

Bisimilarity for non-deterministic systems

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When are two reactive systems the same?

Intuition

Two LTS should be equivalent if they cannot be distinguished by interacting with them.

Equality of functional behaviour

is not preserved by **parallel** composition: non **compositional** semantics, cf,

$$x:=4; x := x+1 \text{ and } x:=5$$

Graph isomorphism

is too strong (why?)

Trace

Definition

Let $T = (S, N, \longrightarrow)$ be a labelled transition system. The set of **traces** $\text{Tr}(s)$, for $s \in S$ is the minimal set satisfying

$$(1) \quad \epsilon \in \text{Tr}(s)$$

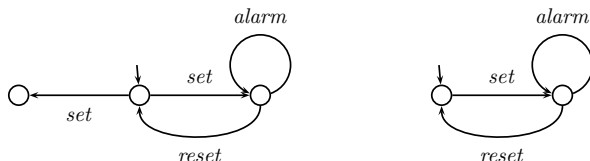
$$(3) \quad a\sigma \in \text{Tr}(s) \Rightarrow \langle \exists s' : s' \in S : s \xrightarrow{a} s' \wedge \sigma \in \text{Tr}(s') \rangle$$

Trace equivalence

Definition

Two states s, r are **trace equivalent** iff $\text{Tr}(s) = \text{Tr}(r)$
 (i.e. they can perform the same finite sequences of transitions)

Example



Trace equivalence applies when one can neither interact with a system, nor distinguish a slow system from one that has come to a stand still.

Simulation

the quest for a **behavioural equality**:
able to identify states that cannot be distinguished by any **realistic**
form of observation

Simulation

A state q **simulates** another state p if every transition from q is corresponded by a transition from p and this capacity is kept along the whole life of the system to which state space q belongs to.

Simulation

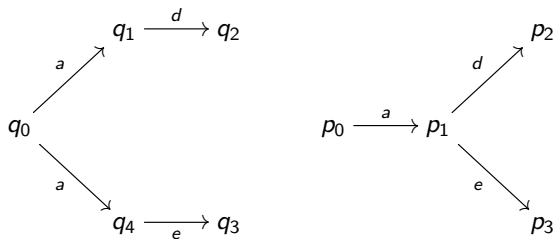
Definition

Given $(S_1, N, \longrightarrow_1)$ and $(S_2, N, \longrightarrow_2)$ over N , relation $R \subseteq S_1 \times S_2$ is a **simulation** iff, for all $(p, q) \in R$ and $a \in N$,

$$(2) \quad p \xrightarrow{a}_1 p' \Rightarrow \langle \exists q' : q' \in S_2 : q \xrightarrow{a}_2 q' \wedge (p', q') \in R \rangle$$

$$\begin{array}{ccc}
 p & R & q \\
 \downarrow a & & \\
 p' & &
 \end{array}
 \quad \Rightarrow \quad
 \begin{array}{ccc}
 & & q \\
 & & \downarrow a \\
 p' & R & q'
 \end{array}$$

Example



$$q_0 \lesssim p_0 \quad \text{cf.} \quad \{(q_0, p_0), (q_1, p_1), (q_4, p_1), (q_2, p_2), (q_3, p_3)\}$$

Similarity

Definition

$$p \lesssim q \equiv \langle \exists R :: R \text{ is a simulation and } (p, q) \in R \rangle$$

Lemma

The similarity relation is a preorder
(i.e. reflexive and transitive)

Bisimulation

Definition

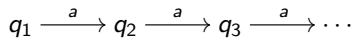
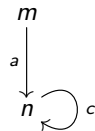
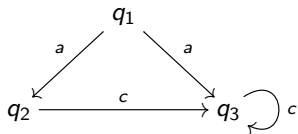
Given $(S_1, N, \longrightarrow_1)$ and $(S_2, N, \longrightarrow_2)$ over N , relation $R \subseteq S_1 \times S_2$ is a **bisimulation** iff both R and its converse R° are simulations.

I.e. whenever $(p, q) \in R$ and $a \in N$,

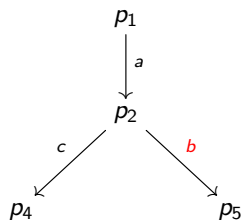
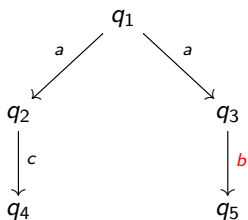
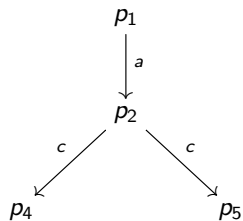
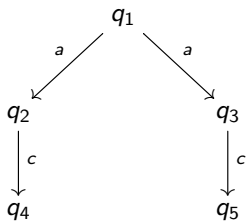
$$(1) \quad p \xrightarrow{a}_1 p' \Rightarrow \langle \exists q' : q' \in S_2 : q \xrightarrow{a}_2 q' \wedge (p', q') \in R \rangle$$

$$(2) \quad q \xrightarrow{a}_2 q' \Rightarrow \langle \exists p' : p' \in S_1 : p \xrightarrow{a}_1 p' \wedge (p', q') \in R \rangle$$

Examples



Examples



After thoughts

- Follows a \forall, \exists pattern: p in all its transitions challenge q which is called to find a match to each of those (and conversely)
- Tighter correspondence with transitions
- Based on the information that the transitions convey, rather than on the shape of the LTS
- Local checks on states
- Lack of hierarchy on the pairs of the bisimulation (no temporal order on the checks is required)

which means bisimilarity can be used to reason about infinite or circular behaviours.

After thoughts

Compare the definition of bisimilarity with

$p == q$ if, for all $a \in N$

$$(1) \quad p \xrightarrow{a}_1 p' \Rightarrow \langle \exists q' : q' \in S_2 : q \xrightarrow{a}_2 q' \wedge p' == q' \rangle$$

$$(2) \quad q \xrightarrow{a}_2 q' \Rightarrow \langle \exists p' : p' \in S_1 : p \xrightarrow{a}_1 p' \wedge p' == q' \rangle$$

- The meaning of $==$ on the pair (p, q) requires having already established the meaning of $==$ on the derivatives
- ... therefore the definition is **ill-founded** if the state space reachable from (p, q) is infinite or contain loops
- ... this is a **local** but **inherently inductive** definition (to revisit later)

After thoughts

Proof method

To prove that two behaviours are bisimilar, find a bisimulation containing them ...

- ... **impredicative** character
- **coinductive** vs **inductive** definition

Properties

Definition

$$p \sim q \equiv \langle \exists R :: R \text{ is a bisimulation and } (p, q) \in R \rangle$$

Lemma

1. The identity relation id is a bisimulation
2. The empty relation \perp is a bisimulation
3. The converse R° of a bisimulation is a bisimulation
4. The composition $S \cdot R$ of two bisimulations S and R is a bisimulation
5. The $\bigcup_{i \in I} R_i$ of a family of bisimulations $\{R_i \mid i \in I\}$ is a bisimulation

Properties

Lemma

The bisimilarity relation is an equivalence relation (i.e. reflexive, symmetric and transitive)

Lemma

The class of all bisimulations between two LTS has the structure of a **complete lattice**, ordered by set inclusion, whose top is the **bisimilarity** relation \sim .

Properties

Lemma

In a **deterministic** labelled transition system, two states are bisimilar iff they are trace equivalent, i.e.,

$$s \sim s' \Leftrightarrow \text{Tr}(s) = \text{Tr}(s')$$

Hint: define a relation R as

$$(x, y) \in R \Leftrightarrow \text{Tr}(x) = \text{Tr}(y)$$

and show R is a bisimulation.

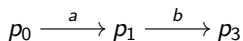
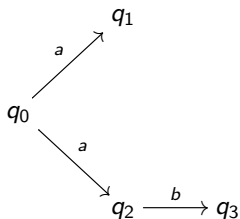
Properties

Warning

The bisimilarity relation \sim is not the symmetric closure of \lesssim

Example

$$q_0 \lesssim p_0, p_0 \lesssim q_0 \quad \text{but} \quad p_0 \not\sim q_0$$



Notes

Similarity as the **greatest simulation**

$$\lesssim \hat{=} \bigcup \{S \mid S \text{ is a simulation}\}$$

Bisimilarity as the **greatest bisimulation**

$$\sim \hat{=} \bigcup \{S \mid S \text{ is a bisimulation}\}$$