Quantum Systems

(Lecture 3: The principles of quantum computation)

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The principles

Quantum computation explores the laws of quantum theory as computational resources.

Thus, the principles of the former are directly derived from the postulates of the latter.

- The state space postulate
- The state evolution postulate
- The state composition postulate
- The state measurement postulate

The underlying maths is that of Hilbert spaces.

The underlying maths: Hilbert spaces

Complex, inner-product vector space

A complex vector space with inner product

$$\langle - | - \rangle : V \times V \longrightarrow \mathbb{C}$$

such that

$$\begin{aligned}
&(1) \quad \langle v | \sum_i \lambda_i \cdot |w_i\rangle\rangle \;=\; \sum_i \lambda_i \langle v | w_i \rangle \\
&(2) \quad \langle v | w \rangle = \overline{\langle w | v \rangle} \\
&(3) \quad \langle v | v \rangle \geq 0 \;\; \text{(with equality iff } |v\rangle = 0\text{)}
\end{aligned}$$

Note: $\langle - | - \rangle$ is conjugate linear in the first argument:

$$\langle \sum_i \lambda_i \cdot |w_i\rangle | v \rangle \;=\; \sum_i \overline{\lambda_i} \langle w_i | v \rangle$$

Notation: $\langle v | w \rangle \;\equiv\; \langle v, w \rangle \equiv (|v\rangle, |w\rangle)$

Dirac's notation

Dirac's bra/ket notation is a handy way to represent elements and constructions on an Hilbert space, amenable to calculations and with direct correspondence to diagrammatic (categorial) representations of process theories

$|u\rangle$ A ket stands for a vector in an Hilbert space V. In $\mathcal{C}^n$, a column vector of complex entries. The identity for $+$ (the zero vector) is just written 0.

$\langle u|$ A bra is a vector in the dual space $V^\dagger$, i.e. scalar-valued linear maps in V — a row vector in $\mathcal{C}^n$.

There is a bijective correspondence between $|u\rangle$ and $\langle u|$

$$|u\rangle = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \Leftrightarrow \begin{bmatrix} \overline{u}_1 \cdots \overline{u}_n \end{bmatrix} = \langle u|$$

Inner product: examples

In $\mathcal{C}$

$$\langle a+bi|c+di\rangle = (a-bi)(c+di) = ac+adi-bci+bd$$

In $\mathcal{C}^n$: The dot product

A useful example of a inner product is the dot product

$$\langle u|v\rangle = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \underbrace{\begin{bmatrix} \overline{u_1} & \overline{u_2} & \cdots & \overline{u_n} \end{bmatrix}}_{\langle u|} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \sum_{i=1}^{n} \overline{u_i} v_i$$

where $\overline{c} = a - ib$ is the complex conjugate of $c = a + ib$

$\langle u|$ is the adjoint of vector $|u\rangle$, i.e a vector in the dual vector space $V^\dagger$.

Old friends: The dual space

$V^\dagger$

If V is a Hilbert space, $V^\dagger$ is the space of linear maps from V to $\mathcal{C}$.

Elements of $V^\dagger$ are denoted by

$$\langle u| : V \longrightarrow \mathcal{C} \text{ defined by } \langle u|(|v\rangle) = \langle u|v\rangle$$

In a matricial representation $\langle u|$ is obtained as the Hermitian conjugate (i.e. the transpose of the vector composed by the complex conjugate of each element) of $|u\rangle$, therefore the dot product of $|u\rangle$ and $|v\rangle$.

Old friends: Norms and orthogonality

Old friends

- $|v\rangle$ and $|w\rangle$ are orthogonal if $\langle v|w\rangle = 0$
- norm: $\| \, |v\rangle \, \| = \sqrt{\langle v|v\rangle}$
- normalization: $\frac{|v\rangle}{\| |v\rangle \|}$
- $|v\rangle$ is a unit vector if $\| \, |v\rangle \, \| = 1$
- A set of vectors $\{|i\rangle, |j\rangle, \cdots, \}$ is orthonormal if each $|i\rangle$ is a unit vector and

$$\langle i|j\rangle \;=\; \delta_{i,j} \;=\; \begin{cases} i = j & \Rightarrow 1 \\ \text{otherwise} & \Rightarrow 0 \end{cases}$$

Old friends: Bases

Orthonormal basis

A orthonormal basis for a Hilbert space V of dimension n is a set $B = \{|i\rangle\}$ of n linearly independent elements of V st

- $\langle i|j\rangle = \delta_{i,j}$ for all $|i\rangle, |j\rangle \in B$
- and B spans V, i.e. every $|v\rangle$ in V can be written as

$$|v\rangle \;=\; \sum_i \alpha_i |i\rangle \quad \text{for some } \alpha_i \in \mathcal{C}$$

Note that the amplitude or coefficient of $|v\rangle$ wrt $|i\rangle$ satisfies

$$\alpha_i \;=\; \langle i|v\rangle$$

Why?

Bases

$\alpha_i \;=\; \langle i|v\rangle$ because

$$\begin{aligned}
\langle i|v\rangle &= \langle i| \sum_j \alpha_j j\rangle \\
&= \sum_j \alpha_j \langle i|j\rangle \\
&= \sum_j \alpha_j \delta_{i,j} \\
&= \alpha_i
\end{aligned}$$

Note

If $|v\rangle$ is expressed wrt any orthonormal basis $\{|i\rangle\}$, i.e. $|v\rangle = \sum_i \alpha_i |i\rangle$, then

$$\| \, |v\rangle \, \| = \sum_i \| \, \alpha_i \, \|^2$$

Example: The Hadamard basis

One of the infinitely many orthonormal bases for a space of dimension 2:

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$
$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

Check e. g.

$$\langle+|-\rangle \;=\; \frac{1}{2}(|0\rangle + |1\rangle, |0\rangle - |1\rangle) \;=\; \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\-1\end{bmatrix} \;=\; \frac{1}{2}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} \;=\; 0$$

$$\|\,|+\rangle\,\| \;=\; \sqrt{\langle+|+\rangle} \;=\; \sqrt{\frac{1}{2}(|0\rangle + |1\rangle, |0\rangle + |1\rangle)} \;=\; \sqrt{\frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}} \;=\; 1$$

Bases

A basis for $V^\dagger$

If $\{|i\rangle\}$ is an orthonormal basis for V, then

$$\{\langle i|\}$$

is an orthonormal basis for $V^\dagger$

Hilbert spaces

The complete picture

An Hilbert space is an inner-product space V st the metric defined by its norm turns V into a complete metric space, i.e.any Cauchy sequence

$$|v_1\rangle, |v_2\rangle, \cdots$$

$$\forall_{\epsilon>0}\, \exists_N\, \forall_{m,n>N} \quad \| \, |v_m - v_n\rangle \, \| \leq \epsilon$$

converges
(i.e. there exists an element $|s\rangle$ in V st $\forall_{\epsilon>0}\, \exists_N\, \forall_{n>N} \quad \| \, |s - v_n\rangle \, \| \leq \epsilon$)

The completeness condition is trivial in finite dimensional vector spaces

The state space postulate

Postulate 1

The state space of a quantum system is described by a unit vector in a Hilbert space

- In practice, with finite resources, one cannot distinguish between a continuous state space from a discrete one with arbitrarily small minimum spacing between adjacente locations.
- One may, then, restrict to finite-dimensional (complex) Hilbert spaces.

The state space postulate

A quantum (binary) state is represented as a superposition, i.e. a linear combination of vectors $|0\rangle$ and $|1\rangle$ with complex coeficients:

$$|\phi\rangle \; = \; \alpha|0\rangle + \beta|1\rangle \; = \; \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

When state $|\phi\rangle$ is measured (i.e. observed) one of the two basic states $|0\rangle, |1\rangle$ is returned with probability

$$\| \alpha \|^2 \quad \text{and} \quad \| \beta \|^2$$

respectively.
Being probabilities, the norm squared of coefficients must satisfy

$$\| \alpha \|^2 + \| \beta \|^2 = \; 1$$

which enforces quantum states to be represented by unit vectors.

The state space of a qubit

Global phase

Unit vectors equivalent up to multiplication by a complex number of modulus one, i.e. a phase factor $e^{i\theta}$, represent the same state.

Let

$$|v\rangle = \alpha|u\rangle + \beta|u'\rangle$$

$$\| e^{i\theta}\alpha \|^2 = (\overline{e^{i\theta}\alpha})(e^{i\theta}\alpha) = (e^{-i\theta}\overline{\alpha})(e^{i\theta}\alpha) = \overline{\alpha}\alpha = \| \alpha \|^2$$

and similarly for β.

As the probabilities $\| \alpha \|^2$ and $\| \beta \|^2$ are the only measurable quantities, global phase has no physical meaning.

Representation redundancy

qubit state space $\neq$ complex vector space used for representation

The state space of a qubit

Relative phase

It is a measure of the angle between the two complex numbers.
Thus, it cannot be discarded!

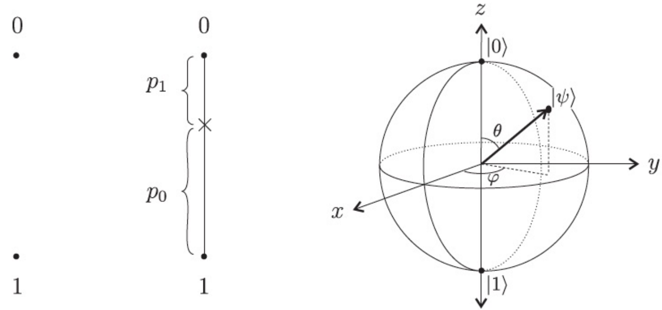
Those are different states

$$\frac{1}{\sqrt{2}}(|u\rangle + |u'\rangle) \quad \frac{1}{\sqrt{2}}(|u\rangle - |u'\rangle) \quad \frac{1}{\sqrt{2}}(e^{i\theta}|u\rangle + |u'\rangle)$$

...

The Bloch sphere

Deterministic, probabilistic and quantum bits



(from [Kaeys *et al*, 2007])

The Bloch sphere: Representing $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

- Express $|\psi\rangle$ in polar form

$$|\psi\rangle = \rho_1 e^{i\varphi_1}|0\rangle + \rho_2 e^{i\varphi_2}|1\rangle$$

- Eliminate one of the four real parameters multiplying by $e^{-i\varphi_1}$

$$|\psi\rangle = \rho_1|0\rangle + \rho_2 e^{i(\varphi_2-\varphi_1)}|1\rangle = \rho_1|0\rangle + \rho_2 e^{i\varphi}|1\rangle$$

making $\varphi = \varphi_2 - \varphi_1$,

which is possible because global phase factors are physically meaningless.

The Bloch sphere: Representing $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

- Switching back the coefficient of $|1\rangle$ to Cartesian coordinates

$$|\psi\rangle = \rho_1|0\rangle + (a + bi)|1\rangle$$

the normalization constraint

$$\|\rho_1\|^2 + \|a{+}ib\|^2 = \|\rho_1\|^2 + (a{-}ib)(a{+}ib) \ = \ \boxed{\|\rho_1\|^2 + a^2 + b^2 \ = \ 1}$$

yields the equation of a unit sphere in the real tridimensional space with Cartesian coordinates: (a, b, ρ_1).

The Bloch sphere: Representing $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

- The polar coordinates (ρ, θ, φ) of a point in the surface of a sphere relate to Cartesian ones through the correspondence

$$\begin{aligned} x =& \rho \sin\theta \cos\varphi \\ y =& \rho \sin\theta \sin\varphi \\ z =& \rho \cos\theta \end{aligned}$$

- Recalling $r = 1$ (cf unit sphere),

$$\begin{aligned} |\psi\rangle &= \rho_1|0\rangle + (a + ib)|1\rangle \\ &= \cos\theta|0\rangle + \sin\theta(\cos\varphi + i\sin\varphi)|1\rangle \\ &= \cos\theta|0\rangle + e^{i\varphi}\sin\theta|1\rangle \end{aligned}$$

which, with two parameters, defines a point in the sphere's surface.

The Bloch sphere

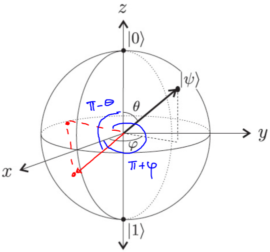
Actually, one may just focus on the upper hemisphere ($0 \leq \theta' \leq \frac{\pi}{2}$) as opposite points in the lower one differ only by a phase factor of -1, as suggested by

$$\begin{array}{rcl} \theta' = 0 & \Rightarrow & |\psi\rangle \; = \; \cos 0|0\rangle + e^{i\varphi}\sin 0|1\rangle \; = \; |0\rangle \\ \theta' = \dfrac{\pi}{2} & \Rightarrow & |\psi\rangle \; = \; \cos\dfrac{\pi}{2}|0\rangle + e^{i\varphi}\sin\dfrac{\pi}{2}|1\rangle \; = \; e^{i\varphi}|1\rangle \; = \; |1\rangle \end{array}$$

Note that longitude (φ) is irrelevant in a pole!

The Bloch sphere

Indeed, let $|\psi'\rangle$ be the opposite point on the sphere with polar coordinates $(1, \pi - \theta, \varphi + \pi)$:



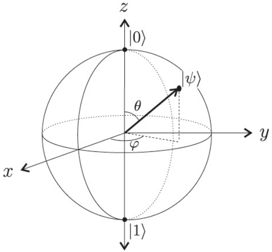
$$
\begin{aligned}
{\color{blue}|\psi'\rangle} &= \cos(\pi - \theta)|0\rangle + e^{i(\varphi+\pi)}\sin(\pi-\theta)|1\rangle \\
&= -\cos\theta|0\rangle + e^{i\varphi}e^{i\pi}\sin\theta|1\rangle \\
&= -\cos\theta|0\rangle + e^{i\varphi}\sin\theta|1\rangle \\
&= -{\color{blue}|\psi\rangle}
\end{aligned}
$$

The Bloch sphere

which leads to

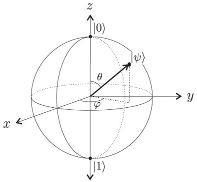
$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

where $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$



The map $\frac{\theta}{2} \mapsto \theta$ is one-to-one at any point but at $\frac{\theta}{2}$:
all points on the equator are mapped into a single point: the south pole.

The Bloch sphere



- The poles represent the classical bits. In general, orthogonal states correspond to antipodal points and every diameter to a basis for the single-qubit state space.
- Once measured a qubit collapses to one of the two poles. Which pole depends exactly on the arrow direction: The angle θ measures that probability: If the arrow points at the equator, there is 50-50 chance to collapse to any of the two poles.
- Rotating a vector wrt the z-axis results into a phase change (φ), and does not affect which state the arrow will collapse to, when measured.

The state evolution postulate

If a quantum state is a ray (i.e. a unit vector in a Hilbert space H up to a global phase), its evolution is specified a certain kind of linear operators $U : H \longrightarrow H$.

Linearity

$$U\left(\sum_j \alpha_j \, |v_j\rangle\right) \;=\; \sum_j \alpha_j \, U(|v_j\rangle)$$

just by itself has an important consequence: quantum states cannot be cloned

The no-cloning theorem

Linearity implies that quantum states cannot be cloned

Let $U(|a\rangle|0\rangle) = |a\rangle|a\rangle$ be a 2-qubit operator and $|c\rangle = \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle)$ for $|a\rangle$, $|b\rangle$ orthogonal. Then,

$$\begin{aligned}
U(|c\rangle|0\rangle) &= \frac{1}{\sqrt{2}}(U(|a\rangle|0\rangle) + U(|b\rangle|0\rangle)) \\
&= \frac{1}{\sqrt{2}}(|a\rangle|a\rangle + |b\rangle|b\rangle) \\
&\neq \frac{1}{\sqrt{2}}(|a\rangle|a\rangle + |a\rangle|b\rangle + |b\rangle|a\rangle + |b\rangle|b\rangle) \\
&= |c\rangle|c\rangle \\
&= U(|c\rangle|0\rangle)
\end{aligned}$$

As already seen, $|x\rangle|y\rangle = |xy\rangle = |x\rangle \otimes |y\rangle$

The adjoint operator

Given an operator $U : H \longrightarrow H$, its adjoint $U^\dagger : H^\dagger \longrightarrow H^\dagger$ is the unique operator satisfying

$$U^\dagger \langle w| \;\; (|v\rangle) \;\; = \;\; \langle w| \;\; (U|v\rangle) \tag{1}$$

Note that $(UV)^\dagger = V^\dagger U^\dagger$ because

$$\begin{aligned}
(UV)^\dagger \langle w| \; (|v\rangle) \; &= \langle w| \; (UV|v\rangle) \\
&= U^\dagger \langle w| \; (V|v\rangle) \\
&= V^\dagger U^\dagger \langle w| \; (|v\rangle)
\end{aligned}$$

The adjoint operator

Using the definition of the application of a transformation in $H^\dagger$ to an element of H, equation (1), boils down to an equality between inner products:

$$\begin{array}{rcl} U^\dagger\langle w|\,(|v\rangle) & = & ((U^\dagger\langle w|)^\dagger, |v\rangle) \\ & = & (|w\rangle U, |v\rangle) \\ & = & (|w\rangle, U|v\rangle) \\ & = & \langle w|\,(U|v\rangle) \end{array}$$

The inner product $(|w\rangle U, |v\rangle) = (|w\rangle, U|v\rangle)$ can be written without any ambiguity as

$$\langle u|U|v\rangle$$

The matrix representation of $U^\dagger$ is the conjugate transpose of that of U

Exercise: Prove that $\overline{\langle w|U|v\rangle} = \langle v|U^\dagger|w\rangle$

The state evolution postulate

Postulate 2

The evolution over time of the state of a closed quantum system is described by a unitary operator.

The evolution is linear

$$U\left(\sum_j \alpha_j \left|v_j\right\rangle\right) = \sum_j \alpha_j \, U(\left|v_j\right\rangle)$$

and preserves the normalization constraint

$$\text{If } \sum_j \alpha_j \, U(\left|v_j\right\rangle) = \sum_j \alpha'_j \left|v_j\right\rangle \text{ then } \sum_j \| \alpha'_j \|^2 = 1$$

The state evolution postulate

Preservation of the normalization constraint means that unit length vectors (and thus orthogonal subspaces) are mapped by U to unit length vectors (and thus to orthogonal subspaces).

It also means that applying a transformation followed by a measurement in the transformed basis is equivalent to a measurement followed by a transformation.

This entails a condition on valid quantum operators: they must preserve the inner product, i.e.

$$(U|v\rangle, U|w\rangle) \; = \; \langle v|U^\dagger U|w\rangle \; = \; \langle v|w\rangle$$

which is the case iff U is unitary, i.e. $U^\dagger = U^{-1}$:

$$U^\dagger U = \; UU^\dagger \; = \; I$$

Unitarity

- Preserving the inner product means that a unitary operator maps orthonormal bases to orthonormal bases.
- Conversely, any operator with this property is unitary.
- If given in matrix form, being unitary means that the set of columns of its matrix representation are orthonormal (because the jth column is the image of $U|j\rangle$). Equivalently, rows are orthonormal (why?)

Unitarity

Unitarity is the only constraint on quantum operators: Any unitary matrix specifies a valid quantum operator.

This means that there are many non-trivial operators on a single qubit (in contrast with the classical case where the only non-trivial operation on a bit is complement.

Finally, because the inverse of a unitary matrix is also a unitary matrix, a quantum operator can always be inverted by another quantum operator

Unitary transformations are reversible

Building larger states from smaller

Operator U in the no-cloning theorem acts on a 2-dimensional state, i.e. over the composition of two qubits.

What does composition mean?

Postulate 3

The state space of a combined quantum system is the tensor product $V \otimes W$ of the state spaces V and W of its components.

Composing quantum states

State spaces in a quantum system combine through tensor: $\otimes$

$$\boxed{n\ m\text{-dimensional vectors} \quad \rightsquigarrow \quad \text{a vector in } m^n\text{-dimensional space}}$$

i.e. the state space of a quantum system grows exponentially with the number of particles: cf, Feyman's original motivation

Example

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \otimes \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a \begin{bmatrix} d \\ e \\ f \end{bmatrix} \\ b \begin{bmatrix} d \\ e \\ f \end{bmatrix} \\ c \begin{bmatrix} d \\ e \\ f \end{bmatrix} \end{bmatrix} = \begin{bmatrix} ad \\ ae \\ af \\ bd \\ be \\ bf \\ cd \\ ce \\ cf \end{bmatrix}$$

Composing quantum states

Tensor $V \otimes W$

- $B_{V \otimes W}$ is a set of elements of the form $|v_i\rangle \otimes |w_j\rangle$, for each $|v_i\rangle \in B_V$, $|w_i\rangle \in B_W$ and $\mathrm{dim(V \otimes W) = dim(V) \times dim(W)}$
- $(|u_1\rangle + |u_2\rangle) \otimes |z\rangle \;=\; |u_1\rangle \otimes |z\rangle + |u_2\rangle \otimes |z\rangle$
- $|z\rangle \otimes (|u_1\rangle + |u_2\rangle) \;=\; |z\rangle \otimes |u_1\rangle + |z\rangle \otimes |u_2\rangle$
- $(\alpha|u\rangle) \otimes |z\rangle \;=\; |u\rangle \otimes (\alpha|z\rangle) \;=\; \alpha(|u\rangle \otimes |z\rangle)$
- $\langle(|u_2\rangle \otimes |z_2\rangle)|(|u_1\rangle \otimes |z_1\rangle)\rangle \;=\; \langle u_2|u_1\rangle\langle z_2|z_1\rangle$

Composing quantum states

Clearly, every element of $V \otimes W$ can be written as

$$\alpha_1(|v_1\rangle \otimes |w_1\rangle) + \alpha_2(|v_2\rangle \otimes |w_1\rangle) + \cdots + \alpha_{nm}(|v_n\rangle \otimes |w_m\rangle)$$

Example

The basis of $V \otimes W$, for V, W qubits with the computational basis is

$$\{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$$

Thus, the tensor of $\alpha_1|0\rangle + \alpha_2|1\rangle$ and $\beta_1|0\rangle + \beta_2|1\rangle$ is

$$\alpha_1\beta_1|0\rangle \otimes |0\rangle \ + \ \alpha_1\beta_2|0\rangle \otimes |1\rangle \ + \ \alpha_2\beta_1|1\rangle \otimes |0\rangle \ + \ \alpha_2\beta_2|1\rangle \otimes |1\rangle$$

i.e., in a simplified notation,

$$\alpha_1\beta_1|00\rangle \ + \ \alpha_1\beta_2|01\rangle \ + \ \alpha_2\beta_1|10\rangle \ + \ \alpha_2\beta_2|11\rangle$$

Bases

The computational basis for a vector space

$$\underbrace{V \otimes V \otimes \cdots \otimes V}_{n}$$

corresponding to the composition of n qubits (each living in V) is the set

$$\{\underbrace{|0\rangle \cdots |0\rangle|0\rangle}_{n},\ \underbrace{|0\rangle \cdots |0\rangle|1\rangle}_{n},\ \underbrace{|0\rangle \cdots |1\rangle|0\rangle}_{n},\ \cdots\ \underbrace{|1\rangle \cdots |1\rangle|1\rangle}_{n}\}$$

$$\overset{abv}{=}$$

$$\{\underbrace{|0 \cdots 00\rangle}_{n},\ \underbrace{|0 \cdots 01\rangle}_{n},\ \underbrace{|0 \cdots 10\rangle}_{n},\ \cdots\ \underbrace{|1 \cdots 11\rangle}_{n}\}$$

which may be written in a compressed (decimal) way as

$$\{|0\rangle, |1\rangle, |2\rangle, |3\rangle, \cdots |2^n - 1\rangle\}$$

Bases

The computational basis for a two qubit system would be

$$\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$$

with

$$|0\rangle = |00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad |1\rangle = |01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad |2\rangle = |10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad |3\rangle = |11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Bases

There are of course other bases ... besides the standard one, e.g.

The Bell basis

$$\begin{aligned}
|\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
|\Phi^-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\
|\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\
|\Psi^-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)
\end{aligned}$$

Compare with the Hadamard basis for the single qubit systems

Representing multi-qubit states

Any unit vector in a 2^n Hilbert space represents a possible n-qubit state, but for

… a certain level of redundancy

- As before, vectors that differ only in a global phase represent the same quantum state
- but also the same phase factor in different qubits of a tensor product represent the same state:

$$|u\rangle \otimes (e^{i\Phi}|z\rangle) \;=\; e^{i\Phi}(|u\rangle \otimes |z\rangle) \;=\; (e^{i\Phi}|u\rangle) \otimes |z\rangle$$

Actually, phase factors in qubits of a single term of a superposition can always be factored out into a coefficient for that term, i.e. phase factors distribute over tensors

Representing multi-qubit states

Representation

- Relative phases still matter (of course!)

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \quad \text{differs from} \quad \frac{1}{\sqrt{2}}(e^{i\Phi}|00\rangle + |11\rangle)$$

even if

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \;=\; \frac{1}{\sqrt{2}}(e^{i\Phi}|00\rangle + e^{i\Phi}|11\rangle) \;=\; \frac{e^{i\Phi}}{\sqrt{2}}(|00\rangle + |11\rangle)$$

- The complex projective space of dimension 1 (depicted in the Block sphere) generalises to higher dimensions, although in practice linearity makes Hilbert spaces easier to use.

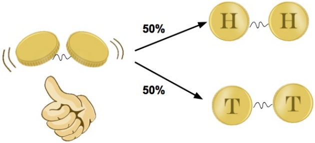
Entanglement

Most states in $V \otimes W$ cannot be written as $|u\rangle \otimes |z\rangle$

For example, the Bell state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \; = \; \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

is entangled



Entanglement

Actually, to make $|\Phi^+\rangle$ equal to

$$(\alpha_1|0\rangle+\beta_1|1\rangle)\otimes(\alpha_2|0\rangle+\beta_2|1\rangle) \;=\; \alpha_1\alpha_2|00\rangle+\alpha_1\beta_2|01\rangle+\beta_1\alpha_2|10\rangle+\beta_1\beta_2|11\rangle$$

would require that $\alpha_1\beta_2 = \beta_1\alpha_2 = 0$ which implies that either

$$\alpha_1\alpha_2 = 0 \;\text{ or }\; \beta_1\beta_2 = 0$$

Note

Entanglement can also be observed in simpler structures, e.g. relations:

$$\{(a, a), (b, b)\} \subseteq A \times A$$

cannot be separated, i.e. written as a Cartesian product of subsets of A.

The measurement postulate

Postulate 4

For a given orthonormal basis $B = \{|v_1\rangle, |v_2\rangle, \cdots\}$, a measurement of a state space $|v\rangle = \sum_i \alpha_i |v_i\rangle$ wrt B, outputs the label i with probability $\| \alpha_i \|^2$ and leaves the system in state $|v_i\rangle$.

- Given a state
$$|v\rangle \; = \; \sum_i \alpha_i |v_i\rangle$$
the probability of collapsing to base state $|v_i\rangle$ is $\| \langle v_i | v \rangle \|^2$.
- Measurements are made through projectors which identify the 'data' (i.e. the subspace of the relevant Hilbert space where the quntum system lives) one wants to measure.

Outer product

- inner product $\langle w|v\rangle$: multiplying $|v\rangle$ on the left by the dual $\langle w|$, yields a scalar.
- outer product $|w\rangle\langle v|$: multiplies on the right, yielding an operator:

$$|w\rangle\langle v| \, (|u\rangle) \;\; = \;\; |w\rangle\langle v|u\rangle \;\; = \;\; \langle v|u\rangle|w\rangle$$

Clearly

$$|v\rangle\langle v| \, (|u\rangle) \;\; = \;\; \langle v|u\rangle|v\rangle$$

which projects $|u\rangle$ to the 1-dimensional subspace of H spanned by $|v\rangle$

Projectors

Any projector P identifies in the state space V a subspace V_P of all vectors $|\phi\rangle$ that are left unchanged by P, i.e. such that

$$P|\phi\rangle = |\phi\rangle$$

Examples

- The identity I projects onto the whole space V.
- The zero operator projects onto the space $\{0\}$ consisting only of the zero vector.
- $|v\rangle\langle v|$ is the projector onto the subspace spanned by $|v\rangle$.

Projectors

Examples

- Projector $|0\rangle\langle 0|$ projects onto the subspace generated by $|0\rangle$, i.e.

$$|0\rangle\langle 0|\,(\alpha|0\rangle + \beta|1\rangle) \;=\; \alpha|0\rangle\langle 0|(|0\rangle) + \beta|0\rangle\langle 0|(|1\rangle) \;=\; \alpha|0\rangle$$

- Similarly, $|10\rangle\langle 10|$ acts on a two-qubit state

$$v \;=\; \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

yielding

$$|10\rangle\langle 10|\,(|v\rangle) \;=\; \alpha_{10}|10\rangle$$

and

$$|00\rangle\langle 00| + |10\rangle\langle 10|(|v\rangle) \;=\; \alpha_{00}|00\rangle + \alpha_{10}|10\rangle$$

Projectors

A projector $P : V \to V_P$ is an operator such that

$$P^2 \ = \ P$$

Additionally, we require P to be Hermitian, i.e.

$$P \ = \ P^\dagger$$

Note that the combination of both properties yields

$$\| \, P|v\rangle \, \|^2 = \ (\langle v|P^\dagger)(\mathrm{P}|v\rangle) \ = \ \langle v|P|v\rangle$$

Example

The probability of getting state $|0\rangle$ when measuring $\alpha|0\rangle + \beta|1\rangle$ with $P = |0\rangle\langle 0|$ is computed as

$$\| \, P|v\rangle \, \|^2 = \ \langle v|P|v\rangle \ = \ \langle v||0\rangle\langle 0||v\rangle \ = \ \langle v|0\rangle \, \langle 0|v\rangle \ = \ \overline{\alpha}\alpha \ = \| \, \alpha \, \|^2$$

Projectors

Two projectors P, Q are orthogonal if $PQ = 0$.

The sum of any collection of orthogonal projectors $\{P_1, P_2, \cdots\}$ is still a projector (verify!).

A projector P has a decomposition if it can be written as a sum of orthogonal projectors:

$$P = \sum_i P_i$$

Such projectors yield measurements wrt to the corresponding decomposition.

Examples

- Complete measurement in the computational basis wrt to decomposition

  $$I = \sum_{i \in 2^n} |i\rangle\langle i|$$

  in a state with n qubits.
- Incomplete measurement: e.g.

  $$\sum_{\{i \in 2^n \mid i \text{ even}\}} |i\rangle\langle i|$$

Projectors

Example: measuring up to (bit equality)

$$V = S_e \oplus S_n$$

with S_e the subspace generated by $\{|00\rangle, |11\rangle\}$ in which the two bits are equal, and S_n its complement. P_e and P_n, are the corresponding projectors.

When measuring

$$v \;=\; \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

with this device, yields a state in which the two bit values are equal with probability

$$\langle v|P_e|v\rangle \;=\; (\sqrt{\|\,\alpha_{00}\,\|^2 + \|\,\alpha_{11}\,\|^2}) \;=\|\,\alpha_{00}\,\|^2 + \|\,\alpha_{11}\,\|^2$$

Of course, the measurement does not determine the value of the two bits, only whether the two bits are equal

Projectors

Any orthonormal collection of vectors $B = \{|v_1\rangle, |v_2\rangle, \cdots\}$ defines a projector

$$P = \sum_i |v_i\rangle\langle v_i|$$

If B spans the entire Hilbert space V, it forms a basis for V and $P = I$, i.e. B provides a decompostion for the identity.

Is there a standard way to provide a decomposition for P?

Yes, if P is a Hermitian operator, because of the

Spectral theorem
Any Hermitian operator on a finite Hilbert space V provides a basis for V consisting of its eigenvectors.

Projectors are Hermitian

Hermitian operators

- define a unique orthogonal subspace decomposition, their eigenspace decomposition, and
- for every such decomposition, there exists a corresponding Hermitian operator whose eigenspace decomposition coincides with it

Properties

Every eigenvalue λ with eigenvector $|r\rangle$ is real, because

$$\lambda\langle r|r\rangle = \langle r|\lambda|r\rangle = \langle r|\left(P|r\rangle\right) = \left(\langle r|P^{\dagger}\right)|r\rangle = \bar{\lambda}\langle r|r\rangle$$

Projectors are Hermitian

Properties

For any P Hermitian, two distinct eigenvalues have disjoint eigenspaces, because, for any unit vector $|v\rangle$,

$$P|v\rangle = \lambda|v\rangle \ \text{ and } \ P|v\rangle = \lambda'|v\rangle \ \text{ and } \ (\lambda - \lambda')|v\rangle = 0$$

and thus $\lambda = \lambda'$.

Moreover, the eigenvectors for distinct eigenvalues must be orthogonal, because

$$\lambda\langle v|w\rangle \ = \ (\langle v|P^{\dagger})\,|w\rangle \ = \ \langle v|\,(P|w\rangle) \ = \ \mu\langle v|w\rangle$$

for any pairs $(\lambda, |v\rangle), (\mu, |w\rangle)$ with $\lambda \neq \mu$.
Thus, $\langle v|w\rangle = 0$, because $\lambda \neq \mu$, and the corresponding subspaces are orthogonal.

Projectors are Hermitian

Eigenspace decomposition of V for P

Any Hermitian P determines a unique decomposition for V

$$V \;=\; \oplus_{\lambda_i} S_{\lambda_i}$$

and any decomposition $V \;=\; \oplus_{i=1}^{k} S_i$ can be realized as the eigenspace decomposition of a Hermitian operator

$$P = \sum_i \lambda_i \mathrm{P}_i$$

where each P_i is the projector onto S_{λ_i}

Projectors are Hermitian

A decomposition can be specified by a Hermitian operator

- Any measurement is specified by a Hermitian operator P
- The possible outcomes of measuring a state $|v\rangle$ with P are labeled by the eigenvalues of P
- The probability of obtaining the outcome labelled by λ_i is

$$\| P_i|v\rangle \|^2$$

- The state after measurement is the normalized projection

$$\frac{P_i|v\rangle}{\| \mathrm{P}_i|v\rangle \|}$$

  onto the λ_i-eigenspace S_i. Thus, the state after measurement is a unit length eigenvector of P with eigenvalue λ_i

Projectors are Hermitian

Notes

- A measurement is not modelled by the action of a Hermitian operator on a state, but of the corresponding projectors.
- Actually, Hermitian operators are only a bookeeping trick
- A Hermitian operator uniquely specifies a subspace decomposition
- For a given subspace decomposition there are many Hermitian operators whose eigenspace decomposition is that decomposition.

Projectors are Hermitian

Example: Measuring a single qubit in the Hadamard basis

Operator

$$X \;=\; |0\rangle\langle 1| + |1\rangle\langle 0| \;=\; \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is Hermitian, with eigenvalues $\lambda_+ = 1$ and $\lambda_- = -1$, and $|+\rangle, |-\rangle$ the corresponding eigenvectors, thus yielding the following projectors:

$$\begin{aligned} P_+ ;= \; |+\rangle\langle +| \;&=\; \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|) \\ P_- \;= \; |-\rangle\langle -| \;&=\; \frac{1}{2}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|) \end{aligned}$$