Simply Typed Lambda-calculus

Renato Neves

Universidade do Minho

HASLab
HIGH-ASSURANCE
SOFTWARE LABORATORY

Table of Contents

Deductive Reasoning

The essence

Knowledge obtained via assumptions and logical rules

Deductive Reasoning

The essence

Knowledge obtained via assumptions and logical rules

Studied since Aristotle . . .

. . . long before the age of artificial computers

What does it have to do with programming ?

A Basic Deductive System

$\mathbb{A}, \mathbb{B} \ldots$ denote propositions and
1 a proposition that always holds

If $\mathbb{A}$ and $\mathbb{B}$ are propositions then

- $\mathbb{A} \times \mathbb{B}$ is a proposition – conjunction of $\mathbb{A}$ and $\mathbb{B}$
- $\mathbb{A} \to \mathbb{B}$ is a proposition – implication of $\mathbb{B}$ from $\mathbb{A}$

A Basic Deductive System

Γ denotes a list of propositions (often called context)

$\Gamma \vdash \mathbb{A}$ reads *"if the propositions in Γ hold then $\mathbb{A}$ also holds"*

$$\frac{\mathbb{A} \in \Gamma}{\Gamma \vdash \mathbb{A}} \text{(ass)} \qquad \frac{}{\Gamma \vdash 1} \text{(trv)} \qquad \frac{\Gamma \vdash \mathbb{A} \times \mathbb{B}}{\Gamma \vdash \mathbb{A}} (\pi_1) \qquad \frac{\Gamma \vdash \mathbb{A} \times \mathbb{B}}{\Gamma \vdash \mathbb{B}} (\pi_2)$$

$$\frac{\Gamma \vdash \mathbb{A} \qquad \Gamma \vdash \mathbb{B}}{\Gamma \vdash \mathbb{A} \times \mathbb{B}} \text{(prd)} \qquad \frac{\Gamma, \mathbb{A} \vdash \mathbb{B}}{\Gamma \vdash \mathbb{A} \to \mathbb{B}} \text{(cry)} \qquad \frac{\Gamma \vdash \mathbb{A} \to \mathbb{B} \quad \Gamma \vdash \mathbb{A}}{\Gamma \vdash \mathbb{B}} \text{(app)}$$

Exercise

Show that $\mathbb{A} \times \mathbb{B} \vdash \mathbb{B} \times \mathbb{A}$

New Knowledge From Old

The rules below are derivable from the previous system

$$\frac{\Gamma, \mathbb{A}, \mathbb{B}, \Delta \vdash \mathbb{C}}{\Gamma, \mathbb{B}, \mathbb{A}, \Delta \vdash \mathbb{C}} \text{ (exchange)} \qquad \frac{\Gamma \vdash \mathbb{A}}{\Gamma, \mathbb{B} \vdash \mathbb{A}} \text{ (weakening)}$$

$$\frac{\Gamma, \mathbb{A} \vdash \mathbb{B} \qquad \Gamma \vdash \mathbb{A}}{\Gamma \vdash \mathbb{B}} \text{ (cut elimination)}$$

Proofs (again) by an appeal to your old friend . . . induction :-)

Exercises

Derive the following judgements

- $\mathbb{A} \to \mathbb{B}, \mathbb{B} \to \mathbb{C} \vdash \mathbb{A} \to \mathbb{C}$
- $\mathbb{A} \to \mathbb{B}, \mathbb{A} \to \mathbb{C} \vdash \mathbb{A} \to \mathbb{B} \times \mathbb{C}$

Back to programming . . .

The Bare Essentials of Programming

We should think of what are the basic features of programming ...

- variables
- function application and creation
- pairing ...

and base our study on the simplest language with such features ...

Simply-typed λ-calculus

The basis of `Haskell`, `ML`, `Eff`, `F#`, `Agda`, `Elm` and many other programming languages

Simply-typed λ-Calculus

Types are defined by $\mathbb{A} ::= 1 \mid \mathbb{A} \times \mathbb{A} \mid \mathbb{A} \to \mathbb{A}$

Γ now a non-repetitive list of typed variables $(x_1 : \mathbb{A}_1 \ldots x_n : \mathbb{A}_n)$

Programs built according to the following deduction rules

$$\frac{x : \mathbb{A} \in \Gamma}{\Gamma \vdash x : \mathbb{A}} \text{(ass)} \qquad \frac{}{\Gamma \vdash * : 1} \text{(triv)} \qquad \frac{\Gamma \vdash t : \mathbb{A} \times \mathbb{B}}{\Gamma \vdash \pi_1\, t : \mathbb{A}} (\pi_1)$$

$$\frac{\Gamma \vdash t : \mathbb{A} \qquad \Gamma \vdash s : \mathbb{B}}{\Gamma \vdash \langle t, s \rangle : \mathbb{A} \times \mathbb{B}} \text{(prd)} \qquad \frac{\Gamma, x : \mathbb{A} \vdash t : \mathbb{B}}{\Gamma \vdash \lambda x : \mathbb{A}.\, t : \mathbb{A} \to \mathbb{B}} \text{(cry)}$$

$$\frac{\Gamma \vdash t : \mathbb{A} \to \mathbb{B} \quad \Gamma \vdash s : \mathbb{A}}{\Gamma \vdash t\, s : \mathbb{B}} \text{(app)}$$

Examples of λ-terms

$x : \mathbb{A} \vdash x : \mathbb{A}$ (identity)

$x : \mathbb{A} \vdash \langle x, x \rangle : \mathbb{A} \times \mathbb{A}$ (duplication)

$x : \mathbb{A} \times \mathbb{B} \vdash \langle \pi_2\ x, \pi_1\ x \rangle : \mathbb{B} \times \mathbb{A}$ (swap)

$f : \mathbb{A} \to \mathbb{B}, g : \mathbb{B} \to \mathbb{C} \vdash \lambda x : \mathbb{A}.\ g(f\ x) : \mathbb{A} \to \mathbb{C}$ (composition)

Exercises

Recall the derivations that lead to the judgement

$$\mathbb{A} \to \mathbb{B}, \mathbb{A} \to \mathbb{C} \vdash \mathbb{A} \to \mathbb{B} \times \mathbb{C}$$

Build the corresponding program

Derive as well the judgement

$$\mathbb{A} \to \mathbb{B} \vdash \mathbb{A} \times \mathbb{C} \to \mathbb{B} \times \mathbb{C}$$

and subsequently build the corresponding program

Table of Contents

A Semantics for Simply Typed λ-calculus

We wish to assign a mathematical meaning to λ-terms

$$[\![-]\!] : \lambda\text{-terms} \longrightarrow \ldots$$

so that we can reason about them rigorously, and take advantage of known mathematical theories

A Semantics for Simply Typed λ-calculus

We wish to assign a mathematical meaning to λ-terms

$$[\![-]\!] : \lambda\text{-terms} \longrightarrow \ldots$$

so that we can reason about them rigorously, and take advantage of known mathematical theories

This is the goal of the next slides. But first . . .

Functions: Basic Facts

For every set X there exists a 'trivial' function

$$! : X \longrightarrow \{\star\} = 1 \qquad !\,(x) = \star$$

We can always pair two functions into $f : X \to A$, $g : X \to B$

$$\langle f, g \rangle : X \to A \times B \qquad \langle f, g \rangle(x) = (f\,x, g\,x)$$

There exist projection functions

$$\pi_1 : X \times Y \to X \qquad \pi_1(x, y) = x$$
$$\pi_2 : X \times Y \to Y \qquad \pi_2(x, y) = y$$

Functions: Basic Facts

We can always 'curry' a function $f : X \times Y \to Z$ into

$$\lambda f : X \to Z^Y \qquad \lambda f(x) = (y \mapsto f(x, y))$$

Consider sets X, Y, Z. There exists an application function

$$\mathrm{app} : Z^Y \times Y \to Z \qquad \mathrm{app}(f, y) = f\ y$$

Denotational Semantics

Types $\mathbb{A}$ interpreted as sets $[\![\mathbb{A}]\!]$

$$[\![1]\!] = \{\star\}$$
$$[\![\mathbb{A} \times \mathbb{B}]\!] = [\![\mathbb{A}]\!] \times [\![\mathbb{B}]\!]$$
$$[\![\mathbb{A} \to \mathbb{B}]\!] = [\![\mathbb{B}]\!]^{[\![\mathbb{A}]\!]}$$

Typing contexts Γ interpreted as Cartesian products

$$[\![\Gamma]\!] = [\![x_1 : \mathbb{A}_1, \ldots, x_n : \mathbb{A}_n]\!] = [\![\mathbb{A}_1]\!] \times \cdots \times [\![\mathbb{A}_n]\!]$$

λ-terms $\Gamma \vdash t : \mathbb{A}$ interpreted as functions

$$[\![\Gamma \vdash t : \mathbb{A}]\!] : [\![\Gamma]\!] \longrightarrow [\![\mathbb{A}]\!]$$

Denotational Semantics

λ-term $\Gamma \vdash t : \mathbb{A}$ interpreted as a function

$$[\![\Gamma \vdash t : \mathbb{A}]\!] : [\![\Gamma]\!] \longrightarrow [\![\mathbb{A}]\!]$$

$$\frac{x_i : \mathbb{A} \in \Gamma}{[\![\Gamma \vdash x_i : \mathbb{A}]\!] = \pi_i} \qquad \frac{}{[\![\Gamma \vdash * : 1]\!] = !} \qquad \frac{[\![\Gamma \vdash t : \mathbb{A} \times \mathbb{B}]\!] = f}{[\![\Gamma \vdash \pi_1 t : \mathbb{A}]\!] = \pi_1 \cdot f}$$

$$\frac{[\![\Gamma \vdash t : \mathbb{A}]\!] = f \qquad [\![\Gamma \vdash s : \mathbb{B}]\!] = g}{[\![\Gamma \vdash \langle t, s\rangle : \mathbb{A} \times \mathbb{B}]\!] = \langle f, g\rangle} \qquad \frac{[\![\Gamma, x : \mathbb{A} \vdash t : \mathbb{B}]\!] = f}{[\![\Gamma \vdash \lambda x : \mathbb{A}.\, t : \mathbb{A} \to \mathbb{B}]\!] = \lambda f}$$

$$\frac{[\![\Gamma \vdash t : \mathbb{A} \to \mathbb{B}]\!] = f \quad [\![\Gamma \vdash s : \mathbb{A}]\!] = g}{[\![\Gamma \vdash t\, s : \mathbb{B}]\!] = \mathrm{app} \cdot \langle f, g\rangle}$$

The Unravelling

$$
\begin{aligned}
[\![x \vdash \langle \pi_2\, x,\ \pi_1\, x \rangle]\!] &= \ldots \\
[\![- \vdash \lambda x.\, \langle \pi_2\, x,\ \pi_1\, x \rangle]\!] &= \ldots \\[1em]
[\![f, g, x \vdash g\, f\, x]\!] &= \ldots \\
[\![f, g \vdash \lambda x.\, g\, f\, x]\!] &= \ldots \\[1em]
[\![f, x \vdash \langle f\, \pi_1\, x,\ \pi_2\, x \rangle]\!] &= \ldots \\
[\![f \vdash \lambda x.\, \langle f\, \pi_1\, x,\ \pi_2\, x \rangle]\!] &= \ldots \\
[\![- \vdash \lambda f.\, \lambda x.\, \langle f\, \pi_1\, x,\ \pi_2\, x \rangle]\!] &= \ldots
\end{aligned}
$$

(**N.B.** all types omitted for simplicity)

Denotational Semantics and Equivalence Revisited

Show that the following equations hold

$$
\begin{aligned}
[\![x, y \vdash \pi_1 \langle x, y \rangle]\!] &= [\![x, y \vdash x]\!] \\
[\![\Gamma \vdash t]\!] &= [\![\Gamma \vdash \langle \pi_1 \ t, \pi_2 \ t \rangle]\!] \\
[\![x \vdash (\lambda y. \langle x, y \rangle) \ x]\!] &= [\![x \vdash \langle x, x \rangle]\!]
\end{aligned}
$$

Denotational Semantics and Equivalence Revisited

Show that the following equations hold

$$\begin{aligned} [\![x, y \vdash \pi_1 \langle x, y \rangle]\!] &= [\![x, y \vdash x]\!] \\ [\![\Gamma \vdash t]\!] &= [\![\Gamma \vdash \langle \pi_1 \ t, \pi_2 \ t \rangle]\!] \\ [\![x \vdash (\lambda y.\, \langle x, y \rangle) \ x]\!] &= [\![x \vdash \langle x, x \rangle]\!] \end{aligned}$$

Show that the (complicated) λ-term below is really just the identity

$$z \vdash \lambda x.\, \langle \pi_2\, x, \pi_1\, x \rangle \left(\lambda y.\, \langle \pi_2\, y, \pi_1\, y \rangle \ z \right)$$

Denotational Semantics and Equivalence Revisited

Show that the following equations hold

$$\llbracket x, y \vdash \pi_1 \langle x, y \rangle \rrbracket \;=\; \llbracket x, y \vdash x \rrbracket$$

$$\llbracket \Gamma \vdash t \rrbracket \;=\; \llbracket \Gamma \vdash \langle \pi_1\ t, \pi_2\ t \rangle \rrbracket$$

$$\llbracket x \vdash (\lambda y.\, \langle x, y \rangle)\ x \rrbracket \;=\; \llbracket x \vdash \langle x, x \rangle \rrbracket$$

Show that the (complicated) λ-term below is really just the identity

$$z \vdash \lambda x.\, \langle \pi_2\, x, \pi_1\, x \rangle \left(\lambda y.\, \langle \pi_2\, y, \pi_1\, y \rangle\ z \right)$$

Hard ?

Table of Contents

Logic to the Rescue !

Recall that the rules below are derivable from our logical system

$$\frac{\Gamma, \mathbb{A}, \mathbb{B}, \Delta \vdash \mathbb{C}}{\Gamma, \mathbb{B}, \mathbb{A}, \Delta \vdash \mathbb{C}} \text{ (exchange)} \qquad \frac{\Gamma \vdash \mathbb{A}}{\Gamma, \mathbb{B} \vdash \mathbb{A}} \text{ (weakening)}$$

$$\frac{\Gamma, \mathbb{A} \vdash \mathbb{B} \qquad \Gamma \vdash \mathbb{A}}{\Gamma \vdash \mathbb{B}} \text{ (cut elimination)}$$

Via the Programming Lens

$$\frac{\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \vdash t : \mathbb{C}}{\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \vdash t : \mathbb{C}} \text{ (exch)}$$

$$\frac{\Gamma \vdash t : \mathbb{A}}{\Gamma, x : \mathbb{B} \vdash t : \mathbb{A}} \text{ (weak)}$$

$$\frac{\Gamma, x : \mathbb{A} \vdash t : \mathbb{B} \qquad \Gamma \vdash s : \mathbb{A}}{\Gamma \vdash \cdots : \mathbb{B}} \text{ (cut elimination)}$$

Via the Programming Lens

$$\frac{\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \vdash t : \mathbb{C}}{\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \vdash t : \mathbb{C}} \text{ (exch)} \qquad \frac{\Gamma \vdash t : \mathbb{A}}{\Gamma, x : \mathbb{B} \vdash t : \mathbb{A}} \text{ (weak)}$$

$$\frac{\Gamma, x : \mathbb{A} \vdash t : \mathbb{B} \qquad \Gamma \vdash s : \mathbb{A}}{\Gamma \vdash \cdots : \mathbb{B}} \text{ (cut elimination)}$$

Filling up the dots will lead us to a fundamental concept

Via the Programming Lens

$$\frac{\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \vdash t : \mathbb{C}}{\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \vdash t : \mathbb{C}} \text{ (exch)} \qquad \frac{\Gamma \vdash t : \mathbb{A}}{\Gamma, x : \mathbb{B} \vdash t : \mathbb{A}} \text{ (weak)}$$

$$\frac{\Gamma, x : \mathbb{A} \vdash t : \mathbb{B} \qquad \Gamma \vdash s : \mathbb{A}}{\Gamma \vdash \cdots : \mathbb{B}} \text{ (cut elimination)}$$

Filling up the dots will lead us to a fundamental concept

Substitution

Substitution

The essence

Substitution of variables in a λ-term t by another λ-term s

$t[s/x]$ reads *"replace every occurrence of x in t by s"*

Substitution

The essence

Substitution of variables in a λ-term t by another λ-term s

$t[s/x]$ reads *"replace every occurrence of x in t by s"*

Example

$$\langle x, x \rangle[s/x] = \langle s, s \rangle$$
$$\langle x, y \rangle[s/x] = \langle s, y \rangle$$
$$\langle y, z \rangle[s/x] = \langle y, z \rangle$$

Substitution More Formally

We define it by induction

$$x[s/y] = \begin{cases} s & \text{if } x = y \\ x & \text{otherwise} \end{cases}$$

$$*[s/y] = *$$

$$\langle t_1, t_2 \rangle [s/y] = \langle t_1[s/y], t_2[s/y] \rangle$$

$$(t_1\ t_2)[s/y] = t_1[s/y]\ t_2[s/y]$$

$$(\pi_1 t)[s/y] = \pi_1 t[s/y]$$

$$(\pi_2 t)[s/y] = \pi_2 t[s/y]$$

$$(\lambda x.\, t)[s/y] = \ \ldots$$

Variable Captures

$\lambda x.\, y$ is a "constant function" (given x return y)

Variable Captures

$\lambda x.\, y$ is a “constant function" (given x return y)

$(\lambda x.\, y)[z/y]$ is still a “constant function" (given x return z)

Variable Captures

$\lambda x.\, y$ is a "constant function" (given x return y)

$(\lambda x.\, y)[z/y]$ is still a "constant function" (given x return z)

$(\lambda x.\, y)[x/y]$ is now the identity !?

Variable Captures

$\lambda x.\, y$ is a “constant function" (given x return y)

$(\lambda x.\, y)[z/y]$ is still a “constant function" (given x return z)

$(\lambda x.\, y)[x/y]$ is now the identity !?

The problem: variable x "captured" by the construct "$\lambda x.$"

Somehow similar to variable shadowing in programming

Substitution More Formally

$$x[s/x] = \begin{cases} s & \text{if } x = y \\ x & \text{otherwise} \end{cases}$$

$$*[s/y] = *$$

$$\langle t_1, t_2\rangle[s/y] = \langle t_1[s/y], t_2[s/y]\rangle$$

$$(t_1\ t_2)[s/y] = t_1[s/y]\ t_2[s/y]$$

$$(\pi_1\ t)[s/y] = \pi_1\ t[s/y]$$

$$(\pi_2\ t)[s/y] = \pi_2\ t[s/y]$$

$$(\lambda x.\, t)[s/y] = \lambda z.\, t[z/x][s/y]$$

(where z is fresh (*i.e.* new))

Exercise

Compute the following substitutions

$$
\begin{aligned}
* [t/y][s/z] &= \ldots \\
\langle y, z \rangle [t/y][s/z] &= \ldots \\
(\lambda x.\, x)[t/x] &= \ldots \\
(\lambda x.\, \langle x, y \rangle)[z/y] &= \ldots \\
(\lambda x.\, \langle x, y \rangle)[x/y] &= \ldots
\end{aligned}
$$

Via the Programming Lens

$$\frac{\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \vdash t : \mathbb{C}}{\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \vdash t : \mathbb{C}} \text{ (exch)} \qquad \frac{\Gamma \vdash t : \mathbb{A}}{\Gamma, x : \mathbb{B} \vdash t : \mathbb{A}} \text{ (weak)}$$

$$\frac{\Gamma, x : \mathbb{A} \vdash t : \mathbb{B} \qquad \Gamma \vdash s : \mathbb{A}}{\Gamma \vdash \cdots : \mathbb{B}} \text{ (cut elimination)}$$

Via the Programming Lens

$$\frac{\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \vdash t : \mathbb{C}}{\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \vdash t : \mathbb{C}} \text{ (exch)} \qquad \frac{\Gamma \vdash t : \mathbb{A}}{\Gamma, x : \mathbb{B} \vdash t : \mathbb{A}} \text{ (weak)}$$

$$\frac{\Gamma, x : \mathbb{A} \vdash t : \mathbb{B} \qquad \Gamma \vdash s : \mathbb{A}}{\Gamma \vdash t[s/x] : \mathbb{B}} \text{ (cut elimination)}$$

Via the Programming Lens

$$\frac{\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \vdash t : \mathbb{C}}{\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \vdash t : \mathbb{C}} \text{ (exch)} \qquad \frac{\Gamma \vdash t : \mathbb{A}}{\Gamma, x : \mathbb{B} \vdash t : \mathbb{A}} \text{ (weak)}$$

$$\frac{\Gamma, x : \mathbb{A} \vdash t : \mathbb{B} \qquad \Gamma \vdash s : \mathbb{A}}{\Gamma \vdash t[s/x] : \mathbb{B}} \text{ (cut elimination)}$$

Substitution also fundamental in the study of equivalence

An Equational System pt. I

$$
\begin{aligned}
\pi_1\langle t, s\rangle &=_{\beta\eta} t & t &=_{\beta\eta} * & & \text{(if } t : 1\text{)} \\
\pi_2\langle t, s\rangle &=_{\beta\eta} s & \lambda x.\, t\, s &=_{\beta\eta} t[s/x] \\
\langle \pi_1\, t, \pi_2\, t\rangle &=_{\beta\eta} t & \lambda x.\, (t\, x) &=_{\beta\eta} t
\end{aligned}
$$

An Equational System pt. II

$$t =_{\beta\eta} t$$

$$\frac{t =_{\beta\eta} s}{s =_{\beta\eta} t}$$

$$\frac{t =_{\beta\eta} s \qquad s =_{\beta\eta} u}{t =_{\beta\eta} u}$$

$$\frac{t =_{\beta\eta} s}{\pi_1\, t =_{\beta\eta} \pi_1\, s}$$

$$\frac{t =_{\beta\eta} s}{\pi_2\, t =_{\beta\eta} \pi_2\, s}$$

$$\frac{t =_{\beta\eta} s \qquad u =_{\beta\eta} v}{\langle t, u \rangle =_{\beta\eta} \langle s, v \rangle}$$

$$\frac{t =_{\beta\eta} s \qquad u =_{\beta\eta} v}{t\, u =_{\beta\eta} s\, v}$$

$$\frac{t =_{\beta\eta} s}{\lambda x.\, t =_{\beta\eta} \lambda x.\, s}$$

$$\frac{\Gamma \vdash t =_{\beta\eta} s}{\pi\Gamma \vdash t =_{\beta\eta} s}$$

$$\frac{u =_{\beta\eta} v \qquad t =_{\beta\eta} s}{u[t/x] =_{\beta\eta} v[s/x]}$$

Equivalence Re-Revisited

Show that the following equations hold

$$\pi_1\langle x, y\rangle =_{\beta\eta} x$$

$$t =_{\beta\eta} \langle \pi_1\ t, \pi_2\ t\rangle$$

$$(\lambda y.\, \langle x, y\rangle)\ x =_{\beta\eta} \langle x, x\rangle$$

$$\lambda x.\, \langle \pi_2\, x, \pi_1\, x\rangle \left(\lambda y.\, \langle \pi_2\, y, \pi_1\, y\rangle\ z\right) =_{\beta\eta} z$$

Table of Contents

Learning Programming from Logic

If conjunction in logic corresponds to pairing in programming

. . . what does disjunction in logic correspond to ?

Revisiting our Deductive System

$\mathbb{A}, \mathbb{B} \ldots$ denote propositions and
1 a proposition that always holds

If $\mathbb{A}$ and $\mathbb{B}$ are propositions then

- $\mathbb{A} \times \mathbb{B}$ is a proposition – conjunction of $\mathbb{A}$ and $\mathbb{B}$
- $\mathbb{A} \to \mathbb{B}$ is a proposition – implication of $\mathbb{B}$ from $\mathbb{A}$
- $\mathbb{A} + \mathbb{B}$ is a proposition – disjunction of $\mathbb{A}$ and $\mathbb{B}$

Revisiting our Deductive System

$$\frac{\mathbb{A} \in \Gamma}{\Gamma \vdash \mathbb{A}} \text{ (ass)} \qquad \frac{}{\Gamma \vdash 1} \text{ (trv)} \qquad \frac{\Gamma \vdash \mathbb{A} \times \mathbb{B}}{\Gamma \vdash \mathbb{A}} \; (\pi_1) \qquad \frac{\Gamma \vdash \mathbb{A} \times \mathbb{B}}{\Gamma \vdash \mathbb{B}} \; (\pi_2)$$

$$\frac{\Gamma \vdash \mathbb{A} \qquad \Gamma \vdash \mathbb{B}}{\Gamma \vdash \mathbb{A} \times \mathbb{B}} \text{ (prd)} \quad \frac{\Gamma, \mathbb{A} \vdash \mathbb{B}}{\Gamma \vdash \mathbb{A} \to \mathbb{B}} \text{ (cry)} \quad \frac{\Gamma \vdash \mathbb{A} \to \mathbb{B} \quad \Gamma \vdash \mathbb{A}}{\Gamma \vdash \mathbb{B}} \text{ (app)}$$

. . . .

$$\frac{\Gamma \vdash \mathbb{A}}{\Gamma \vdash \mathbb{A} + \mathbb{B}} \text{ (inl)} \qquad \frac{\Gamma \vdash \mathbb{B}}{\Gamma \vdash \mathbb{A} + \mathbb{B}} \text{ (inr)}$$

$$\frac{\Gamma \vdash \mathbb{A} + \mathbb{B} \qquad \Gamma, \mathbb{A} \vdash \mathbb{C} \qquad \Gamma, \mathbb{B} \vdash \mathbb{C}}{\Gamma \vdash \mathbb{C}} \text{ (coprd)}$$

Conditionals Enter the Scene !

$$\frac{x : \mathbb{A} \in \Gamma}{\Gamma \vdash x : \mathbb{A}}\ (\text{ass}) \qquad \frac{}{\Gamma \vdash * : 1}\ (\text{triv}) \qquad \frac{\Gamma \vdash t : \mathbb{A} \times \mathbb{B}}{\Gamma \vdash \pi_1\, t : \mathbb{A}}\ (\pi_1)$$

$$\frac{\Gamma \vdash t : \mathbb{A} \qquad \Gamma \vdash s : \mathbb{B}}{\Gamma \vdash \langle t, s \rangle : \mathbb{A} \times \mathbb{B}}\ (\text{prd}) \qquad \frac{\Gamma, x : \mathbb{A} \vdash t : \mathbb{B}}{\Gamma \vdash \lambda x : \mathbb{A}.\, t : \mathbb{A} \to \mathbb{B}}\ (\text{cry})$$

$$\frac{\Gamma \vdash t : \mathbb{A} \to \mathbb{B} \quad \Gamma \vdash s : \mathbb{A}}{\Gamma \vdash t\, s : \mathbb{B}}\ (\text{app})$$

. . . .

$$\frac{\Gamma \vdash t : \mathbb{A}}{\Gamma \vdash \text{inl}_{\mathbb{B}}\, t : \mathbb{A} + \mathbb{B}}\ (\text{inl}) \qquad \frac{\Gamma \vdash t : \mathbb{B}}{\Gamma \vdash \text{inr}_{\mathbb{A}}\, t : \mathbb{A} + \mathbb{B}}\ (\text{inr})$$

$$\frac{\Gamma \vdash t : \mathbb{A} + \mathbb{B} \qquad \Gamma, x : \mathbb{A} \vdash s : \mathbb{C} \qquad \Gamma, y : \mathbb{B} \vdash u : \mathbb{C}}{\Gamma \vdash \text{case } t \text{ of inl}(x) \Rightarrow s; \text{inr}(y) \Rightarrow u : \mathbb{C}}\ (\text{coprd})$$

Exercises

Derive the following judgements

- $\mathbb{A} + \mathbb{B} \vdash \mathbb{B} + \mathbb{A}$
- $\mathbb{A} \times (\mathbb{B} + \mathbb{C}) \vdash \mathbb{A} \times \mathbb{B} + \mathbb{A} \times \mathbb{C}$
- $\mathbb{A} \times \mathbb{B} + \mathbb{A} \times \mathbb{C} \vdash \mathbb{A}$
- $\mathbb{A} \times \mathbb{B} + \mathbb{A} \times \mathbb{C} \vdash \mathbb{B} + \mathbb{C}$
- $\mathbb{A} \times \mathbb{B} + \mathbb{A} \times \mathbb{C} \vdash \mathbb{A} \times (\mathbb{B} + \mathbb{C})$

Then build the corresponding programs

Revisiting our Denotational Semantics

Types $\mathbb{A}$ interpreted as sets $[\![\mathbb{A}]\!]$

$$[\![1]\!] = \{\star\}$$

$$[\![\mathbb{A} \times \mathbb{B}]\!] = [\![\mathbb{A}]\!] \times [\![\mathbb{B}]\!]$$

$$[\![\mathbb{A} \to \mathbb{B}]\!] = [\![\mathbb{B}]\!]^{[\![\mathbb{A}]\!]}$$

. . . .

$$[\![\mathbb{A} + \mathbb{B}]\!] = [\![\mathbb{A}]\!] + [\![\mathbb{B}]\!]$$

Judgements $\Gamma \vdash t : \mathbb{A}$ interpreted as functions

$$[\![\Gamma \vdash t : \mathbb{A}]\!] : [\![\Gamma]\!] \longrightarrow [\![\mathbb{A}]\!]$$

Functions: Basic Facts

There exist injection functions

$$i_1 : X \to X + Y \qquad x \mapsto i_1(x)$$
$$i_2 : Y \to X + Y \qquad y \mapsto i_2(y)$$

We can always 'co-pair' two functions into $f : A \to X$, $g : B \to X$

$$[f, g] : A + B \to X \qquad [f, g](i_1(x)) = f(x), \quad [f, g](i_2(y)) = g(y)$$

Revisiting our Denotational Semantics

$$\frac{x_i : \mathbb{A} \in \Gamma}{[\![\Gamma \vdash x_i : \mathbb{A}]\!] = \pi_i} \qquad \frac{}{[\![\Gamma \vdash * : 1]\!] = \,!} \qquad \frac{[\![\Gamma \vdash t : \mathbb{A} \times \mathbb{B}]\!] = f}{[\![\Gamma \vdash \pi_1 t : \mathbb{A}]\!] = \pi_1 \cdot f}$$

$$\frac{[\![\Gamma \vdash t : \mathbb{A}]\!] = f \qquad [\![\Gamma \vdash s : \mathbb{B}]\!] = g}{[\![\Gamma \vdash \langle t, s\rangle : \mathbb{A} \times \mathbb{B}]\!] = \langle f, g\rangle} \qquad \frac{[\![\Gamma, x : \mathbb{A} \vdash t : \mathbb{B}]\!] = f}{[\![\Gamma \vdash \lambda x : \mathbb{A}.\, t : \mathbb{A} \to \mathbb{B}]\!] = \lambda f}$$

$$\frac{[\![\Gamma \vdash t : \mathbb{A} \to \mathbb{B}]\!] = f \quad [\![\Gamma \vdash s : \mathbb{A}]\!] = g}{[\![\Gamma \vdash t\, s : \mathbb{B}]\!] = \mathrm{app} \cdot \langle f, g\rangle}$$

. . . .

$$\frac{[\![\Gamma \vdash t : \mathbb{A}]\!] = f}{[\![\Gamma \vdash \mathrm{inl}_{\mathbb{B}}\ t : \mathbb{A} + \mathbb{B}]\!] = i_1 \cdot f} \qquad \frac{[\![\Gamma \vdash t : \mathbb{B}]\!] = f}{[\![\Gamma \vdash \mathrm{inr}_{\mathbb{A}}\ t : \mathbb{A} + \mathbb{B}]\!] = i_2 \cdot f}$$

$$\frac{[\![\Gamma \vdash t : \mathbb{A} + \mathbb{B}]\!] = f \qquad [\![\Gamma, x : \mathbb{A} \vdash s : \mathbb{C}]\!] = g \qquad [\![\Gamma, y : \mathbb{B} \vdash u : \mathbb{C}]\!] = h}{[\![\Gamma \vdash \mathrm{case}\ t\ \mathrm{of}\ \mathrm{inl}(x) \Rightarrow s; \mathrm{inr}(y) \Rightarrow u : \mathbb{C}]\!] = [g, h] \cdot \mathrm{dist} \cdot \langle \mathrm{id}, f\rangle}$$

Exercises

$$
\begin{aligned}
[\![x \vdash \mathrm{case}\, x \,\mathrm{of}\, \mathrm{inl}(y) \Rightarrow \mathrm{inr}(y); \mathrm{inr}(z) \Rightarrow \mathrm{inl}(z)]\!] &= \ldots \\
[\![x \vdash \mathrm{case}\, x \,\mathrm{of}\, \mathrm{inl}(y) \Rightarrow \pi_1\, y; \mathrm{inr}(z) \Rightarrow \pi_1\, z]\!] &= \ldots \\
[\![x \vdash \mathrm{case}\, x \,\mathrm{of}\, \mathrm{inl}(y) \Rightarrow \langle \pi_1\, y, \mathrm{inl}\, \pi_2\, y\rangle; \mathrm{inr}(z) \Rightarrow \langle \pi_1\, z, \mathrm{inl}\, \pi_2\, z\rangle]\!] &= \ldots
\end{aligned}
$$

Revisiting our Equational System

$$\pi_1\langle t, s\rangle =_{\beta\eta} t \qquad t =_{\beta\eta} * \qquad \text{(if } t : 1)$$

$$\pi_2\langle t, s\rangle =_{\beta\eta} s \qquad \lambda x.\, t\, s =_{\beta\eta} t[s/x]$$

$$\langle \pi_1\, t, \pi_2\, t\rangle =_{\beta\eta} t \qquad \lambda x.\, (t\, x) =_{\beta\eta} t$$

. . . .

$$\mathrm{case}\ \mathrm{inl}\ t\ \mathrm{of}\ \mathrm{inl}(x) \Rightarrow s; \mathrm{inr}(y) \Rightarrow u =_{\beta\eta} s[t/x]$$

$$\mathrm{case}\ \mathrm{inr}\ t\ \mathrm{of}\ \mathrm{inl}(x) \Rightarrow s; \mathrm{inr}(y) \Rightarrow u =_{\beta\eta} u[t/y]$$

$$\mathrm{case}\ x\ \mathrm{of}\ \mathrm{inl}(y) \Rightarrow t[\mathrm{inl}(y)/x]; \mathrm{inr}(z) \Rightarrow t[\mathrm{inr}(z)/x] =_{\beta\eta} t$$

Exercises

Show that

$$\Big(\lambda x.\, \mathrm{case}\, x \,\mathrm{of}\, \mathrm{inl}(y) \Rightarrow \mathrm{inr}(y);\, \mathrm{inr}(z) \Rightarrow \mathrm{inl}(z)\Big)\, \mathrm{inl}(a) =_{\beta\eta} \mathrm{inr}(a)$$

$$\Big(\lambda x.\, \mathrm{case}\, x \,\mathrm{of}\, \mathrm{inl}(y) \Rightarrow \mathrm{inr}(y);\, \mathrm{inr}(z) \Rightarrow \mathrm{inl}(z)\Big)\, \mathrm{inr}(a) =_{\beta\eta} \mathrm{inl}(a)$$

Prove the following implication

$$\begin{cases} (\lambda x.\, t)\, \mathrm{inl}(y) =_{\beta\eta} (\lambda x.\, s)\, \mathrm{inl}(y) \\ (\lambda x.\, t)\, \mathrm{inr}(z) =_{\beta\eta} (\lambda x.\, s)\, \mathrm{inr}(z) \end{cases} \implies \lambda x.\, t =_{\beta\eta} \lambda x.\, s$$

What can logic teach us more about programming ?

Table of Contents

Eager vs. Lazy

Lazy Evaluation (*e.g.* Haskell)

$\pi_2 \langle \text{divergence}, 0 \rangle = 0$

Strict Evaluation (*e.g.* Python)

$\pi_2 \langle \text{divergence}, 0 \rangle = \text{divergence}$

Eager vs. Lazy

Lazy Evaluation (*e.g.* Haskell)

$\pi_2 \langle \text{divergence}, 0 \rangle = 0$

Strict Evaluation (*e.g.* Python)

$\pi_2 \langle \text{divergence}, 0 \rangle = \text{divergence}$

Strict evaluation breaks product laws

Quantum Computation: No-cloning and Entanglement

Forbidden to write down $\langle x, x \rangle$

Quantum Computation: No-cloning and Entanglement

Forbidden to write down $\langle x, x \rangle$

Certainly false that $\langle \pi_1\, x, \pi_2\, x \rangle = x$

Quantum Computation: No-cloning and Entanglement

Forbidden to write down $\langle x, x \rangle$

Certainly false that $\langle \pi_1\, x, \pi_2\, x \rangle = x$

Last case also holds in probabilistic programming

Beyond Cartesianism

Cartesian structures thus often non-adequate

We will explore a more general approach

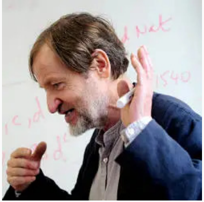
Linear λ-calculus

Controlled use of resources (no duplication, no discarding)

Product laws need not hold

Broader range of applications than ‘Cartesian λ-calculus’

A Linear Deductive System



$\mathbb{A}, \mathbb{B} \ldots$ denote propositions and $\mathbb{I}$ a trivial one

If $\mathbb{A}$ and $\mathbb{B}$ are propositions then

- $\mathbb{A} \otimes \mathbb{B}$ is a proposition – 'linear conjunction' of $\mathbb{A}$ and $\mathbb{B}$
- $\mathbb{A} \multimap \mathbb{B}$ is a proposition – 'linear implication' of $\mathbb{B}$ from $\mathbb{A}$

A Linear Deductive System

$\Gamma, \Delta \ldots$ denote lists of propositions

$$\frac{}{\mathbb{A} \vdash \mathbb{A}} \text{ (ass)} \qquad \frac{}{(-) \vdash \mathbb{I}} \text{ (trv)} \qquad \frac{\Gamma \vdash \mathbb{I} \qquad \Delta \vdash \mathbb{A}}{\Gamma, \Delta \vdash \mathbb{A}} \text{ (dsc)}$$

$$\frac{\Gamma \vdash \mathbb{A} \qquad \Delta \vdash \mathbb{B}}{\Gamma, \Delta \vdash \mathbb{A} \otimes \mathbb{B}} \text{ (prd)} \qquad \frac{\Gamma \vdash \mathbb{A} \otimes \mathbb{B} \qquad \Delta, \mathbb{A}, \mathbb{B} \vdash \mathbb{C}}{\Gamma, \Delta \vdash \mathbb{C}} \text{ (prj)}$$

$$\frac{\Gamma, \mathbb{A} \vdash \mathbb{B}}{\Gamma \vdash \mathbb{A} \multimap \mathbb{B}} \text{ (cry)} \qquad \frac{\Gamma \vdash \mathbb{A} \multimap \mathbb{B} \quad \Delta \vdash \mathbb{A}}{\Gamma, \Delta \vdash \mathbb{B}} \text{ (app)}$$

Linear λ-calculus

$$\frac{}{x : \mathbb{A} \vdash x : \mathbb{A}} \qquad \frac{}{(-) \vdash * : \mathbb{I}} \qquad \frac{\Gamma \vdash t : \mathbb{I} \qquad \Delta \vdash s : \mathbb{A}}{\Gamma, \Delta \vdash t \text{ to } *.\ s : \mathbb{A}}$$

$$\frac{\Gamma \vdash t : \mathbb{A} \qquad \Delta \vdash s : \mathbb{B}}{\Gamma, \Delta \vdash t \otimes s : \mathbb{A} \otimes \mathbb{B}} \qquad \frac{\Gamma \vdash t : \mathbb{A} \otimes \mathbb{B} \qquad \Delta, x : \mathbb{A}, y : \mathbb{B} \vdash s : \mathbb{C}}{\Gamma, \Delta \vdash \text{pm } t \text{ to } x \otimes y.\, s : \mathbb{C}}$$

$$\frac{\Gamma, x : \mathbb{A} \vdash t : \mathbb{B}}{\Gamma \vdash \lambda x : \mathbb{A}.\ t : \mathbb{A} \multimap \mathbb{B}} \qquad \frac{\Gamma \vdash t : \mathbb{A} \multimap \mathbb{B} \quad \Delta \vdash s : \mathbb{A}}{\Gamma, \Delta \vdash t\, s : \mathbb{B}}$$

Examples of Linear λ-terms

$x : \mathbb{A} \vdash x : \mathbb{A}$ (identity)

$x : \mathbb{A} \otimes \mathbb{B} \vdash \text{pm } x \text{ to } a \otimes b.\, b \otimes a : \mathbb{B} \otimes \mathbb{A}$ (swap)

$(-) \vdash \lambda x.\, \text{pm } x \text{ to } a \otimes b.\, b \otimes a : \mathbb{A} \otimes \mathbb{B} \multimap \mathbb{B} \otimes \mathbb{A}$ (swap curried)

$x : \mathbb{I} \otimes \mathbb{A} \vdash \text{pm } x \text{ to } i \otimes a.\, (i \text{ to } * \,.\, a) : \mathbb{A}$ (discard triv)

Examples of Linear λ-terms in Quantum

$$x : \mathbb{B}, y : \mathbb{B} \vdash \mathrm{cnot}(\mathrm{had}(x), y) : \mathbb{Q} \otimes \mathbb{Q} \qquad \text{(EPR pair)}$$

$$x : \mathbb{B}, y : \mathbb{B} \vdash \Big(\lambda x.\, \mathsf{pm}\ x\ \mathsf{to}\ a \otimes b.\, b \otimes a\Big)\Big(\mathrm{cnot}(\mathrm{had}(x), y)\Big) \qquad \text{(EPR swapped)}$$

Examples of Linear λ-terms in Quantum

$$x : \mathbb{B}, y : \mathbb{B} \vdash \mathrm{cnot}(\mathrm{had}(x), y) : \mathbb{Q} \otimes \mathbb{Q} \qquad \text{(EPR pair)}$$

$$x : \mathbb{B}, y : \mathbb{B} \vdash \Big(\lambda x.\, \mathsf{pm}\ x\ \mathsf{to}\ a \otimes b.\, b \otimes a\Big)\Big(\mathrm{cnot}(\mathrm{had}(x), y)\Big) \qquad \text{(EPR swapped)}$$

Does swapping actually have any effect on the pair ?

Next Steps

Answer to previous question calls for semantics

Next Steps

Answer to previous question calls for semantics

More generally a full study of linear λ-calculus calls for semantics

Next Steps

Answer to previous question calls for semantics

More generally a full study of linear λ-calculus calls for semantics

. . . which we will obtain via Category Theory :-)