Introduction to the modal μ-calculus

Luís Soares Barbosa

Interaction & Concurrency Course Unit (Lcc)

Universidade do Minho

Is Hennessy-Milner logic expressive enough?

Is Hennessy-Milner logic expressive enough?

- it cannot detect deadlock in an arbitrary process
- or general safety: all reachable states verify ϕ
- or general liveness: there is a reachable states which verifies ϕ
- ...

... essentially because

formulas in this logic cannot see deeper than their modal depth

Is Hennessy-Milner logic expressive enough?

Example

$$\phi \; = \; \text{a taxi eventually returns to its Central}$$

$$\phi \; = \; \langle reg \rangle true \vee \langle - \rangle \langle reg \rangle true \vee \langle - \rangle \langle - \rangle \langle reg \rangle true \vee \langle - \rangle \langle - \rangle \langle - \rangle \langle reg \rangle true \vee \; \ldots$$

Revisiting Hennessy-Milner logic

Allowing regular expressions within modalities

$$\rho \ ::= \epsilon \mid \alpha \mid \rho.\rho \mid \rho + \rho \mid \rho^* \mid \rho^+$$

where

- α is an action formula and ϵ is the empty word
- concatenation $\rho.\rho$, choice $\rho + \rho$ and closures ρ^* and ρ^+

Laws

$$\begin{aligned}
\langle \rho_1 + \rho_2 \rangle \phi &= \langle \rho_1 \rangle \phi \vee \langle \rho_2 \rangle \phi \\
[\rho_1 + \rho_2] \phi &= [\rho_1] \phi \wedge [\rho_2] \phi \\
\langle \rho_1 . \rho_2 \rangle \phi &= \langle \rho_1 \rangle \langle \rho_2 \rangle \phi \\
[\rho_1 . \rho_2] \phi &= [\rho_1][\rho_2] \phi
\end{aligned}$$

Revisiting Hennessy-Milner logic

Examples of properties

- $\langle \epsilon \rangle \phi \;=\; [\epsilon]\phi \;=\; \phi$
- $\langle a.a.b \rangle \phi \;=\; \langle a \rangle \langle a \rangle \langle b \rangle \phi$
- $\langle a.b + g.d \rangle \phi \;=\; \langle a.b \rangle \phi \vee \langle g.d \rangle \phi$

Safety

- $[-^*]\phi$
- it is impossible to do two consecutive enter actions without a leave action in between:
  $[-^*.enter. - leave^*.enter]\mathit{false}$
- absence of deadlock:
  $[-^*]\langle - \rangle \mathit{true}$

Revisiting Hennessy-Milner logic

Examples of properties

Liveness

- $\langle -^* \rangle \phi$
- after sending a message, it can eventually be received: $[send]\langle -^*.receive \rangle true$
- after a send, a receive is possible as long as an exception does not happen: $[send.-excp^*]\langle (-^*.receive) + (-^*.excp) \rangle true$

The modal μ-calculus

- modalities with regular expressions are not enough in general
- ... but correspond to a subset of the modal μ-calculus [Kozen83]

Add explicit minimal/maximal fixed point operators to Hennessy-Milner logic

$$\phi \;::=\; X \mid \mathit{true} \mid \mathit{false} \mid \neg\phi \mid \phi\wedge\phi \mid \phi\vee\phi \mid \phi\rightarrow\phi \mid \langle a\rangle\phi \mid [a]\phi \mid \mu X\,.\,\phi \mid \nu X\,.\,\phi$$

The modal μ-calculus

The modal μ-calculus (intuition)

- $\mu X \,.\, \phi$ is valid for all those states in the smallest set X that satisfies the equation $X = \phi$ (finite paths, liveness)
- $\nu X \,.\, \phi$ is valid for the states in the largest set X that satisfies the equation $X = \phi$ (infinite paths, safety)

Warning

In order to be sure that a fixed point exists, X must occur positively in the formula, i.e. preceded by an even number of negations.

Temporal properties as limits

Example

$$A \widehat{=} \sum_{i \geq 0} A_i \quad \text{with} \quad A_0 \widehat{=} \mathbf{0} \text{ e } A_{i+1} \widehat{=} a.A_i$$
$$A' \widehat{=} A + D \quad \text{with} \quad D \widehat{=} a.D$$

- $A \not\sim A'$
- but there is no modal formula to distinguish A from A'
- notice $A' \models \langle a \rangle^{i+1} true$ which A_i fails
- a distinguishing formula would require infinite conjunction
- what we want to express is the possibility of doing a in the long run

Temporal properties as limits

idea: introduce recursion in formulas

$$X \mathrel{\widehat{=}} \langle a \rangle X$$

meaning?

- the recursive formula is interpreted as a fixed point of function

  $$\| \langle a \rangle \|$$

  in $\mathcal{P}\mathbb{P}$
- i.e., the solutions $S \subseteq \mathbb{P}$, such that of

  $$S = \| \langle a \rangle \| (S)$$
- how do we solve this equation?

Solving equations ...

over natural numbers

$$\begin{aligned} x &= 3x && \text{one solution } (x = 0) \\ x &= 1 + x && \text{no solutions} \\ x &= 1x && \text{many solutions (every natural } x) \end{aligned}$$

over sets of integers

$$\begin{aligned} x &= \{22\} \cap x && \text{one solution } (x = \{22\}) \\ x &= \mathbb{N} \setminus x && \text{no solutions} \\ x &= \{22\} \cup x && \text{many solutions (every } x \text{ st } \{22\} \subseteq x) \end{aligned}$$

Solving equations ...

In general, for a monotonic function f, i.e.

$$X \subseteq Y \;\Rightarrow\; f\,X \subseteq f\,Y$$

Knaster-Tarski Theorem [1928]

A monotonic function f in a complete lattice has a

- unique maximal fixed point:

$$\nu_f \;=\; \bigcup\{X \in \mathcal{P}\mathbb{P} \mid X \subseteq f\,X\}$$

- unique minimal fixed point:

$$\mu_f \;=\; \bigcap\{X \in \mathcal{P}\mathbb{P} \mid f\,X \subseteq X\}$$

- moreover the space of its solutions forms a complete lattice

Back to the example ...

$S \in \mathcal{P}\mathbb{P}$ is a pre-fixed point of $\|\langle a \rangle\|$
iff

$$\|\langle a \rangle\|(S) \;\subseteq\; S$$

Recalling,

$$\|\langle a \rangle\|(S) \;=\; \{E \in \mathbb{P} \mid \exists_{E' \in S} \,.\, E \xrightarrow{a} E'\}$$

the set of sets of processes we are interested in is

$$\boxed{\begin{aligned} Pre &= \{S \subseteq \mathbb{P} \mid \{E \in \mathbb{P} \mid \exists_{E' \in S} \,.\, E \xrightarrow{a} E'\} \subseteq S\} \\ &= \{S \subseteq \mathbb{P} \mid \forall_{Z \in \mathbb{P}} \,.\, (Z \in \{E \in \mathbb{P} \mid \exists_{E' \in S} \,.\, E \xrightarrow{a} E'\} \Rightarrow Z \in S)\} \\ &= \{S \subseteq \mathbb{P} \mid \forall_{E \in \mathbb{P}} \,.\, ((\exists_{E' \in S} \,.\, E \xrightarrow{a} E') \Rightarrow E \in S)\} \end{aligned}}$$

which can be characterized by predicate

$$\text{(PRE)} \qquad (\exists_{E' \in S} \,.\, E \xrightarrow{a} E') \Rightarrow E \in S \qquad \text{(for all } E \in \mathbb{P}\text{)}$$

Back to the example ...

The set of pre-fixed points of

$$\|\langle a\rangle\|$$

is

$$\begin{aligned} Pre &= \{S \subseteq \mathbb{P} \mid \|\langle a\rangle\|(S) \subseteq S\} \\ &= \{S \subseteq \mathbb{P} \mid \forall_{E \in \mathbb{P}} \,.\, ((\exists_{E' \in S} \,.\, E \xrightarrow{a} E') \Rightarrow E \in S)\} \end{aligned}$$

- Clearly, $\{A \mathrel{\widehat{=}} a.A\} \in Pre$
- but $\emptyset \in Pre$ as well

Therefore, its least solution is

$$\bigcap Pre = \emptyset$$

Conclusion: taking the meaning of $X = \langle a\rangle X$ as the least solution of the equation leads us to equate it to *false*

... but there is another possibility ...

$S \in \mathcal{P}\mathbb{P}$ is a post-fixed point of

$$\|\langle a \rangle\|$$

iff

$$S \;\subseteq\; \|\langle a \rangle\|(S)$$

leading to the following set of post-fixed points

$$\boxed{\begin{aligned} \mathit{Post} &= \{S \subseteq \mathbb{P} \mid S \subseteq \{E \in \mathbb{P} \mid \exists_{E' \in S} \,.\, E \xrightarrow{a} E'\}\} \\ &= \{S \subseteq \mathbb{P} \mid \forall_{Z \in \mathbb{P}} \,.\, (Z \in S \Rightarrow Z \in \{E \in \mathbb{P} \mid \exists_{E' \in S} \,.\, E \xrightarrow{a} E'\})\} \\ &= \{S \subseteq \mathbb{P} \mid \forall_{E \in \mathbb{P}} \,.\, (E \in S \Rightarrow \exists_{E' \in S} \,.\, E \xrightarrow{a} E')\} \end{aligned}}$$

(POST) If $E \in S$ then $E \xrightarrow{a} E'$ for some $E' \in S$ (for all $E \in P$)

... but there is another possibility ...

Therefore, its greatest solution

$$\bigcup Post$$

is the greatest subset of $\mathbb{P}$ of processes with at least an infinite computation verifying

(POST) If $E \in S$ then $E \xrightarrow{a} E'$ for some $E' \in S$ (for all $E \in P$)

- i.e. if $E \in S$ it can perform a and this ability is maintained in its continuation

Conclusion: taking the meaning of $X = \langle a \rangle X$ as the greatest solution of the equation characterizes the property occurrence of a is possible

The general case

The meaning (i.e. set of processes) of a formula $X \mathrel{\widehat{=}} \phi X$ where X occurs free in ϕ is a solution of equation

$$X = f(X) \qquad \text{with} \quad f(S) = |\{S/X\}\phi|$$

in $\mathcal{P}\mathbb{P}$, where $|.|$ is extended to formulae with variables by $|X| = X$

The general case

The Knaster-Tarski theorem gives precise characterizations of the

- smallest solution: the intersection of all S such that

$$\text{(PRE)} \qquad \text{If} \quad E \in f(S) \quad \text{then} \quad E \in S$$

  to be denoted by

$$\mu X \,.\, \phi$$

- greatest solution: the union of all S such that

$$\text{(POST)} \qquad \text{If} \quad E \in S \quad \text{then} \quad E \in f(S)$$

  to be denoted by

$$\nu X \,.\, \phi$$

In the previous example:

$$\nu X \,.\, \langle a \rangle true \qquad\qquad \mu X \,.\, \langle a \rangle true$$

The general case

The Knaster-Tarski theorem gives precise characterizations of the

- smallest solution: the intersection of all S such that

  $$\text{(PRE)} \qquad \text{If} \quad E \in f(S) \quad \text{then} \quad E \in S$$

  to be denoted by

  $$\mu X \,.\, \phi$$

- greatest solution: the union of all S such that

  $$\text{(POST)} \qquad \text{If} \quad E \in S \quad \text{then} \quad E \in f(S)$$

  to be denoted by

  $$\nu X \,.\, \phi$$

In the previous example:

$$\nu X \,.\, \langle a \rangle \mathit{true} \qquad\qquad \mu X \,.\, \langle a \rangle \mathit{true}$$

The modal μ-calculus: syntax

... Hennessy-Milner + recursion (i.e. fixed points):

$$\phi \ ::= \ X \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \langle K \rangle \phi \mid [K]\phi \mid \mu X \,.\, \phi \mid \nu X \,.\, \phi$$

where $K \subseteq Act$ and X is a set of propositional variables

- Note that

$$true \stackrel{abv}{=} \nu X \,.\, X \qquad \text{and} \qquad false \stackrel{abv}{=} \mu X \,.\, X$$

The modal μ-calculus: denotational semantics

- Presence of variables requires models parametric on valuations:

$$V : X \longrightarrow \mathcal{P}\mathbb{P}$$

- Then,

$$
\begin{aligned}
|X|_V &= V(X) \\
|\phi_1 \wedge \phi_2|_V &= |\phi_1|_V \cap |\phi_2|_V \\
|\phi_1 \vee \phi_2|_V &= |\phi_1|_V \cup |\phi_2|_V \\
|[K]\phi|_V &= |[K]|(|\phi|_V) \\
|\langle K \rangle\phi|_V &= |\langle K \rangle|(|\phi|_V)
\end{aligned}
$$

- and add

$$
\begin{aligned}
|\nu X \,.\, \phi|_V &= \bigcup\{S \in \mathbb{P} \mid S \subseteq |\{S/X\}\phi|_V\} \\
|\mu X \,.\, \phi|_V &= \bigcap\{S \in \mathbb{P} \mid |\{S/X\}\phi|_V \subseteq S\}
\end{aligned}
$$

Notes

where

$$\color{blue}{\|[K]\|}\, X \;=\; \{F \in \mathbb{P} \mid \text{if } F \xrightarrow{a} F' \;\wedge\; a \in K \;\text{ then }\; F' \in X\}$$

$$\color{blue}{\|\langle K\rangle\|}\, X \;=\; \{F \in \mathbb{P} \mid \exists_{F' \in X, a \in K} \,.\, F \xrightarrow{a} F'\}$$

Modal μ-calculus

Intuition

- looks at modal formulas as set-theoretic combinators,
- introduces mechanisms to specify their fixed points,
- leading to a generalisation of Hennessy-Milner logic for processes to capture enduring properties.

References

- Original reference: *Results on the propositional μ-calculus*, D. Kozen, 1983.
- Introductory text: *Modal and temporal logics for processes*, C. Stirling, 1996

Notes

The modal μ-calculus [Kozen, 1983] is

- decidable
- strictly more expressive than PDL and CTL*

Moreover

- The correspondence theorem of the induced temporal logic with bisimilarity is kept

Example 1: $X \mathrel{\widehat{=}} \phi \vee \langle a \rangle X$

Look for fixed points of

$$f(X) \mathrel{\widehat{=}} |\phi| \cup |\langle a \rangle|(X)$$

Example 1: $X \mathrel{\widehat{=}} \phi \vee \langle a \rangle X$

$$
\begin{aligned}
\text{(PRE)} \quad & \text{If} \quad E \in f(X) \quad \text{then} \quad E \in X \\
\equiv \quad & \text{If} \quad E \in (|\phi| \cup |\langle a \rangle|(X)) \quad \text{then} \quad E \in X \\
\equiv \quad & \text{If} \quad E \in \{F \mid F \models \phi\} \cup \{F \in \mathbb{P} \mid \exists_{F' \in X} \,.\, F \xrightarrow{a} F'\} \\
& \qquad \text{then} \quad E \in X \\
\equiv \quad & \text{if} \quad E \models \phi \vee \exists_{E' \in X} \,.\, E \xrightarrow{a} E' \quad \text{then} \quad E \in X
\end{aligned}
$$

The smallest set of processes verifying this condition is composed of processes with at least a computation along which a can occur until ϕ holds. Taking its intersection, we end up with processes in which ϕ holds in a finite number of steps.

Example 1: $X \mathrel{\widehat{=}} \phi \vee \langle a \rangle X$

$$
\begin{array}{lllll}
\text{(POST)} & \text{If} & E \in X & \text{then} & E \in f(X) \\
\equiv & \text{If} & E \in X & \text{then} & E \in (\|\phi\| \cup \|\langle a \rangle\|(X)) \\
\equiv & \text{If} & E \in X & \text{then} & E \in \{F \mid F \models \phi\} \cup \{F \in X \mid \exists_{F' \in X} \,.\, F \xrightarrow{a} F'\} \\
\equiv & \text{If} & E \in X & \text{then} & E \models \phi \vee \exists_{E' \in X} \,.\, E \xrightarrow{a} E'
\end{array}
$$

The greatest fixed point also includes processes which keep the possibility of doing a without ever reaching a state where ϕ holds.

Example 1: $X \triangleq \phi \vee \langle a \rangle X$

- strong until:

$$\mu X \,.\, \phi \vee \langle a \rangle X$$

- weak until

$$\nu X \,.\, \phi \vee \langle a \rangle X$$

Relevant particular cases:

- ϕ holds after internal activity:

$$\mu X \,.\, \phi \vee \langle \tau \rangle X$$

- ϕ holds in a finite number of steps

$$\mu X \,.\, \phi \vee \langle - \rangle X$$

Example 2: $X \mathrel{\widehat{=}} \phi \wedge \langle a\rangle X$

$$\text{(PRE)} \quad \text{If} \quad E \models \phi \wedge \exists_{E' \in X} \,.\, E \xrightarrow{a} E' \quad \text{then} \quad E \in X$$

implies that

$$\mu X \,.\, \phi \wedge \langle a\rangle X \Leftrightarrow \mathit{false}$$

$$\text{(POST)} \quad \text{If} \quad E \in X \quad \text{then} \quad E \models \phi \wedge \exists_{E' \in X} \,.\, E \xrightarrow{a} E'$$

implies that

$$\nu X \,.\, \phi \wedge \langle a\rangle X$$

denote all processes which verify ϕ and have an infinite computation

Example 2: $X \mathrel{\widehat{=}} \phi \wedge \langle a \rangle X$

Variant:

- ϕ holds along a finite or infinite *a*-computation:

$$\nu X \,.\, \phi \wedge (\langle a \rangle X \vee [a] \mathit{false})$$

In general:

- weak safety:

$$\nu X \,.\, \phi \wedge (\langle K \rangle X \vee [K] \mathit{false})$$

- weak safety, for $K = \mathit{Act}$:

$$\nu X \,.\, \phi \wedge (\langle - \rangle X \vee [-] \mathit{false})$$

Example 3: $X \mathrel{\widehat{=}} [-]X$

(POST) If $E \in X$ then $E \in \|[-]\|(X)$

$\equiv$ If $E \in X$ then (if $E \xrightarrow{x} E'$ and $x \in Act$ then $E' \in X$)

implies $\nu X \,.\, [-]X \Leftrightarrow \mathit{true}$

(PRE) If (if $E \xrightarrow{x} E'$ and $x \in Act$ then $E' \in X$) then $E \in X$

implies $\mu X \,.\, [-]X$ represent finite processes (why?)

Safety and liveness

- weak liveness:

$$\mu X \,.\, \phi \vee \langle - \rangle X$$

- strong safety

$$\nu X \,.\, \psi \wedge [-] X$$

making $\psi = \neg\phi$ both properties are dual:

- there is at least a computation reaching a state s such that $s \models \phi$
- all states s reached along all computations maintain ϕ, ie, $s \models \neg\phi$

Safety and liveness

Qualifiers weak and strong refer to a quatification over computations

- weak liveness:

$$\mu X \,.\, \phi \vee \langle - \rangle X$$

(corresponds to Ctl formula $\mathtt{E}\ \mathtt{F}\ \phi$)

- strong safety

$$\nu X \,.\, \psi \wedge [-] X$$

(corresponds to Ctl formula $\mathtt{A}\ \mathtt{G}\ \psi$)

cf, liner time vs branching time

Duality

$$\neg(\mu X \,.\, \phi) = \nu X \,.\, \neg\phi$$
$$\neg(\nu X \,.\, \phi) = \mu X \,.\, \neg\phi$$

Example:

- divergence:

$$\nu X \,.\, \langle\tau\rangle X$$

- convergence (= all non observable behaviour is finite)

$$\neg(\nu X \,.\, \langle\tau\rangle X) \;=\; \mu X \,.\, \neg(\langle\tau\rangle X) \;=\; \mu X \,.\, [\tau] X$$

Safety and liveness

- weak safety:

$$\nu X \,.\, \phi \wedge (\langle - \rangle X \vee [-] \mathit{false})$$

(there is a computation along which ϕ holds)

- strong liveness

$$\mu X \,.\, \neg \phi \vee ([-] X \wedge \langle - \rangle \mathit{true})$$

(a state where the complement of ϕ holds can be finitely reached)

Conditional properties

ϕ_1 =
After collecting a passenger (*icr*), the taxi drops him at destination (*fcr*)
Second part of ϕ_1 is strong liveness:

$$\mu X \,.\, [-fcr]X \wedge \langle - \rangle true$$

holding only after *icr*.
Is it enough to write:

$$[icr](\mu X \,.\, [-fcr]X \wedge \langle - \rangle true)$$

?
what we want does not depend on the initial state: it is liveness embedded into strong safety:

$$\nu Y \,.\, [icr](\mu X \,.\, [-fcr]X \wedge \langle - \rangle true) \ \wedge \ [-]Y$$

Conditional properties

$\phi_1 \;=$

After collecting a passenger (icr), the taxi drops him at destination (fcr)

Second part of ϕ_1 is strong liveness:

$$\mu X \,.\, [-fcr]X \wedge \langle - \rangle true$$

holding only after icr.

Is it enough to write:

$$[icr](\mu X \,.\, [-fcr]X \wedge \langle - \rangle true)$$

?

what we want does not depend on the initial state: it is liveness embedded into strong safety:

$$\nu Y \,.\, [icr](\mu X \,.\, [-fcr]X \wedge \langle - \rangle true) \;\wedge\; [-]Y$$

Conditional properties

The previous example is conditional liveness but one can also have

- conditional safety:

$$\nu Y\,.\,(\neg\phi \vee (\phi \wedge \nu X\,.\,\psi \wedge [-]X)) \wedge [-]Y$$

(whenever ϕ holds, ψ cannot cease to hold)

Cyclic properties

ϕ = every second action is *out*
is expressed by

$$\nu X \,.\, [-]([-out]\mathit{false} \wedge [-]X)$$

ϕ = *out* follows *in*, but other actions can occur in between

$$\nu X \,.\, [out]\mathit{false} \wedge [in](\mu Y \,.\, [in]\mathit{false} \wedge [out]X \wedge [-out]Y) \wedge [-in]X$$

Note that the use of least fixed points imposes that the amount of computation between *in* and *out* is finite

Cyclic properties

ϕ = a state in which *in* can occur, can be reached an infinite number of times

$$\nu X \,.\, \mu Y \,.\, (\langle in \rangle true \vee \langle - \rangle Y) \;\wedge\; ([-]X \;\wedge\; \langle - \rangle true)$$

ϕ = *in* occurs an infinite number of times

$$\nu X \,.\, \mu Y \,.\, [-in]Y \wedge [-]X \wedge \langle - \rangle true$$

ϕ = *in* occurs an finite number of times

$$\mu X \,.\, \nu Y \,.\, [-in]Y \wedge [in]X$$