Modal logic for concurrent processes

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Motivation

System's correctness wrt a specification

- equivalence checking (between two designs), through $\sim$ and $=$
- unsuitable to check properties such as

  can the system perform action α followed by β?

  which are best answered by exploring the process state space

Which logic?

- Modal logic over transition systems
- The Hennessy-Milner logic (offered in mCRL2)
- The modal μ-calculus (offered in mCRL2)

The language

Syntax

$$\phi \ ::= \ p \mid \mathit{true} \mid \mathit{false} \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \rightarrow \phi_2 \mid \langle m \rangle \phi \mid [m]\phi$$

where $p \in \text{PROP}$ and $m \in \text{MOD}$

Disjunction ($\vee$) and equivalence ($\leftrightarrow$) are defined by abbreviation. The signature of the basic modal language is determined by sets PROP of propositional symbols (typically assumed to be denumerably infinite) and MOD of modality symbols.

The language

Notes

- if there is only one modality in the signature (i.e., MOD is a singleton), write simply $\Diamond\phi$ and $\Box\phi$
- the language has some redundancy: in particular modal connectives are dual (as quantifiers are in first-order logic): $[m]\phi$ is equivalent to $\neg\langle m\rangle\neg\phi$
- define modal depth in a formula ϕ, denoted by $\mathrm{md}\,\phi$ as the maximum level of nesting of modalities in ϕ

The language

Semantics

A model for the language is a pair $\mathfrak{M} = \langle \mathbb{F}, V \rangle$, where

- $\mathfrak{F} = \langle W, \{R_m\}_{m \in \mathrm{MOD}} \rangle$
  is a Kripke frame, ie, a non empty set W and a family of binary relations over W, one for each modality symbol $m \in \mathrm{MOD}$.
  Elements of W are called points, states, worlds or simply vertices in the directed graphs corresponding to the modality symbols.
- $V : \mathrm{PROP} \longrightarrow \mathcal{P}(W)$ is a valuation.

The language

Satisfaction: for a model $\mathfrak{M}$ and a point w

$\mathfrak{M}, w \models true$		
$\mathfrak{M}, w \not\models false$		
$\mathfrak{M}, w \models p$	iff	$w \in V(p)$
$\mathfrak{M}, w \models \neg\phi$	iff	$\mathfrak{M}, w \not\models \phi$
$\mathfrak{M}, w \models \phi_1 \wedge \phi_2$	iff	$\mathfrak{M}, w \models \phi_1$ and $\mathfrak{M}, w \models \phi_2$
$\mathfrak{M}, w \models \phi_1 \rightarrow \phi_2$	iff	$\mathfrak{M}, w \not\models \phi_1$ or $\mathfrak{M}, w \models \phi_2$
$\mathfrak{M}, w \models \langle m \rangle\phi$	iff	there exists $v \in W$ st $wR_m v$ and $\mathfrak{M}, v \models \phi$
$\mathfrak{M}, w \models [m]\phi$	iff	for all $v \in W$ st $wR_m v$ and $\mathfrak{M}, v \models \phi$

The language

Safistaction

A formula ϕ is

- satisfiable in a model $\mathfrak{M}$ if it is satisfied at some point of $\mathfrak{M}$
- globally satisfied in $\mathfrak{M}$ ($\mathfrak{M} \models \phi$) if it is satisfied at all points in $\mathfrak{M}$
- valid ($\models \phi$) if it is globally satisfied in all models
- a semantic consequence of a set of formulas Γ ($\Gamma \models \phi$) if for all models $\mathfrak{M}$ and all points w, if $\mathfrak{M}, w \models \Gamma$ then $\mathfrak{M}, w \models \phi$

Examples

Temporal logic

- W is a set of instants
- there is a unique modality corresponding to the transitive closure of the next-time relation
- origin: Arthur Prior, an attempt to *deal with temporal information from the inside, capturing the situated nature of our experience and the context-dependent way we talk about it*

Examples

Process logic (Hennessy-Milner logic)

- $\text{PROP} = \emptyset$
- $W = \mathbb{P}$ is a set of states, typically process terms, in a labelled transition system
- each subset $K \subseteq Act$ of actions generates a modality corresponding to transitions labelled by an element of K

Assuming the underlying LTS $\mathfrak{F} = \langle \mathbb{P}, \{p \xrightarrow{K} p' \mid K \subseteq Act\} \rangle$ as the modal frame, satisfaction is abbreviated as

$$p \models \langle K \rangle \phi \quad \text{iff} \quad \exists_{q \in \{p' \mid p \xrightarrow{a} p' \,\wedge\, a \in K\}} \,.\, q \models \phi$$

$$p \models [K] \phi \quad \text{iff} \quad \forall_{q \in \{p' \mid p \xrightarrow{a} p' \,\wedge\, a \in K\}} \,.\, q \models \phi$$

Examples

Process logic: The taxi network example

- $\phi_0 =$ *In a taxi network, a car can collect a passenger or be allocated by the Central to a pending service*
- $\phi_1 =$ *This applies only to cars already on service*
- $\phi_2 =$ *If a car is allocated to a service, it must first collect the passenger and then plan the route*
- $\phi_3 =$ *On detecting an emergence the taxi becomes inactive*
- $\phi_4 =$ *A car on service is not inactive*

Examples

Process logic: The taxi network example

- $\phi_0 = \langle rec, alo \rangle true$
- $\phi_1 = [onservice]\langle rec, alo \rangle true$ or
  $\phi_1 = [onservice]\phi_0$
- $\phi_2 = [alo]\langle rec \rangle \langle plan \rangle true$
- $\phi_3 = [sos][-]false$
- $\phi_4 = [onservice]\langle - \rangle true$

Process logic: typical properties

- inevitability of a: $\langle - \rangle \mathit{true} \wedge [-a]\mathit{false}$
- progress: $\langle - \rangle \mathit{true}$
- deadlock or termination: $[-]\mathit{false}$
- what about

$$\langle - \rangle \mathit{false} \quad \text{and} \quad [-]\mathit{true} \quad ?$$

- satisfaction decided by unfolding the definition of $\models$: no need to compute the transition graph

Hennessy-Milner logic

... propositional logic with action modalities

Syntax

$$\phi \ ::= \ true \mid false \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \langle K \rangle \phi \mid [K]\phi$$

Semantics: $E \models \phi$

$$
\begin{array}{lll}
E \models true & & \\
E \not\models false & & \\
E \models \phi_1 \wedge \phi_2 & \text{iff} & E \models \phi_1 \ \wedge \ E \models \phi_2 \\
E \models \phi_1 \vee \phi_2 & \text{iff} & E \models \phi_1 \ \vee \ E \models \phi_2 \\
E \models \langle K \rangle \phi & \text{iff} & \exists_{F \in \{E' \mid E \xrightarrow{a} E' \wedge a \in K\}} \,.\, F \models \phi \\
E \models [K]\phi & \text{iff} & \forall_{F \in \{E' \mid E \xrightarrow{a} E' \wedge a \in K\}} \,.\, F \models \phi
\end{array}
$$

Example

$$
\begin{aligned}
Sem &\mathrel{\widehat{=}} get.put.Sem \\
P_i &\mathrel{\widehat{=}} \overline{get}.c_i.\overline{put}.P_i \\
S &\mathrel{\widehat{=}} (Sem \mid (|_{i \in I}\ P_i)) \backslash_{\{get,put\}}
\end{aligned}
$$

- $Sem \models \langle get \rangle true$ holds because

$$\exists_{F \in \{Sem' \mid Sem \xrightarrow{get} Sem'\}} \, . \, F \models true$$

  with $F = put.Sem$.

- However, $Sem \models [put] false$ also holds, because $T = \{Sem' \mid Sem \xrightarrow{put} Sem'\} = \emptyset$.
  Hence $\forall_{F \in T} \, . \, F \models false$ becomes trivially true.

- The only action initially permmited to S is τ: $\models [-\tau] false$.

Example

$$Sem \mathrel{\widehat{=}} get.put.Sem$$
$$P_i \mathrel{\widehat{=}} \overline{get}.c_i.\overline{put}.P_i$$
$$S \mathrel{\widehat{=}} (Sem \mid (|_{i \in I}\ P_i)) \backslash_{\{get, put\}}$$

- Afterwards, S can engage in any of the critical events $c_1, c_2, ..., c_i$: $[\tau]\langle c_1, c_2, ..., c_i\rangle true$
- After the semaphore initial synchronization and the occurrence of c_j in P_j, a new synchronization becomes inevitable: $S \models [\tau][c_j](\langle - \rangle true \wedge [-\tau] false)$

Exercise

Verify:

$$
\begin{aligned}
\neg\langle a\rangle\phi &= [a]\neg\phi \\
\neg[a]\phi &= \langle a\rangle\neg\phi \\
\langle a\rangle \mathit{false} &= \mathit{false} \\
[a]\mathit{true} &= \mathit{true} \\
\langle a\rangle(\phi \vee \psi) &= \langle a\rangle\phi \vee \langle a\rangle\psi \\
[a](\phi \wedge \psi) &= [a]\phi \wedge [a]\psi \\
\langle a\rangle\phi \wedge [a]\psi &\Rightarrow \langle a\rangle(\phi \wedge \psi)
\end{aligned}
$$

A denotational semantics

Idea: associate to each formula ϕ the set of processes that makes it true

ϕ vs $|\phi| = \{E \in \mathbb{P} \mid E \models \phi\}$

$$|true| = \mathbb{P}$$
$$|false| = \emptyset$$
$$|\phi_1 \wedge \phi_2| = |\phi_1| \cap |\phi_2|$$
$$|\phi_1 \vee \phi_2| = |\phi_1| \cup |\phi_2|$$

$$|[K]\phi| = |[K]|(|\phi|)$$
$$|\langle K\rangle\phi| = |\langle K\rangle|(|\phi|)$$

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$$

$$
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|[K]\phi| &= |[K]|(|\phi|) \\
|\langle K\rangle\phi| &= |\langle K\rangle|(|\phi|)
\end{aligned}
$$

$\|[K]\|$ and $\|\langle K\rangle\|$

Just as $\wedge$ corresponds to $\cap$ and $\vee$ to $\cup$, modal logic combinators correspond to unary functions on sets of processes:

$$\|[K]\|(X) = \{F \in \mathbb{P} \mid \text{if } F \xrightarrow{a} F' \wedge a \in K \text{ then } F' \in X\}$$

$$\|\langle K\rangle\|(X) = \{F \in \mathbb{P} \mid \exists_{F' \in X, a \in K} \,.\, F \xrightarrow{a} F'\}$$

Note

These combinators perform a reduction to the previous state indexed by actions in K

$\|[K]\|$ and $\|\langle K\rangle\|$

Example

$$
\begin{tikzcd}
 & q_1 \arrow[dl, "a"'] \arrow[dr, "a"] & \\
q_2 \arrow[rr, "c"] & & q_3 \arrow[loop right, "c"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
m \arrow[d, "a"'] \\
n \arrow[loop right, "c"]
\end{tikzcd}
$$

$$
\begin{aligned}
|\langle a\rangle|\{q_2, n\} &= \{q_1, m\} \\
|[a]|\{q_2, n\} &= \{q_2, q_3, m, n\}
\end{aligned}
$$

A denotational semantics

$$\boxed{E \models \phi \ \text{ iff } \ E \in |\phi|}$$

Example: $0 \models [-]\mathit{false}$

because

$$\begin{aligned} |[-]\mathit{false}| &= \|[-]\|(|\mathit{false}|) \\ &= \|[-]\|(\emptyset) \\ &= \{F \in \mathbb{P} \mid \text{if } F \xrightarrow{x} F' \wedge x \in Act \ \text{ then } \ F' \in \emptyset\} \\ &= \{0\} \end{aligned}$$

A denotational semantics

$$\boxed{E \models \phi \text{ iff } E \in |\phi|}$$

Example: ?? $\models \langle - \rangle true$

because

$$\begin{aligned} |\langle - \rangle true| &= |\langle - \rangle|(|true|) \\ &= |\langle - \rangle|(\mathbb{P}) \\ &= \{F \in \mathbb{P} \mid \exists_{F' \in \mathbb{P}, a \in K} \,.\, F \xrightarrow{a} F'\} \\ &= \mathbb{P} \setminus \{0\} \end{aligned}$$

A denotational semantics

Complement

Any property ϕ divides $\mathbb{P}$ into two disjoint sets:

$$|\phi| \text{ and } \mathbb{P} - |\phi|$$

The characteristic formula of the complement of $|\phi|$ is ϕ^c:

$$|\phi^c| = \mathbb{P} - |\phi|$$

where ϕ^c is defined inductively on the formulae structure:

$$\mathit{true}^c = \mathit{false} \qquad \mathit{false}^c = \mathit{true}$$
$$(\phi_1 \wedge \phi_2)^c = \phi_1^c \vee \phi_2^c$$
$$(\phi_1 \vee \phi_2)^c = \phi_1^c \wedge \phi_2^c$$
$$(\langle a \rangle \phi)^c = [a]\phi^c$$

... but negation is not explicitly introduced in the logic.

Modal Equivalence

For each (finite or infinite) set Γ of formulae,

$$E \equiv_\Gamma F \quad \Leftrightarrow \quad \forall_{\phi \in \Gamma} \,.\, E \models \phi \Leftrightarrow F \models \phi$$

Examples

$$a.b.0 + a.c.0 \;\; \equiv_\Gamma \;\; a.(b.0 + c.0)$$

for $\Gamma \; = \; \{\langle x_1 \rangle \langle x_2 \rangle ... \langle x_n \rangle true \mid x_i \in Act\}$

(what about $\equiv_\Gamma$ for $\Gamma = \{\langle x_1 \rangle \langle x_2 \rangle \langle x_3 \rangle ... \langle x_n \rangle [-] false \mid x_i \in Act\}$?)

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Modal Equivalence

For each (finite or infinite) set Γ of formulae,

$$E \equiv F \quad \Leftrightarrow \quad E \equiv_\Gamma F \text{for every set } \Gamma \text{ of well-formed formulae}$$

Lemma

$$E \sim F \quad \Rightarrow \quad E \equiv F$$

Note

the converse of this lemma does not hold, e.g. let

- $A \widehat{=} \sum_{i \geq 0} A_i$, where $A_0 \widehat{=} 0$ and $A_{i+1} \widehat{=} a.A_i$
- $A' \widehat{=} A + \underline{fix}\ (X = a.X)$

$$\neg(A \sim A') \quad \text{but} \quad A \equiv A'$$

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Modal Equivalence

Theorem [Hennessy-Milner, 1985]

$$E \sim F \quad \Leftrightarrow \quad E \equiv F$$

for image-finite processes.

Image-finite processes

E is image-finite iff $\{F \mid E \xrightarrow{a} F\}$ is finite for every action $a \in Act$

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Modal Equivalence

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for image-finite processes.

proof

$\Rightarrow$: by induction of the formula structure

$\Leftarrow$: show that $\equiv$ is itself a bisimulation, by contradiction