Quantum Systems

(Lecture 3: The principles of quantum computation)

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The principles

If quantum computation explores the laws of quantum mechanics as computational resources, principles of the former are directly derived from the postulates of the latter.

- The state space postulate
- The state evolution postulate
- The state composition postulate
- The state measurement postulate

The state space postulate

Postulate 1

The state space of a quantum system is described by a unit vector in a Hilbert space

- In practice, with finite resources, one cannot distinguish between a continuous state space from a discrete one with arbitrarily small minimum spacing between adjacente locations.
- One may, then, restrict to finite-dimensional (complex) Hilbert spaces.

The state space postulate

A qubit is encoded in a 2-dimensional such space as a linear combination (superposition) of basis vectors with complex coefficients:

$$|\phi\rangle \;=\; \alpha|0\rangle + \beta|1\rangle \;=\; \begin{bmatrix}\alpha \\ \beta\end{bmatrix}$$

obeying the normalization constraint

$$\| \alpha \|^2 + \| \beta \|^2 = \; 1$$

which enforces quantum states to be represented by unit vectors (to ensure compatibility with the measurement postulate)

Recall that a complex amplitude α can always be presented as a phase factor $e^{i\theta}$, where θ is know the phase

The state space of a qubit

Representation redundancy:

qubit state space $\neq$ complex vector space used for representation

Global phase

Unit vectors equivalent up to multiplication by a complex number of modulus one, i.e. a phase $e^{i\theta}$, represent the same state.

Let

$$|v\rangle = \alpha|u\rangle + \beta|u'\rangle$$

$$\| e^{i\theta}\alpha \|^2 = (\overline{e^{i\theta}\alpha})(e^{i\theta}\alpha) = (e^{-i\theta}\overline{\alpha})(e^{i\theta}\alpha) = \overline{\alpha}\alpha = \| \alpha \|^2$$

and similarly for β.

As the probabilities $\| \alpha \|^2$ and $\| \beta \|^2$ are the only measurable quantities, the global phase has no physical meaning.

The state space of a qubit

Relative phase

It is a measure of the angle between the two complex numbers.
Thus, it cannot be discarded!

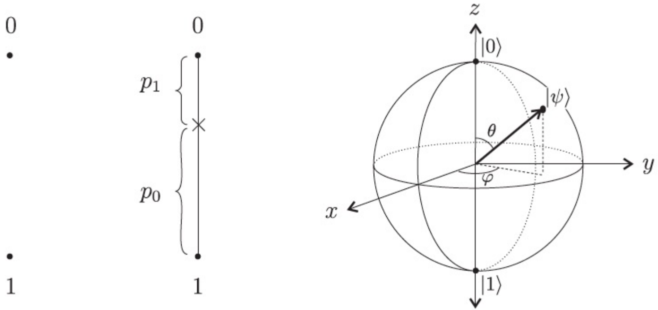
Those are different states

$$\frac{1}{\sqrt{2}}(|u\rangle + |u'\rangle) \quad \frac{1}{\sqrt{2}}(|u\rangle - |u'\rangle) \quad \frac{1}{\sqrt{2}}(e^{i\theta}|u\rangle + |u'\rangle)$$

...

The Bloch sphere

Deterministic, probabilistic and quantum bits



(from [Kaeys *et al*, 2007])

The Bloch sphere

The state of a quantum bit is described by a complex unit vector in a 2-dim Hilbert space, which, up to a physically irrelevant global phase factor, can be written as

$$|\psi\rangle = \underbrace{\cos\frac{\theta}{2}}_{\alpha}|0\rangle + \underbrace{e^{i\varphi}\sin\frac{\theta}{2}}_{\beta}|1\rangle$$

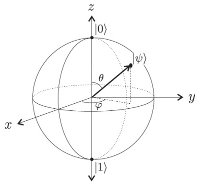
where $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$, and depicted as a point on the surface of a 3-dim Bloch sphere, defined by θ and φ.
The Bloch vector $|\psi\rangle$ has

- Spherical coordinates:
  $x = \rho \sin\theta \cos\varphi \quad y = \rho \sin\theta \sin\varphi \quad = \quad z = \rho\cos\theta$
- Measurement probabilities:

$$\|\alpha\|^2 = \left(\cos\frac{\theta}{2}\right) \quad = \quad \frac{1}{2} + \frac{1}{2}\cos\theta$$

$$\|\beta\|^2 = \left(\sin\frac{\theta}{2}\right) \quad = \quad \frac{1}{2} - \frac{1}{2}\cos\theta$$

The Bloch sphere



- The poles represent the classical bits. In general, orthogonal states correspond to antipodal points and every diameter to a basis for the single-qubit state space.
- Once measured a qubit collapses to one of the two poles. Which pole depends exactly on the arrow direction: The angle θ measures that probability: If the arrow points at the equator, there is 50-50 chance to collapse to any of the two poles.
- Rotating a vector wrt the z-axis results into a phase change (φ), and does not affect which state the arrow will collapse to, when measured.

The Bloch sphere

Representing $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Express $|\psi\rangle$ in polar form

$$|\psi\rangle = \rho_1 e^{i\varphi_1}|0\rangle + \rho_2 e^{i\varphi_2}|1\rangle$$

and eliminate one of the four real parameters multiplying by $e^{-i\varphi_1}$

$$|\psi\rangle = \rho_1|0\rangle + \rho_2 e^{i(\varphi_2 - \varphi_1)}|1\rangle = \rho_1|0\rangle + \rho_2 e^{i\varphi}|1\rangle$$

making $\varphi = \varphi_2 - \varphi_1$.

Switch back the coefficient of $|1\rangle$ to Cartesian coordinates and compute the normalization constraint

$$\| \rho_1 \|^2 + \| a + ib \|^2 = \| \rho_1 \|^2 + (a - ib)(a + ib) \; = \| \rho_1 \|^2 + a^2 + b^2 \; = \; 1$$

which is the equation of a unit sphere in Real 3-dim space with Cartesian coordinates: (a, b, ρ_1).

The Bloch sphere

Back to polar,

$$
\begin{aligned}
x =& \rho \sin\theta \cos\varphi \\
y =& \rho \sin\theta \sin\varphi \\
z =& \rho \cos\theta
\end{aligned}
$$

So, recalling that $\rho = 1$,

$$
\begin{aligned}
|\psi\rangle &= z|0\rangle + (a + ib)|1\rangle \\
&= \cos\theta|0\rangle + \sin\theta(\cos\varphi - i\sin\varphi)|1\rangle \\
&= \cos\theta|0\rangle + e^{i\varphi}\sin\theta|1\rangle
\end{aligned}
$$

which, with two parameters, defines a point in the sphere's surface.

The Bloch sphere

Actually, one may just focus on the upper hemisphere ($0 \leq \theta' \leq \frac{\pi}{2}$) as opposite points in the lower one differ only by a phase factor of -1:

Let $|\psi'\rangle$ be the opposite point on the sphere with polar coordinates $(1, \pi - \theta', \varphi + \pi)$

$$\begin{aligned}
|\psi'\rangle &= \cos(\pi - \theta')|0\rangle + e^{i(\varphi+\pi)} \sin(\pi - \theta')|1\rangle \\
&= -\cos\theta'|0\rangle + e^{i\varphi} e^{i\pi} \sin\theta'|1\rangle \\
&= -\cos\theta'|0\rangle + e^{i\varphi} \sin\theta'|1\rangle \\
&= -|\psi\rangle
\end{aligned}$$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi} \sin\frac{\theta}{2}|1\rangle$$

where $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$

$\mathcal{C}$ compactification

The Bloch sphere is a bijective correspondence between qubits and point in the space; formally, a latitude (ϕ) and longitude (θ) based representation of the state space of a qubit in the complex projective space of dimension 1.

Alternative: $\mathcal{C}$ compactification

Represents a qubit by a complex number in $\mathcal{C} \cup \{\perp\}$ through a correspondence ξ:

$$\xi = \alpha|0\rangle + \beta|1\rangle \mapsto b/a \quad \text{and} \quad |1\rangle \mapsto \perp$$

$$\xi^{-1} = \gamma \mapsto \frac{1}{\sqrt{1+\|\gamma\|^2}}|0\rangle + \frac{\gamma}{\sqrt{1+\|\gamma\|^2}}|1\rangle \quad \text{and} \quad \perp \mapsto |1\rangle$$

The state evolution postulate

Postulate 2

The evolution over time of the state of a closed quantum system is described by a unitary operator.

The evolution is linear

$$U\left(\sum_j \alpha_j |v_j\rangle\right) = \sum_j \alpha_j U(|v_j\rangle)$$

and preserves the normalization constraint

$$\text{If } \sum_j \alpha_j U(|v_j\rangle) = \sum_j \alpha'_j |v_j\rangle \text{ then } \sum_j \| \alpha'_j \|^2 = 1$$

Unitarity

Unitary

This entails a condition on valid quantum operators: they must preserve the inner product, i.e.

$$(U|v\rangle, U|w\rangle) = \langle v|U^\dagger U|w\rangle = \langle v|w\rangle$$

which is the case iff U is unitary

$$U^\dagger U = UU^\dagger = I$$

- Preserving the inner product means that a unitary operator maps orthonormal bases to orthonormal bases.
- Conversely, any operator with this property is unitary.
- If given in matrix form, being unitary means that the set of columns of its matrix representation are orthonormal (because the jth column is the image of $U|j\rangle$). Equivalently, rows are orthonormal (why?)

Unitarity

Unitarity is the only constraint on quantum operators: Any unitary matrix specifies a valid quantum operator.

This means that there are many non-trivial operators on a single qubit (in contrast with the classical case where the only non-trivial operation on a bit is complement.

Finally, because the inverse of a unitary matrix is also a unitary matrix, a quantum operator can always be inverted by another quantum operator

$\boxed{\text{Unitary transformations are reversible}}$

The state evolution postulate

Examples: The Pauli operators

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Operators X, Y and Z correspond to rotations in the Bloch sphere along the x, y and z axis, respectively.
- Any 1-qubit unitary operator can be expressed as a linear combination of Pauli operators.

The no-cloning theorem

Linearity implies that quantum states cannot be cloned

Let $U(|a\rangle|0\rangle) = |a\rangle|a\rangle$ be a 2-qubit operator and $|c\rangle = \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle)$ for $|a\rangle$, $|b\rangle$ orthogonal. Then,

$$
\begin{aligned}
U(|c\rangle|0\rangle) &= \frac{1}{\sqrt{2}}(U(|a\rangle|0\rangle) + U(|b\rangle|0\rangle)) \\
&= \frac{1}{\sqrt{2}}(|a\rangle|a\rangle + |b\rangle|b\rangle) \\
&\neq \frac{1}{\sqrt{2}}(|a\rangle|a\rangle + |a\rangle|b\rangle + |b\rangle|a\rangle + |b\rangle|b\rangle) \\
&= |c\rangle|c\rangle \\
&= U(|c\rangle|0\rangle)
\end{aligned}
$$

This, however, does not preclude the construction of a known quantum state from a known quantum state.

Building larger states from smaller

Operator U in the no-cloning theorem acts on a 2-dimensional state, i.e. over the composition of two qubits.

$\boxed{\text{What does composition mean?}}$

Postulate 3

The state space of a combined quantum system is the tensor product $V \otimes W$ of the state spaces V and W of its components.

Composing classical states

State spaces in a classical system combine through direct sum: $\oplus$

$$\boxed{n\ m\text{-dimensional vectors} \rightsquigarrow \text{a vector in } mn\text{-dimensional space}}$$

Example

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \oplus \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix}$$

Composing classical states

Direct sum $V \oplus W$

- $B_{V \oplus W} = B_V \cup B_W$ and $\dim(\mathrm{V} \oplus \mathrm{W}) = \dim(\mathrm{V}) + \dim(\mathrm{W})$
- Vector addition and scalar multiplication are performed in each component and the results added
- $\langle (|u_2\rangle \oplus |z_2\rangle) | (|u_1\rangle \oplus |z_1\rangle) \rangle \; = \; \langle u_2 | u_1 \rangle + \langle z_2 | z_1 \rangle$
- V and W embed canonically in $V \oplus W$ and the images are orthogonal under the standard inner product

Composing quantum states

State spaces in a quantum system combine through tensor: $\otimes$

$$\boxed{n\ m\text{-dimensional vectors} \quad \rightsquigarrow \quad \text{a vector in } m^n\text{-dimensional space}}$$

i.e. the state space of a quantum system grows exponentially with the number of particles: cf, Feyman's original motivation

Example

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \otimes \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a \begin{bmatrix} d \\ e \\ f \end{bmatrix} \\ b \begin{bmatrix} d \\ e \\ f \end{bmatrix} \\ c \begin{bmatrix} d \\ e \\ f \end{bmatrix} \end{bmatrix} = \begin{bmatrix} ad \\ ae \\ af \\ bd \\ be \\ bf \\ cd \\ ce \\ cf \end{bmatrix}$$

Composing quantum states

Tensor $V \otimes W$

- $B_{V\otimes W}$ is a set of elements of the form $|v_i\rangle \otimes |w_j\rangle$, for each $|v_i\rangle \in B_V$, $|w_i\rangle \in B_W$ and $\dim(\mathrm{V} \otimes \mathrm{W}) = \dim(\mathrm{V}) \times \dim(\mathrm{W})$
- $(|u_1\rangle + |u_2\rangle) \otimes |z\rangle \;=\; |u_1\rangle \otimes |z\rangle + |u_2\rangle \otimes |z\rangle$
- $|z\rangle \otimes (|u_1\rangle + |u_2\rangle) \;=\; |z\rangle \otimes |u_1\rangle + |z\rangle \otimes |u_2\rangle$
- $(\alpha|u\rangle) \otimes |z\rangle \;=\; |u\rangle \otimes (\alpha|z\rangle) \;=\; \alpha(|u\rangle \otimes |z\rangle)$
- $\langle(|u_2\rangle \otimes |z_2\rangle)|(|u_1\rangle \otimes |z_1\rangle)\rangle \;=\; \langle u_2|u_1\rangle\langle z_2|z_1\rangle$

Composing quantum states

Clearly, every element of $V \otimes W$ can be written as

$$\alpha_1(|v_1\rangle \otimes |w_1\rangle) + \alpha_2(|v_2\rangle \otimes |w_1\rangle) + \cdots + \alpha_{nm}(|v_n\rangle \otimes |w_m\rangle)$$

Example

The basis of $V \otimes W$, for V, W qubits with the computational basis is

$$\{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$$

Thus, the tensor of $\alpha_1|0\rangle + \alpha_2|1\rangle$ and $\beta_1|0\rangle + \beta_2|1\rangle$ is

$$\alpha_1\beta_1|0\rangle \otimes |0\rangle \; + \; \alpha_1\beta_2|0\rangle \otimes |1\rangle \; + \; \alpha_2\beta_1|1\rangle \otimes |0\rangle \; + \; \alpha_2\beta_2|1\rangle \otimes |1\rangle$$

i.e., in a simplified notation,

$$\alpha_1\beta_1|00\rangle \; + \; \alpha_1\beta_2|01\rangle \; + \; \alpha_2\beta_1|10\rangle \; + \; \alpha_2\beta_2|11\rangle$$

Bases

The computational basis for a vector space

$$\underbrace{V \otimes V \otimes \cdots \otimes V}_{n}$$

corresponding to the composition of n qubits (each living in V) is the set

$$\{\underbrace{|0\rangle \cdots |0\rangle|0\rangle}_{n}, \underbrace{|0\rangle \cdots |0\rangle|1\rangle}_{n}, \underbrace{|0\rangle \cdots |1\rangle|0\rangle}_{n}, \cdots \underbrace{|1\rangle \cdots |1\rangle|1\rangle}_{n}\}$$

$$\stackrel{\text{abv}}{=}$$

$$\{\underbrace{|0 \cdots 00\rangle}_{n}, \underbrace{|0 \cdots 01\rangle}_{n}, \underbrace{|0 \cdots 10\rangle}_{n}, \cdots \underbrace{|1 \cdots 11\rangle}_{n}\}$$

which may be written in a compressed (decimal) way as

$$\{|0\rangle, |1\rangle, |2\rangle, |3\rangle, \cdots |2^n - 1\rangle\}$$

Bases

The computational basis for a two qubit system would be

$$\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$$

with

$$|0\rangle = |00\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad |1\rangle = |01\rangle = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \quad |2\rangle = |10\rangle = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \quad |3\rangle = |11\rangle = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

Bases

There are of course other bases
... besides the standard one, e.g.

The Bell basis

$$
\begin{aligned}
|\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
|\Phi^-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\
|\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\
|\Psi^-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)
\end{aligned}
$$

Compare with the Hadamard basis for the single qubit systems

Representing multi-qubit states

Any unit vector in a 2^n Hilbert space represents a possible n-qubit state, but for

... a certain level of redundancy

- As before, vectors that differ only in a global phase represent the same quantum state
- but also the same phase factor in different qubits of a tensor product represent the same state:

$$|u\rangle \otimes (e^{i\Phi}|z\rangle) \;=\; e^{i\Phi}(|u\rangle \otimes |z\rangle) \;=\; (e^{i\Phi}|u\rangle) \otimes |z\rangle$$

Actually, phase factors in qubits of a single term of a superposition can always be factored out into a coefficient for that term, i.e. phase factors distribute over tensors

Representing multi-qubit states

Representation

- Relative phases still matter (of course!)

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \quad \text{differs from} \quad \frac{1}{\sqrt{2}}(e^{i\Phi}|00\rangle + |11\rangle)$$

even if

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(e^{i\Phi}|00\rangle + e^{i\Phi}|11\rangle) = \frac{e^{i\Phi}}{\sqrt{2}}(|00\rangle + |11\rangle)$$

- The complex projective space of dimension 1 (depicted in the Block sphere) generalises to higher dimensions, although in practice linearity makes Hilbert spaces easier to use.

Entanglement

Most states in $V \otimes W$ cannot be written as $|u\rangle \otimes |z\rangle$

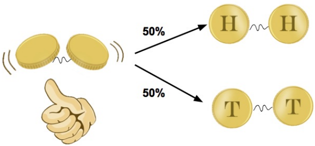
- By $\mathbb{C}$ compactification a single-qubit state can be specified by a single complex number so any tensor product of n qubit states can be specified by n complex numbers. But it takes $2^n - 1$ complex numbers to describe states of an n qubit system.
- Since $2^n \gg n$, the vast majority of n-qubit states cannot be described in terms of the state of n separate qubits.
- Such states, that cannot be written as the tensor product of n single-qubit states, are entangled states.

Entanglement

For example, the Bell state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \;=\; \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

is entangled



Entanglement

Actually, to make $|\Phi^+\rangle$ equal to

$$(\alpha_1|0\rangle+\beta_1|1\rangle)\otimes(\alpha_2|0\rangle+\beta_2|1\rangle) \;=\; \alpha_1\alpha_2|00\rangle+\alpha_1\beta_2|01\rangle+\beta_1\alpha_2|10\rangle+\beta_1\beta_2|11\rangle$$

would require that $\alpha_1\beta_2 = \beta_1\alpha_2 = 0$ which implies that either

$$\alpha_1\alpha_2 = 0 \;\text{ or }\; \beta_1\beta_2 = 0$$

Note
Entanglement can also be observed in simpler structures, e.g. relations:

$$\{(a, a), (b, b)\} \subseteq A \times A$$

cannot be separated, i.e. written as a Cartesian product of subsets of A.

The measurement postulate

Postulate 4

For a given orthonormal basis $B = \{|v_1\rangle, |v_2\rangle, \cdots\}$, a measurement of a state space $|v\rangle = \sum_i \alpha_i |v_i\rangle$ wrt B, outputs the label i with probability $\| \alpha_i \|^2$ and leaves the system in state $|v_i\rangle$.

- Measurements are made through projectors which identify the ‘data’ (i.e. the subspace of the relevant Hilbert space where the quntum system lives) one wants to measure.
- Let us start with a couple of examples ... but for the general notion let us recall the notion of adjoint operator.

Adjoints

Given an operator U, its adjoint is the unique operator satisfying

$$(|w\rangle, U|v\rangle) \;=\; (U^{\dagger}|w\rangle, |v\rangle)$$

where $(|x\rangle, |y\rangle)$ is the 'verbose' representation for the inner product $\langle x, y\rangle$. Thus, in Dirac notation the equality above becomes

$$\langle w|Uv\rangle = (U^{\dagger}|w\rangle)^{\dagger}|v\rangle \;=\; \langle wU|v\rangle$$

or simply

$$\langle w|U|v\rangle$$

The matrix representation of $U^{\dagger}$ is the conjugate transpose of that of U

Exercise: Prove that $\overline{\langle w|U|v\rangle} \;=\; \langle v|U^{\dagger}|w\rangle$

Projectors

Any projector P identifies in the state space V a subspace V_P of all vectors $|\phi\rangle$ that are left unchanged by P, i.e. such that

$$P|\phi\rangle = |\phi\rangle$$

Examples

- The identity I projects onto the whole space V.
- The zero operator projects onto the space $\{0\}$ consisting only of the zero vector.
- $|u\rangle\langle u|$ is the projector onto the subspace spanned by $|u\rangle$.

Projectors

Examples

- Projector $|0\rangle\langle 0|$ projects onto the subspace generated by $|0\rangle$, i.e.

$$|0\rangle\langle 0|\,(\alpha|0\rangle + \beta|1\rangle) \;=\; \alpha|0\rangle\langle 0|(|0\rangle) + \beta|0\rangle\langle 0|(|1\rangle) \;=\; \alpha|0\rangle$$

- Similarly, $|10\rangle\langle 10|$ acts on a two-qubit state

$$v \;=\; \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

yielding

$$|10\rangle\langle 10|\,(|v\rangle) \;=\; \alpha_{10}|10\rangle$$

and

$$|00\rangle\langle 00| + |10\rangle\langle 10|(|v\rangle) \;=\; \alpha_{00}|00\rangle + \alpha_{10}|10\rangle$$

Projectors

A projector $P : V \to V_P$ is an operator such that

$$P^2 = P$$

Additionally, we require P to be Hermitian, i.e.

$$P = P^\dagger$$

Note that the combination of both properties yields

$$\| P|v\rangle \|^2 = (\langle v|P^\dagger)(\mathrm{P}|v\rangle) = \langle v|P|v\rangle$$

Example

The probability of getting state $|0\rangle$ when measuring $\alpha|0\rangle + \beta|1\rangle$ with $P = |0\rangle\langle 0|$ is computed as

$$\| P|v\rangle \|^2 = \langle v|P|v\rangle = \langle v||0\rangle\langle 0||v\rangle = \langle v|0\rangle \langle 0|v\rangle = \overline{\alpha}\alpha = \| \alpha \|^2$$

Projectors

Two projectors P, Q are orthogonal if $PQ = 0$.

The sum of any collection of orthogonal projectors $\{P_1, P_2, \cdots\}$ is still a projector (verify!).

A projector P has a decomposition if it can be written as a sum of orthogonal projectors:

$$P = \sum_i P_i$$

Such projectors yield measurements wrt to the corresponding decomposition.

Examples

- Complete measurement in the computational basis wrt to decomposition

$$I = \sum_{i \in 2^n} |i\rangle\langle i|$$

in a state with n qubits.

- Incomplete measurement: e.g.

$$\sum_{\{i \in 2^n \mid i \text{ even}\}} |i\rangle\langle i|$$

Projectors

Example: measuring up to (bit equality)

$$V = S_e \oplus S_n$$

with S_e the subspace generated by $\{|00\rangle, |11\rangle\}$ in which the two bits are equal, and S_n its complement. P_e and P_n, are the corresponding projectors.

When measuring

$$v \;=\; \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

with this device, yields a state in which the two bit values are equal with probability

$$\langle v|P_e|v\rangle \;=\; (\sqrt{\| \alpha_{00} \|^2 + \| \alpha_{11} \|^2}) \;= \| \alpha_{00} \|^2 + \| \alpha_{11} \|^2$$

Of course, the measurement does not determine the value of the two bits, only whether the two bits are equal

Projectors

Any orthonormal collection of vectors $B = \{|v_1\rangle, |v_2\rangle, \cdots\}$ defines a projector

$$P = \sum_i |v_i\rangle\langle v_i|$$

If B spans the entire Hilbert space V, it forms a basis for V and $P = I$, i.e. B provides a decompostion for the identity.

Is there a standard way to provide a decomposition for P?

Yes, if P is a Hermitian operator, because of the

Spectral theorem
Any Hermitian operator on a finite Hilbert space V provides a basis for V consisting of its eigenvectors.

Projectors are Hermitian

Hermitian operators

- define a unique orthogonal subspace decomposition, their eigenspace decomposition, and
- for every such decomposition, there exists a corresponding Hermitian operator whose eigenspace decomposition coincides with it

Properties

Every eigenvalue λ with eigenvector $|r\rangle$ is real, because

$$\lambda\langle r|r\rangle \;=\; \langle r|\lambda|r\rangle \;=\; \langle r|\left(P|r\rangle\right) \;=\; \left(\langle r|P^\dagger\right)|r\rangle \;=\; \bar{\lambda}\langle r|r\rangle$$

Projectors are Hermitian

Properties

For any P Hermitian, two distinct eigenvalues have disjoint eigenspaces, because, for any unit vector $|v\rangle$,

$$P|v\rangle = \lambda|v\rangle \text{ and } P|v\rangle = \lambda'|v\rangle \text{ and } (\lambda - \lambda')|v\rangle = 0$$

and thus $\lambda = \lambda'$.

Moreover, the eigenvectors for distinct eigenvalues must be orthogonal, because

$$\lambda\langle v|w\rangle \;=\; (\langle v|P^\dagger)\,|w\rangle \;=\; \langle v|\,(P|w\rangle) \;=\; \mu\langle v|w\rangle$$

for any pairs $(\lambda, |v\rangle), (\mu, |w\rangle)$ with $\lambda \neq \mu$.
Thus, $\langle v|w\rangle = 0$, because $\lambda \neq \mu$, and the corresponding subspaces are orthogonal.

Projectors are Hermitian

Eigenspace decomposition of V for P

Any Hermitian P determines a unique decomposition for V

$$V = \oplus_{\lambda_i} S_{\lambda_i}$$

and any decomposition $V = \oplus_{i=1}^{k} S_i$ can be realized as the eigenspace decomposition of a Hermitian operator

$$P = \sum_i \lambda_i \mathrm{P}_i$$

where each P_i is the projector onto S_{λ_i}

Projectors are Hermitian

A decomposition can be specified by a Hermitian operator

- Any measurement is specified by a Hermitian operator P
- The possible outcomes of measuring a state $|v\rangle$ with P are labeled by the eigenvalues of P
- The probability of obtaining the outcome labelled by λ_i is

$$\| P_i|v\rangle \|^2$$

- The state after measurement is the normalized projection

$$\frac{P_i|v\rangle}{\| \mathrm{P}_i|v\rangle \|}$$

  onto the λ_i-eigenspace S_i. Thus, the state after measurement is a unit length eigenvector of P with eigenvalue λ_i