x = f_{a\theta b}(x)$$

2. For identity,

$$f_u(x) = u\theta x = x$$

3. For inverse,

$$f_a \cdot f_{a^{-1}}(x) = a\theta(a^{-1}\theta x) = (a\theta a^{-1})\theta x = u\theta x = x$$

We have proved that T is a group (note that axioms are inherited from the properties of function composition restricted to bijections). It remains to show that T is *isomorphic* to G. Let $h : G \longrightarrow T$ be defined by $h\,(a) = f_a$.

- Cllearly $h(a\theta b) = f_{a\theta b} = f_a \cdot f_b = h(a) \cdot h(b)$ is a homomorphim between both groups.
- T is entirely composed of bijections f_a for every element $a \in G$, thus h is a surjection.
- If $a \neq b$, then $h(a) = f_a \neq f_b = h(b)$; thus h is injective.

Action of a group

A group (G, θ, u) acts over a set X through a function (the action) $\tau : G \times X \longrightarrow X$ which satisfies the following properties: $\tau(u, x) = x$ and $\tau(g\theta f, x) = \tau(g, (\tau(f, x))$.

Exercise 7

Show that i) the group S_n acts over set n (initial fragment of $\mathbb{N}$ with n numbers), and ii) that every group (G, θ, u) acts over itself through the map $(g, x) \mapsto g\theta x\theta g^{-1}$.

Notes.
There are several introductory textbooks on the mathematical background stuff discussed in this lecture. I would recommend Paul Halmos' very well written introductions to set theory [2] and to modern logic from an algebraic perspective [3]. Davey and Priestley textbook [1] on ordered structures is recommended for the second topic in the summary. For a very pleasant and solid, although not elementary, reading on algebraic structures I can't but recommend *the* book [4].

References

[1] D. A. Davey and H. A. Priestley. *Introduction to lattices and order (Second Edition)*. Cambridge University Press, 2002.

[2] P. Halmos. *Naive Set Theory*. Springer (Undergraduate texts in Mathematics), 1974.

[3] P. Halmos and S. Givant. *Logic as Algebra*. The Mathematical Association of America (Dolciani Mathematical Expositions, 21), 1998.

[4] S. Mac Lane and G. Birkhoff. *Algebra (Third Edition)*. Cambridge University Press, 1988.

[5] A. Tarski. A lattice–theoretic fixpoint theorem and its applications. *Pacific Journal of Mathematics*, 5:285–309, 1955.