Bisimilarity

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Looking for suitable notions of equivalence of behaviours

Intuition

Two LTS should be equivalent if they cannot be distinguished by interacting with them.

Equality of functional behaviour

is not preserved by parallel composition: non compositional semantics, cf,

`x:=4; x := x+1` and `x:=5`

Graph isomorphism

is too strong (why?)

Trace

Definition

Let $T = \langle S, N, \longrightarrow \rangle$ be a labelled transition system. The set of traces $\mathrm{Tr}(s)$, for $s \in S$ is the minimal set satisfying

$$(1) \quad \epsilon \in \mathrm{Tr}(s)$$

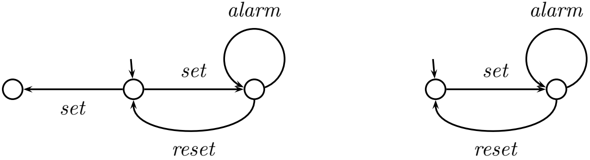
$$(3) \quad a\sigma \in \mathrm{Tr}(s) \Rightarrow \langle \exists\, s' \,:\, s' \in S : \, s \xrightarrow{a} s' \wedge \sigma \in \mathrm{Tr}(s') \rangle$$

Trace equivalence

Definition

Two states s, r are trace equivalent iff $\mathsf{Tr}(s) = \mathsf{Tr}(r)$
(i.e. they can perform the same finite sequences of transitions)

Example



Trace equivalence applies when one can neither interact with a system, nor distinguish a slow system from one that has come to a stand still.

Simulation

the quest for a behavioural equality:
able to identify states that cannot be distinguished by any realistic form of observation

Simulation

A state q is simulated another state p if every transition from q is corresponded by a transition from p and this capacity is kept along the whole life of the system to which state space q belongs to.

Simulation

Definition

Given $\langle S_1, N, \longrightarrow_1 \rangle$ and $\langle S_2, N, \longrightarrow_2 \rangle$ over N, relation $R \subseteq S_1 \times S_2$ is a simulation iff, for all $\langle p, q \rangle \in R$ and $a \in N$,

$$(2) \quad p \xrightarrow{a}_1 p' \;\Rightarrow\; \langle \exists\, q' \;:\; q' \in S_2 : \; q \xrightarrow{a}_2 q' \wedge \langle p', q' \rangle \in R \rangle$$

$$\begin{tikzcd}
p \;\; R \;\; q \arrow[d, "a"] & \Rightarrow & q \arrow[d, "a"] \\
p' & & p' \;\; R \;\; q'
\end{tikzcd}$$

Example

$$
\begin{tikzcd}
 & q_1 \arrow[r, "d"] & q_2 \\
q_0 \arrow[ur, "a"] \arrow[dr, "a"'] & & \\
 & q_4 \arrow[r, "e"'] & q_3
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & & p_2 \\
p_0 \arrow[r, "a"] & p_1 \arrow[ur, "d"] \arrow[dr, "e"'] & \\
 & & p_3
\end{tikzcd}
$$

$$q_0 \lesssim p_0 \qquad \text{cf.} \qquad \{\langle q_0, p_0\rangle, \langle q_1, p_1\rangle, \langle q_4, p_1\rangle, \langle q_2, p_2\rangle, \langle q_3, p_3\rangle\}$$

Similarity

Definition

$$p \lesssim q \;\equiv\; \langle \exists\, R \;::\; R \text{ is a simulation and } \langle p, q \rangle \in R \rangle$$

Lemma

The similarity relation is a preorder
(i.e. reflexive and transitive)

Bisimulation

Definition

Given $\langle S_1, N, \longrightarrow_1\rangle$ and $\langle S_2, N, \longrightarrow_2\rangle$ over N, relation $R \subseteq S_1 \times S_2$ is a bisimulation iff both R and its converse R° are simulations.
I.e. whenever $\langle p, q\rangle \in R$ and $a \in N$,

$$(1)\quad p \xrightarrow{a}_1 p' \ \Rightarrow\ \langle \exists\, q' \ :\ q' \in S_2 :\ q \xrightarrow{a}_2 q' \wedge \langle p', q'\rangle \in R\rangle$$

$$(2)\quad q \xrightarrow{a}_2 q' \ \Rightarrow\ \langle \exists\, p' \ :\ p' \in S_1 :\ p \xrightarrow{a}_1 p' \wedge \langle p', q'\rangle \in R\rangle$$

Bisimulation

The Game characterization

Two players R and I discuss whether the transition structures are mutually corresponding

- R starts by chosing a transition
- I replies trying to match it
- if I succeeds, R plays again
- R wins if I fails to find a corresponding match
- I wins if it replies to all moves from R and the game is in a configuration where all states have been visited or R can't move further. In this case is said that I has a wining strategy

Examples

$$\begin{tikzcd}
 & q_1 \arrow[dl, "a"'] \arrow[dr, "a"] & \\
q_2 \arrow[rr, "c"] & & q_3 \arrow[loop right, "c"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
m \arrow[d, "a"'] \\
n \arrow[loop right, "c"]
\end{tikzcd}$$

$$q_1 \xrightarrow{a} q_2 \xrightarrow{a} q_3 \xrightarrow{a} \cdots \qquad\qquad \begin{tikzcd} h \arrow[loop right, "a"] \end{tikzcd}$$

Examples

$$
\begin{tikzcd}
& q_1 \arrow[dl, "a"'] \arrow[dr, "a"] & \\
q_2 \arrow[d, "c"] & & q_3 \arrow[d, "c"] \\
q_4 & & q_5
\end{tikzcd}
\qquad
\begin{tikzcd}
& p_1 \arrow[d, "a"] & \\
& p_2 \arrow[dl, "c"'] \arrow[dr, "c"] & \\
p_4 & & p_5
\end{tikzcd}
$$

$$
\begin{tikzcd}
& q_1 \arrow[dl, "a"'] \arrow[dr, "a"] & \\
q_2 \arrow[d, "c"] & & q_3 \arrow[d, "\color{red}b"] \\
q_4 & & q_5
\end{tikzcd}
\qquad
\begin{tikzcd}
& p_1 \arrow[d, "a"] & \\
& p_2 \arrow[dl, "c"'] \arrow[dr, "\color{red}b"] & \\
p_4 & & p_5
\end{tikzcd}
$$

After thoughts

- Follows a $\forall, \exists$ pattern: p in all its transitions challenge q which is called to find a matchh to each of those (and conversely)
- Tighter correspondence with transitions
- Based on the information that the transitions convey, rather than on the shape of the LTS
- Local checks on states
- Lack of hierarchy on the pairs of the bisimulation (no temporal order on the checks is required)

which means bisimilarity can be used to reason about infinite or circular behaviours.

After thoughts

Compare the definition of bisimilarity with

$p == q$ if, for all $a \in N$

$$(1)\quad p \xrightarrow{a}_1 p' \;\Rightarrow\; \langle \exists\, q' \;:\; q' \in S_2 : \; q \xrightarrow{a}_2 q' \wedge p' == q' \rangle$$

$$(2)\quad q \xrightarrow{a}_2 q' \;\Rightarrow\; \langle \exists\, p' \;:\; p' \in S_1 : \; p \xrightarrow{a}_1 p' \wedge p' == q' \rangle$$

After thoughts

$p == q$ if, for all $a \in N$

$$(1) \quad p \downarrow_1 \; \Leftrightarrow q \downarrow_2$$

$$(2.1) \quad p \xrightarrow{a}_1 p' \; \Rightarrow \; \langle \exists \, q' \; : \; q' \in S_2 : \; q \xrightarrow{a}_2 q' \wedge p' == q' \rangle$$

$$(2.1) \quad q \xrightarrow{a}_2 q' \; \Rightarrow \; \langle \exists \, p' \; : \; p' \in S_1 : \; p \xrightarrow{a}_1 p' \wedge p' == q' \rangle$$

- The meaning of $==$ on the pair $\langle p, q \rangle$ requires having already established the meaning of $==$ on the derivatives
- ... therefore the definition is ill-founded if the state space reachable from $\langle p, q \rangle$ is infinite or contain loops
- ... this is a local but inherently inductive definition (to revisit later)

After thoughts

Proof method

To prove that two behaviours are bisimilar, find a bisimulation containing them ...

- ... impredicative character
- coinductive vs inductive definition

Properties

Definition

$$p \sim q \;\equiv\; \langle \exists\, R \;::\; R \text{ is a bisimulation and } \langle p, q \rangle \in R \rangle$$

Lemma

1. The identity relation *id* is a bisimulation
2. The empty relation $\perp$ is a bisimulation
3. The converse R° of a bisimulation is a bisimulation
4. The composition $S \cdot R$ of two bisimulations S and R is a bisimulation
5. The $\bigcup_{i \in I} R_i$ of a family of bisimulations $\{R_i \mid i \in I\}$ is a bisimulation

Properties

Lemma

The bisimilarity relation is an equivalence relation
(i.e. reflexive, symmetric and transitive)

Lemma

The class of all bisimulations between two LTS has the structure of a complete lattice, ordered by set inclusion, whose top is the bisimilarity relation $\sim$.

Properties

Lemma

In a deterministic labelled transition system, two states are bisimilar iff they are trace equivalent, i.e.,

$$s \sim s' \iff \mathrm{Tr}(s) = \mathrm{Tr}(s')$$

Hint: define a relation R as

$$\langle x, y \rangle \in R \iff \mathrm{Tr}(x) = \mathrm{Tr}(y)$$

and show R is a bisimulation.

Properties

Warning

$\boxed{\text{The bisimilarity relation } \sim \text{ is not the symmetric closure of } \lesssim}$

Example

$$q_0 \lesssim p_0, \;\; p_0 \lesssim q_0 \quad \text{but} \quad p_0 \not\sim q_0$$

$$\begin{tikzcd}
 & q_1 & \\
q_0 \arrow[ur, "a"] \arrow[dr, "a"'] & & \\
 & q_2 \arrow[r, "b"] & q_3
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
p_0 \arrow[r, "a"] & p_1 \arrow[r, "b"] & p_3
\end{tikzcd}$$

Notes

Similarity as the greatest simulation

$$\lesssim \;\;\widehat{=}\;\; \bigcup\{S \mid S \text{ is a simulation}\}$$

Bisimilarity as the greatest bisimulation

$$\sim \;\;\widehat{=}\;\; \bigcup\{S \mid S \text{ is a bisimulation}\}$$