Contexts and tensorial strength

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Architecture and Calculi Course Unit

Sharing contexts

It is very useful to have two programs M, N in sequential composition $x \leftarrow M \,;\, N$ that are able share contexts

In other words, it is 1 useful to have the following rule for sequential composition

$$\frac{\Gamma \vdash_{\mathsf{c}} M : \mathbb{A} \qquad \Gamma, x : \mathbb{A} \vdash_{\mathsf{c}} N : \mathbb{B}}{\Gamma \vdash_{\mathsf{c}} x \leftarrow M \,;\, N : \mathbb{B}}$$

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This would allow us to solve the previous exercise quite easily

$$f : \mathbb{A} \to \mathbb{A}, x : \mathbb{A} \vdash_{\mathsf{c}} y \leftarrow f(x) \,;\, f(y) : \mathbb{A}$$

Sharing contexts

The natural way of interpreting the rule would be

$$\frac{[\![\Gamma \vdash_{\mathsf{c}} M : \mathbb{A}]\!] = f \qquad [\![\Gamma, x : \mathbb{A} \vdash_{\mathsf{c}} N : \mathbb{B}]\!] = g}{[\![\Gamma \vdash_{\mathsf{c}} x \leftarrow M\,;\, N : \mathbb{B}]\!] = g^{\star} \cdot \langle \mathrm{id}, f \rangle}$$

but $\langle \mathrm{id}, f \rangle : [\![\Gamma]\!] \longrightarrow [\![\Gamma]\!] \times T[\![\mathbb{A}]\!]$ and $g^{\star} : T([\![\Gamma]\!] \times [\![\mathbb{A}]\!]) \longrightarrow T[\![\mathbb{B}]\!]$

Thus we need to find a suitable function

$$\mathrm{str} : [\![\Gamma]\!] \times T[\![\mathbb{A}]\!] \longrightarrow T([\![\Gamma]\!] \times [\![\mathbb{A}]\!])$$

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There is a natural way of doing this ...

Tensorial strength

For every monad T and function $f : X \to Y$ we can build a function

$$Tf = (\eta \cdot f)^\star : TX \to TY$$

Note also that for every $x \in X$ we can define

$$\mathsf{id}_x : Y \to X \times Y, \quad y \mapsto (x, y)$$

From these we define the so-called strength of T

$$\mathrm{str} : X \times TY \to T(X \times Y), \quad (x, t) \mapsto (T\mathsf{id}_x)(t)$$

Finally,

$$\frac{[\![\Gamma \vdash_{\mathsf{c}} M : \mathbb{A}]\!] = f \qquad [\![\Gamma, x : \mathbb{A} \vdash_{\mathsf{c}} N : \mathbb{B}]\!] = g}{[\![\Gamma \vdash_{\mathsf{c}} x \leftarrow M \,;\, N : \mathbb{B}]\!] = g^\star \cdot \mathrm{str} \cdot \langle \mathsf{id}, f \rangle}$$

Exercises

Given an explicit definition for the tensorial strength of

- the monad of exceptions,
- the monad of durations

Consider the λ-term,

$$f : \mathbb{A} \to \mathbb{A}, x : \mathbb{A} \vdash_{\mathsf{c}} y \leftarrow f(x) \,;\, f(y) : \mathbb{A}$$

What is its execution time when,

$$f \text{ is given by } (v \mapsto (1, v))$$

Semantics of effectul λ-calculus with shared contexts

$$\frac{x_i : \mathbb{A} \in \Gamma}{[\![\Gamma \vdash x_i]\!] = \pi_i} \qquad \frac{}{[\![\Gamma \vdash *]\!] = !} \qquad \frac{[\![\Gamma \vdash V : \mathbb{A}]\!] = f \qquad [\![\Gamma \vdash U : \mathbb{B}]\!] = g}{[\![\Gamma \vdash \langle V, U \rangle : \mathbb{A} \times \mathbb{B}]\!] = \langle f, g \rangle}$$

$$\frac{[\![\Gamma, x : \mathbb{A} \vdash_{\mathsf{c}} M : \mathbb{B}]\!] = f}{[\![\Gamma \vdash \lambda x : \mathbb{A}.\, M : \mathbb{A} \to \mathbb{B}]\!] = \lambda f} \qquad \frac{[\![\Gamma \vdash V : \mathbb{A} \times \mathbb{B}]\!] = f}{[\![\Gamma \vdash \pi_1 V : \mathbb{A}]\!] = \pi_1 \cdot f}$$

. . . .

$$\frac{[\![\Gamma \vdash V : \mathbb{A}]\!] = f}{[\![\Gamma \vdash_{\mathsf{c}} \mathtt{return}\ V : \mathbb{A}]\!] = \eta \cdot f} \qquad \frac{[\![\Gamma \vdash_{\mathsf{c}} M : \mathbb{A}]\!] = f \qquad [\![x : \mathbb{A} \vdash_{\mathsf{c}} N : \mathbb{B}]\!] = g}{[\![\Gamma \vdash_{\mathsf{c}} x \leftarrow M\, ;\, N : \mathbb{B}]\!] = g^\star \cdot \mathrm{str} \cdot \langle \mathsf{id}, f \rangle}$$

$$\frac{[\![\Gamma \vdash V : \mathbb{A} \to \mathbb{B}]\!] = f \quad [\![\Gamma \vdash U : \mathbb{A}]\!] = g}{[\![\Gamma \vdash_{\mathsf{c}} V\ U : \mathbb{B}]\!] = \mathrm{app} \cdot \langle f, g \rangle}$$

$$\frac{(\sigma, n) \in \Sigma \qquad \forall i \leq n.\ [\![\Gamma \vdash_{\mathsf{c}} M_i : \mathbb{A}]\!] = f_i}{[\![\Gamma \vdash_{\mathsf{c}} \sigma(M_1, \ldots M_n)]\!] = [\![\sigma]\!]_{[\![\mathbb{A}]\!]} \cdot \langle f_1, \ldots, f_n \rangle}$$