Quantum Systems

(Lecture 5: Quantum algorithms — first examples and techniques)

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Computing: A probabilistic machine

States: Given a set of possible configurations, states are vectors of probabilities in $\mathcal{R}^n$ which express indeterminacy about the exact physical configuration, e.g. $\left[p_0 \cdots p_n\right]^T$ st $\sum_i p_1 = 1$
Operator: double stochastic matrix (*must come (go) from (to) somewhere*), where $M_{i,j}$ specifies the probability of evolution from configuration j to i
Evolution: computed through matrix multiplication with a vector $|u\rangle$ of current probabilities

- $M|u\rangle$ (next state)

Measurement: the system is always in some configuration — if found in i, the new state will be a vector $|t\rangle$ st $t_j = \delta_{j,i}$

Computing: A probabilistic machine

Composition:

$$p \otimes q = \begin{bmatrix} p_1 \\ 1 - p_1 \end{bmatrix} \otimes \begin{bmatrix} q_1 \\ 1 - q_1 \end{bmatrix} = \begin{bmatrix} p_1 q_1 \\ p_1(1 - q_1) \\ (1 - p_1) q_1 \\ (1 - p_1)(1 - q_1) \end{bmatrix}$$

- correlated states: cannot be expressed as $p \otimes q$, e.g.

$$\begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0.5 \end{bmatrix}$$

- Operators are also composed by $\otimes$ (Kronecker product):

$$M \otimes N = \begin{bmatrix} M_{1,1}N & \cdots & M_{1,n}N \\ \vdots & & \vdots \\ M_{m,1}N & \cdots & M_{m,n}N \end{bmatrix}$$

Computing: A quantum machine

States: given a set of possible configurations, states are unit vectors of (complex) amplitudes in $\mathcal{C}^n$
Operator: unitary matrix ($M^\dagger M = I$). The norm squared of a unitary matrix forms a double stochastic one.
Evolution: computed through matrix multiplication with a vector $|u\rangle$ of current amplitudes (wave function)

- $M|u\rangle$ (next state)
- $|u\rangle^T M^T$ (previous state)

Measurement: configuration i is observed with probability $\| \alpha_i \|^2$ if found in i, the new state will be a vector $|t\rangle$ st $t_j = \delta_{j,i}$
Composition: also by a tensor on the complex vector space; may exist entangled states

Computing: Algorithms

Quantum algorithms

1. State preparation (fix initial setting)
2. Transformation
   (combination of unitary transformations)
3. Measurement
   (projection onto a basis vector associated with a measurement tool)

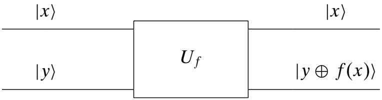
What's next?

1. Study a number of algorithmic techniques
2. and their application to the development of quantum algorithms

The Deutsch problem (from Lecture 1)

Is $f : \mathbf{2} \longrightarrow \mathbf{2}$ constant, with a unique evaluation?

Oracle



where $\oplus$ stands for exclusive or, i.e. addition module 2.

- The oracle takes input $|x\rangle|y\rangle$ to $|x\rangle|y \oplus f(x)\rangle$
- Fixing $y = 0$ the output is $|x\rangle|f(x)\rangle$

The Deutsch problem (from Lecture 1)

Preparing the first qubit as $|x\rangle$ is the (quantum version of) input x:

$$\begin{aligned} |0\rangle|0\rangle &\mapsto |0\rangle|f(0)\rangle \\ |1\rangle|0\rangle &\mapsto |1\rangle|f(1)\rangle \end{aligned}$$

But in the quantum world, one can better: input a superposition of $|0\rangle$ and $|1\rangle$ to get

$$|\frac{|0\rangle+|1\rangle}{\sqrt{2}},0\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle+\frac{1}{\sqrt{2}}|1\rangle\right)|0\rangle = \frac{1}{\sqrt{2}}|0\rangle\,|0\rangle+\frac{1}{\sqrt{2}}|1\rangle\,|0\rangle \mapsto \cdots$$

The Deutsch problem (from Lecture 1)

. . .

$$\begin{aligned} U_f\left(\frac{1}{\sqrt{2}}|0\rangle\,|0\rangle+\frac{1}{\sqrt{2}}|1\rangle\,|0\rangle\right) &= \frac{1}{\sqrt{2}}U_f|0\rangle|0\rangle+\frac{1}{\sqrt{2}}U_f|1\rangle|0\rangle \\ &= \frac{1}{\sqrt{2}}|0\rangle|0\oplus f(0)\rangle+\frac{1}{\sqrt{2}}|1\rangle|0\oplus f1\rangle \\ &= \frac{1}{\sqrt{2}}|0\rangle|f(0)\rangle+\frac{1}{\sqrt{2}}|1\rangle|f1\rangle \end{aligned}$$

- The value of f on both possible inputs (0 and 1) was computed simultaneously in superposition
- Double evaluation — the bottleneck in a classical solution — was avoided by superposition

Is such **quantum parallelism** useful? (from Lecture 1)

NO

Although both values have been computed simultaneously, only one of them is retrieved upon measurement in the computational basis: Actually, 0 or 1 will be retrieved with identical probability (why?).

YES

The Deutsch problem is not interested on the concrete values f may take, but on a global property of f: whether it is constant or not, technically on the value of

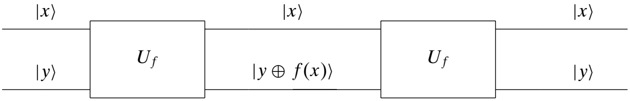
$$f(0) \oplus f(1)$$

The Deutsch algorithm explores another quantum resource — interference — to obtain that global information on f

Is the oracle a **quantum gate**?

First of all, one must prove that

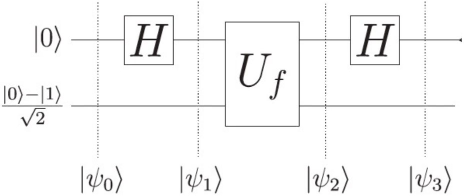
- The oracle is a unitary, i.e. reversible gate



$$|x\rangle|(y \oplus f(x)) \oplus f(x)\rangle \;=\; |x\rangle|y \oplus (f(x) \oplus f(x))\rangle \;=\; |x\rangle|y \oplus 0\rangle \;=\; |x\rangle|y\rangle$$

Deutsch algorithm (from Lecture 1)

Idea: Avoid double evaluation by superposition and interference



The circuit computes:

$$|\varphi_1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2}$$

Deutsch algorithm (from Lecture 1)

After the oracle, at φ_2, one obtains

$$|x\rangle \frac{|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle}{\sqrt{2}} = \begin{cases} |x\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}} & \Leftarrow f(x) = 0 \\ |x\rangle \frac{|1\rangle - |0\rangle}{\sqrt{2}} & \Leftarrow f(x) = 1 \end{cases}$$

$$= (-1)^{f(x)} |x\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

For $|x\rangle$ a superposition:

$$|\varphi_2\rangle = \left(\frac{(-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle}{\sqrt{2}} \right) \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

$$= \begin{cases} (\pm 1) \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) & \Leftarrow f \text{ constant} \\ (\pm 1) \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) & \Leftarrow f \text{ not constant} \end{cases}$$

Deutsch algorithm (from Lecture 1)

$$
\begin{aligned}
|\sigma_3\rangle &= H|\sigma_2\rangle \\
&= \begin{cases} (\pm 1)\,|0\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) & \Leftarrow f \text{ constant} \\ (\pm 1)\,|1\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) & \Leftarrow f \text{ not constant} \end{cases}
\end{aligned}
$$

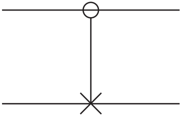
To answer the original problem is now enough to measure the first qubit: if it is in state $|0\rangle$, then f is constant.

Note

As the initial state in the second qubit can be prepared as $H|1\rangle$, the circuit is equivalent to

$$(H \otimes I)\, U_f\, (H \otimes H)(|01\rangle)$$

Recalling the *CNOT* gate



$$\overbrace{\begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix}}^{CNOT}$$

$$\begin{aligned} CNOT|0\rangle|\varphi\rangle &= |0\rangle I|\varphi\rangle \\ CNOT|1\rangle|\varphi\rangle &= |1\rangle X|\varphi\rangle \end{aligned}$$

Recall its effect when applied in the Hadamard basis, e.g.

$$\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) \mapsto \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

The phase jumps, or is kicked back, from the second to the first qubit.

The phase 'kick back' technique

This happens because $\frac{|0\rangle-|1\rangle}{\sqrt{2}}$ is an eigenvector of

- X (with $\lambda = -1$) and of I (with $\lambda = 1$)
- and, thus, $X\frac{|0\rangle-|1\rangle}{\sqrt{2}} = -1\frac{|0\rangle-|1\rangle}{\sqrt{2}}$ and $I\frac{|0\rangle-|1\rangle}{\sqrt{2}} = 1\frac{|0\rangle-|1\rangle}{\sqrt{2}}$

Thus,

$$\begin{aligned} CNOT\,|1\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) &= |1\rangle\left(X\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\right) \\ &= |1\rangle\left((-1)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\right) \\ &= -|1\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) \end{aligned}$$

while $CNOT\,|0\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) = |0\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$

The phase 'kick back' technique

The phase has been kicked back to the first (control) qubit:

$$CNOT\,|i\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) = (-1)^i|i\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

for $i \in \{0,1\}$, yielding, when the first (control) qubit is in a superposition of $|0\rangle$ and $|1\rangle$,

$$CNOT\,(\alpha|0\rangle+\beta|1\rangle)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) = (\alpha|0\rangle-\beta|1\rangle)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

The phase 'kick back' technique

Input an eigenvector to the target qubit of operator $\widehat{U}_{f(x)}$, and associate the eigenvalue with the state of the control qubit

Phase 'kick back' in the Deutsch algorithm

Instead of *CNOT*, an oracle U_f for an arbitrary Boolean function $f : \mathbf{2} \longrightarrow \mathbf{2}$, presented as a controlled-gate, i.e. a 1-gate $\widehat{U}_{f(x)}$ acting on the second qubit and controlled by the state $|x\rangle$ of the first one, mapping

$$|y\rangle \mapsto |y \oplus f(x)\rangle$$

The critical issue is that state $|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$ is an eigenvector of $\widehat{U}_{f(x)}$

Phase ‘kick back’ in the Deutsch algorithm

$$\begin{aligned}
U_f\,|x\rangle|-\rangle &= |x\rangle\,\widehat{U}_{f(x)}|-\rangle \\
&= \left(\frac{|x\rangle\,\widehat{U}_{f(x)}\,|0\rangle - |x\rangle\,\widehat{U}_{f(x)}\,|1\rangle}{\sqrt{2}}\right) \\
&= \left(\frac{|x\rangle|0 \oplus f(x)\rangle - |x\rangle|1 \oplus f(x)\rangle}{\sqrt{2}}\right) \\
&= |x\rangle\left(\frac{|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle}{\sqrt{2}}\right) \\
&= |x\rangle(-1)^{f(x)}\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = |x\rangle(-1)^{f(x)}|-\rangle
\end{aligned}$$

Thus, when the control qubit is in a superposition of $|0\rangle$ and $|1\rangle$,

$$U_f\,(\alpha|0\rangle + \beta|1\rangle)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = \left((-1)^{f(0)}\alpha|0\rangle + (-1)^{f(1)}\beta|1\rangle\right)|-\rangle$$

Generalizing Deutsch ...

Generalizing Deutsch's algorithm to functions whose domain is an

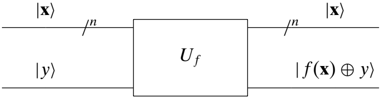
initial segment n of $\mathbb{N}$ encoded into a binary string

i.e. the set of natural numbers from 0 to $\mathbf{2}^n - 1$

The Deutsch-Jozsa problem

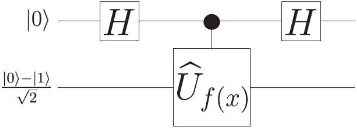
Assuming $f : \mathbf{2}^n \longrightarrow \mathbf{2}$ is either balanced or constant, determine which is the case with a unique evaluation

The oracle

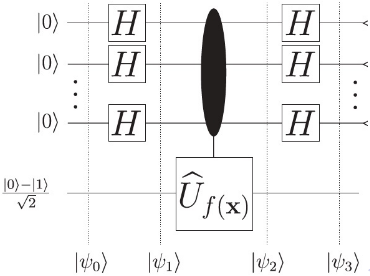


Generalizing Deutsch ...

The Deutsch circuit



The Deutsch-Joza circuit



The Deutsch-Jozsa Algorithm

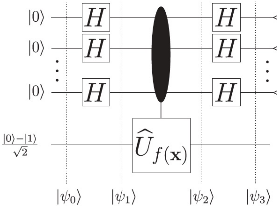
The crucial step is to compute $H^{\otimes n}$ over n qubits:

$$\begin{aligned} H^{\otimes n}|0\rangle^{\otimes n} &= \left(\frac{1}{\sqrt{2}}\right)^n \underbrace{(|0\rangle + |1\rangle) \otimes \cdots \otimes (|0\rangle + |1\rangle)}_{n} \\ &= \frac{1}{\sqrt{2^n}} \sum_{\mathbf{x} \in \mathbf{2}^n} |\mathbf{x}\rangle \end{aligned}$$

Thus

$$\begin{aligned} \varphi_0 &= |0\rangle^{\otimes n} \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \\ \varphi_1 &= \frac{1}{\sqrt{2^n}} \sum_{\mathbf{x} \in \mathbf{2}^n} |\mathbf{x}\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \end{aligned}$$

The Deutsch-Jozsa Algorithm



The phase kick-back effect

$$
\begin{aligned}
\varphi_2 &= \frac{1}{\sqrt{2^n}} U_f \left(\sum_{\mathbf{x}\in\mathbf{2}^n} |\mathbf{x}\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \right) \\
&= \frac{1}{\sqrt{2^n}} \sum_{\mathbf{x}\in\mathbf{2}^n} (-1)^{f(x)} |\mathbf{x}\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)
\end{aligned}
$$

The Deutsch-Jozsa Algorithm

Finally, we have to compute the last stage of $H^{\otimes}$ application.

$$H|x\rangle \;=\; \frac{1}{\sqrt{2}}(|0\rangle + (-1)^{x}|1\rangle) \;=\; \frac{1}{\sqrt{2}}\sum_{z\in\mathbf{2}}(-1)^{xz}|z\rangle$$

$$\begin{aligned}
H^{\otimes}|x\rangle &= H^{\otimes}(|x_1\rangle,\cdots,|x_n\rangle) \\
&= H|x_1\rangle \otimes \cdots \otimes H|x_n\rangle \\
&= \frac{1}{\sqrt{2}}(|0\rangle + (-1)^{x_1}|1\rangle)\,\frac{1}{\sqrt{2}}(|0\rangle + (-1)^{x_2}|1\rangle)\,\cdots\,\frac{1}{\sqrt{2}}(|0\rangle + (-1)^{x_n}|1\rangle) \\
&= \frac{1}{\sqrt{2^n}}\sum_{z_1 z_2\cdots z_n\in\mathbf{2}}(-1)^{x_1 z_1 + x_2 z_2 + \cdots + x_n z_n}|z_1\rangle|z_2\rangle\cdots|z_n\rangle \\
&= \frac{1}{\sqrt{2^n}}\sum_{z\in\mathbf{2}^n}(-1)^{x.z}|z\rangle
\end{aligned}$$

The Deutsch-Jozsa Algorithm

$$\begin{aligned}
|\varphi_3\rangle &= \frac{\sum_{\mathbf{x}\in\mathbf{2}^n}(-1)^{f(\mathbf{x})}\sum_{\mathbf{z}\in\{0,1\}^n}(-1)^{\mathbf{z}.\mathbf{x}}|\mathbf{z}\rangle}{2^n}\ \frac{|0\rangle-|1\rangle}{\sqrt{2}} \\
&= \frac{\sum_{\mathbf{x},\mathbf{z}\in\mathbf{2}^n}(-1)^{f(\mathbf{x})}(-1)^{\mathbf{z}.\mathbf{x}}|\mathbf{z}\rangle}{2^n}\ \frac{|0\rangle-|1\rangle}{\sqrt{2}} \\
&= \frac{\sum_{\mathbf{x},\mathbf{z}\in\mathbf{2}^n}(-1)^{f(\mathbf{x})+\mathbf{z}.\mathbf{x}}|\mathbf{z}\rangle}{2^n}\ \frac{|0\rangle-|1\rangle}{\sqrt{2}}
\end{aligned}$$

Now, consider the amplitude for state $|z\rangle = |0\rangle^{\otimes}$:

$$\frac{1}{2^n}\sum_{\mathbf{x}\in\mathbf{2}^n}(-1)^{f(x)}$$

The Deutsch-Jozsa Algorithm

Thus

$$\boxed{f \text{ is constant at } 1} \quad \rightsquigarrow \quad \frac{-(2^n)|\mathbf{0}\rangle}{2^n} = -|\mathbf{0}\rangle$$

$$\boxed{f \text{ is constant at } 0} \quad \rightsquigarrow \quad \frac{(2^n)|\mathbf{0}\rangle}{2^n} = |\mathbf{0}\rangle$$

As $|\varphi_3\rangle$ has unit length, all other amplitudes must be 0 and the top qubits collapse to $|\mathbf{0}\rangle$

$$\boxed{f \text{ is balanced}} \quad \rightsquigarrow \quad \frac{0|\mathbf{0}\rangle}{2^n} = 0|\mathbf{0}\rangle$$

because half of the $\mathbf{x}$ will cancel the other half. The top qubits collapse to some other basis state, as $|\mathbf{0}\rangle$ has zero amplitude

$$\boxed{\text{The top qubits collapse to } |\mathbf{0}\rangle \text{ iff } f \text{ is constant}}$$

Quantum Algorithms

The Deutsch-Jozsa algorithm: Lessons learnt

- Exponential speed up: f was evaluated once rather than $2^n - 1$ times
- The quantum state encoded global properties of function f
- ... that can be extracted by exploiting cleverly such non local correlations.

Quantum Algorithms

The Deutsch-Jozsa algorithm

Exponential speed up: f was evaluated once rather than $2^n - 1$ times

Classes of quantum algorithm

- Based on the quantum Fourier transform: The Deutsch-Jozsa is a simple example; Phase estimation; Shor algorithm; etc.
- Based on amplitude amplification: Variants of Grover algorithm for search processes.
- Quantum simulation.