Small-step Semantics

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Table of Contents

Semantics for every season

Operational semantics	How a program operates
Denotational semantics	What a program is
Axiomatic semantics	Which logical properties a program satisfies

Small-step operational semantics

How a program operates step-by-step

Example

$\langle \mathtt{x} := 1; \mathtt{x} := \mathtt{x} + 1, v \rangle \longrightarrow \langle \mathtt{x} := \mathtt{x} + 1, 1 \rangle \longrightarrow 2$

A machine with an 'evaluation stack' that is processed at each step

Its uses

Describes how a program operates step-by-step

Describes evaluation techniques (*e.g.* short-circuiting)

A basis for tracing/debugging

Foundations of concurrency, complexity, ...

Rich notions of equivalence

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Thus an essential tool for understanding a programming language

Table of Contents

A propositional language

$$\mathtt{b} ::= \mathtt{x} \mid \mathtt{b} \wedge \mathtt{b} \mid \neg \mathtt{b}$$

Every $\mathtt{x}$ is a proposition (*i.e.* it has either value $\mathtt{tt}$ or $\mathtt{ff}$)

A propositional language

$$\mathtt{b} ::= \mathtt{x} \mid \mathtt{b} \wedge \mathtt{b} \mid \neg \mathtt{b}$$

Every $\mathtt{x}$ is a proposition (*i.e.* it has either value $\mathtt{tt}$ or $\mathtt{ff}$)

Can we provide a small-step semantics to this language ?

Key points

Uses a memory $\sigma : X \to \mathrm{Bool}$ that assigns to every proposition $\mathtt{x}$ its truth-value $\sigma(\mathtt{x})$

A term $\mathtt{b}$ is evaluated step-by-step until a truth-value v is reached

$$\bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow v$$

Focus is on the next step (of the evaluation)

The semantics – a universe of laws

$$\frac{}{\langle \mathtt{x}, \sigma \rangle \longrightarrow \sigma(\mathtt{x})}\ (\mathsf{var}) \qquad \frac{\langle \mathtt{b}, \sigma \rangle \longrightarrow v}{\langle \neg \mathtt{b}, \sigma \rangle \longrightarrow \neg v}\ (\mathsf{neg}_1)$$

The semantics – a universe of laws

$$\frac{}{\langle \mathtt{x}, \sigma \rangle \longrightarrow \sigma(\mathtt{x})}\ (\mathsf{var})$$

$$\frac{\langle \mathtt{b}, \sigma \rangle \longrightarrow v}{\langle \neg \mathtt{b}, \sigma \rangle \longrightarrow \neg v}\ (\mathsf{neg}_1)$$

$$\frac{\langle \mathtt{b}, \sigma \rangle \longrightarrow \langle \mathtt{b}', \sigma' \rangle}{\langle \neg \mathtt{b}, \sigma \rangle \longrightarrow \langle \neg \mathtt{b}', \sigma' \rangle}\ (\mathsf{neg}_2)$$

$$\frac{\langle \mathtt{b}_1, \sigma \rangle \longrightarrow \mathtt{ff}}{\langle \mathtt{b}_1 \wedge \mathtt{b}_2, \sigma \rangle \longrightarrow \mathtt{ff}}\ (\mathsf{and}_1)$$

$$\frac{\langle \mathtt{b}_1, \sigma \rangle \longrightarrow \mathtt{tt}}{\langle \mathtt{b}_1 \wedge \mathtt{b}_2, \sigma \rangle \longrightarrow \langle \mathtt{b}_2, \sigma \rangle}\ (\mathsf{and}_2)$$

$$\frac{\langle \mathtt{b}_1, \sigma \rangle \longrightarrow \langle \mathtt{b}_1', \sigma' \rangle}{\langle \mathtt{b}_1 \wedge \mathtt{b}_2, \sigma \rangle \longrightarrow \langle \mathtt{b}_1' \wedge \mathtt{b}_2, \sigma' \rangle}\ (\mathsf{and}_3)$$

An example

$\neg\neg \mathbf{x} \longrightarrow ?$

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$$\dfrac{\dfrac{\dfrac{}{\langle \mathrm{x}, \sigma\rangle \longrightarrow \sigma(\mathrm{x})}\,(\mathrm{var})}{\langle \neg \mathrm{x}, \sigma\rangle \longrightarrow \neg\sigma(\mathrm{x})}\,(\mathrm{neg}_1)}{\langle \neg\neg \mathrm{x}, \sigma\rangle \longrightarrow \neg\neg\sigma(\mathrm{x})}\,(\mathrm{neg}_1)$$

Another example

$(\mathtt{x} \wedge \mathtt{b}_1) \wedge \mathtt{b}_2 \longrightarrow ?$

If $\sigma(\mathtt{x}) = \mathtt{ff}$:

$$\dfrac{\dfrac{\dfrac{}{\langle \mathtt{x}, \sigma\rangle \longrightarrow \mathtt{ff}}\,(\text{var})}{\langle \mathtt{x} \wedge \mathtt{b}_1, \sigma\rangle \longrightarrow \mathtt{ff}}\,(\text{and}_1)}{\langle (\mathtt{x} \wedge \mathtt{b}_1) \wedge \mathtt{b}_2, \sigma\rangle \longrightarrow \mathtt{ff}}\,(\text{and}_1)$$

Yet another example

$(\mathtt{x} \wedge \mathtt{b}_1) \wedge \mathtt{b}_2 \longrightarrow ?$

If $\sigma(\mathtt{x}) = \mathtt{tt}$:

$$\dfrac{\dfrac{\dfrac{}{\langle \mathtt{x}, \sigma\rangle \longrightarrow \mathtt{tt}}\,(\mathsf{var})}{\langle \mathtt{x} \wedge \mathtt{b}_1, \sigma\rangle \longrightarrow \langle \mathtt{b}_1, \sigma\rangle}\,(\mathsf{and}_2)}{\langle (\mathtt{x} \wedge \mathtt{b}_1) \wedge \mathtt{b}_2, \sigma\rangle \longrightarrow \langle \mathtt{b}_1 \wedge \mathtt{b}_2, \sigma\rangle}\,(\mathsf{and}_3)$$

Now you try !

$\mathtt{x} \wedge \neg \mathtt{x} \longrightarrow ?$

$\neg(\neg \mathtt{x} \wedge \neg \mathtt{y}) \longrightarrow ?$

Provide semantics to the Boolean implication $\mathtt{b} \Rightarrow \mathtt{b}$

From one step to many . . .

One often is uninterested on the next step . . .

. . . and rather on the output (that the sequence of steps leads to)

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This multi-step transition $\longrightarrow^n$ is defined by the rules

$$\frac{\langle \mathsf{b}, \sigma \rangle \longrightarrow v}{\langle \mathsf{b}, \sigma \rangle \longrightarrow^1 v} \text{ (stp)} \qquad \frac{\langle \mathsf{b}, \sigma \rangle \longrightarrow \langle \mathsf{b}', \sigma' \rangle \qquad \langle \mathsf{b}', \sigma' \rangle \longrightarrow^n v}{\langle \mathsf{b}, \sigma \rangle \longrightarrow^{n+1} v} \text{ (nxt)}$$

What's next ?

Fine, we have an operational semantics; so what ?

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We can now prove cool properties about our language !!

Example (Termination)

It is always the case that $\langle \mathrm{b}, \sigma \rangle \longrightarrow^n v$ for some v and n

Exercise 1

Define a 'complexity function'

$$\begin{aligned} \mathrm{compl}(\mathtt{x}) &= 1 \\ \mathrm{compl}(\neg \mathtt{b}) &= \mathrm{compl}(\mathtt{b}) \\ \mathrm{compl}(\mathtt{b}_1 \wedge \mathtt{b}_2) &= \mathrm{compl}(\mathtt{b}_1) + \mathrm{compl}(\mathtt{b}_2) \end{aligned}$$

Show by induction that $\mathrm{compl}(\mathtt{b}) \geq 1$ for every $\mathtt{b}$

Exercise 2

Show by induction the following implication

$$\text{If } \langle \mathtt{b}, \sigma \rangle \longrightarrow \langle \mathtt{b}', \sigma' \rangle \text{ then } \mathrm{compl}(\mathtt{b}) > \mathrm{compl}(\mathtt{b}')$$

Recognising the pattern

Our induction proofs relied on

- a 'base' (or terminating) case
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Often hard to see on which structure should induction be founded

- natural numbers
- syntactic structure of programs
- derivation trees
- ...

Keep in mind . . .

Induction is a basic tool of every programming theorist

Exercise 3

Show by induction the following implication

$$\text{If } \langle \mathsf{b}, \sigma \rangle \longrightarrow^{n} v \text{ then } \mathrm{compl}(\mathsf{b}) \geq n$$

Table of Contents

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One often is uninterested on the number of steps . . .

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This multi-step transition $\longrightarrow^\star$ is defined by the rules

$$\frac{\langle \mathsf{b}, \sigma \rangle \longrightarrow v}{\langle \mathsf{b}, \sigma \rangle \longrightarrow^\star v}\ (\mathsf{stp}) \qquad \frac{\langle \mathsf{b}, \sigma \rangle \longrightarrow \langle \mathsf{b}', \sigma' \rangle \qquad \langle \mathsf{b}', \sigma' \rangle \longrightarrow^\star v}{\langle \mathsf{b}, \sigma \rangle \longrightarrow^\star v}\ (\mathsf{nxt})$$

Exercise 4

Show by induction the following equivalence

$$\langle \mathrm{b}, \sigma \rangle \longrightarrow^{n} v \text{ (for some } n\text{) iff } \langle \mathrm{b}, \sigma \rangle \longrightarrow^{\star} v$$

Table of Contents

A simple while-language

Arithmetic expressions

$$\mathtt{e} ::= \mathtt{n} \mid \mathtt{e} \cdot \mathtt{e} \mid \mathtt{x} \mid \mathtt{e} + \mathtt{e}$$

Programs

$$\mathtt{p} ::= \mathtt{x} := \mathtt{e} \mid \mathtt{p}\ ;\ \mathtt{p} \mid \texttt{if}\ \mathtt{b}\ \texttt{then}\ \mathtt{p}\ \texttt{else}\ \mathtt{p} \mid \texttt{while}\ \mathtt{b}\ \texttt{do}\ \{\ \mathtt{p}\ \}$$

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Homework: provide semantics to the arithmentic expressions

Key points

Similar to before but now with assignments, conditionals . . .

Unlike before memory can be altered throughout the computation

The output values will now be memories

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Unlike before memory can be altered throughout the computation

The output values will now be memories

We will use $\sigma[v/\mathtt{x}]$ to denote the memory that is like σ except for the fact that $\mathtt{x}$ has now value v

A while-language and its semantics

$$\frac{\langle \mathtt{e}, \sigma\rangle \longrightarrow^{\star} v}{\langle \mathtt{x} := \mathtt{e}, \sigma\rangle \longrightarrow \sigma[v/\mathtt{x}]}\ (\mathsf{asg}) \qquad \frac{\langle \mathtt{p}, \sigma\rangle \longrightarrow \sigma'}{\langle \mathtt{p}\ ;\ \mathtt{q}, \sigma\rangle \longrightarrow \langle \mathtt{q}, \sigma'\rangle}\ (\mathsf{seq}_1)$$

$$\frac{\langle \mathtt{p}, \sigma\rangle \longrightarrow \langle \mathtt{p}', \sigma'\rangle}{\langle \mathtt{p}\ ;\ \mathtt{q}, \sigma\rangle \longrightarrow \langle \mathtt{p}'\ ;\ \mathtt{q}, \sigma'\rangle}\ (\mathsf{seq}_2) \qquad \frac{\langle \mathtt{b}, \sigma\rangle \longrightarrow^{\star} \mathtt{tt}}{\langle \mathtt{if}\ \mathtt{b}\ \mathtt{then}\ \mathtt{p}\ \mathtt{else}\ \mathtt{q}, \sigma\rangle \longrightarrow \langle \mathtt{p}, \sigma\rangle}\ (\mathsf{if}_1)$$

$$\frac{\langle \mathtt{b}, \sigma\rangle \longrightarrow^{\star} \mathtt{ff}}{\langle \mathtt{if}\ \mathtt{b}\ \mathtt{then}\ \mathtt{p}\ \mathtt{else}\ \mathtt{q}, \sigma\rangle \longrightarrow \langle \mathtt{q}, \sigma\rangle}\ (\mathsf{if}_2) \qquad \frac{\langle \mathtt{b}, \sigma\rangle \longrightarrow^{\star} \mathtt{ff}}{\langle \mathtt{while}\ \mathtt{b}\ \mathtt{do}\ \{\ \mathtt{p}\ \}, \sigma\rangle \longrightarrow^{\star} \sigma}\ (\mathsf{wh}_2)$$

$$\frac{\langle \mathtt{b}, \sigma\rangle \longrightarrow^{\star} \mathtt{tt}}{\langle \mathtt{while}\ \mathtt{b}\ \mathtt{do}\ \{\ \mathtt{p}\ \}, \sigma\rangle \longrightarrow \langle \mathtt{p}\ ;\ \mathtt{while}\ \mathtt{b}\ \mathtt{do}\ \{\ \mathtt{p}\ \}, \sigma\rangle}\ (\mathsf{wh}_1)$$

Exercises

1. Write down the sequence of steps that originates from

$$\langle \texttt{while}\ \texttt{tt}\ \texttt{do}\ \{\ \texttt{x} := \texttt{x} + 1\ \}, \sigma \rangle$$

2. Conclude that our previous termination property was lost

Table of Contents

A simple concurrent language

Arithmetic expressions

$$\mathtt{e} ::= \mathtt{n} \mid \mathtt{e} \cdot \mathtt{e} \mid \mathtt{x} \mid \mathtt{e} + \mathtt{e}$$

Programs

$$\mathtt{p} ::= \mathtt{x} := \mathtt{e} \mid \mathtt{p} \,;\, \mathtt{p} \mid \texttt{if}\ \mathtt{b}\ \texttt{then}\ \mathtt{p}\ \texttt{else}\ \mathtt{p} \mid \texttt{while}\ \mathtt{b}\ \texttt{do}\ \{\, \mathtt{p} \,\} \mid \mathtt{p} \parallel \mathtt{q}$$

A concurrent language and its semantics

$$\frac{\langle \mathrm{p}, \sigma \rangle \longrightarrow \sigma'}{\langle \mathrm{p} \parallel \mathrm{q}, \sigma \rangle \longrightarrow \langle \mathrm{q}, \sigma' \rangle} \ (\mathsf{par}_1) \qquad \frac{\langle \mathrm{q}, \sigma \rangle \longrightarrow \sigma'}{\langle \mathrm{p} \parallel \mathrm{q}, \sigma \rangle \longrightarrow \langle \mathrm{p}, \sigma' \rangle} \ (\mathsf{par}_2)$$

$$\frac{\langle \mathrm{p}, \sigma \rangle \longrightarrow \langle \mathrm{p}', \sigma' \rangle}{\langle \mathrm{p} \parallel \mathrm{q}, \sigma \rangle \longrightarrow \langle \mathrm{p}' \parallel \mathrm{q}, \sigma' \rangle} \ (\mathsf{par}_3) \qquad \frac{\langle \mathrm{q}, \sigma \rangle \longrightarrow \langle \mathrm{q}', \sigma' \rangle}{\langle \mathrm{p} \parallel \mathrm{q}, \sigma \rangle \longrightarrow \langle \mathrm{p} \parallel \mathrm{q}', \sigma' \rangle} \ (\mathsf{par}_4)$$

Exercises

1. Write down the possible outputs of

$$\langle (\mathtt{y} := \mathtt{y} + 1 \,;\, \mathtt{x} := \mathtt{x} + 1) \parallel \mathtt{x} := 0, \sigma \rangle$$

2. Conclude that our previous determinacy property was lost

Conclusions

We (briefly) studied our first style of semantics

The gist: it describes how a program operates step-by-step

We also saw how valuable induction is in our context

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Further details about small-step semantics and induction can be consulted *e.g.* in [Rey98, Chapter 6] and [Win93, Chapter 3] respectively

 John C Reynolds, *Theories of programming languages*, Cambridge University Press, 1998.

 Glynn Winskel, *The formal semantics of programming languages - an introduction*, Foundation of computing series, MIT Press, 1993.