Buh !

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Next Stop: Category Theory

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In a Nutshell

The Mathematics of Mathematics

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Abstract non-sense

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El Libertadora

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The Mathematics of Mathematics

Abstract non-sense

El Libertadora

You have actually met her before

A Category

Consists of a 'set' of things (a.k.a. objects) $A, B, C \ldots$

arrows (a.k.a. morphisms) $f : A \to B$ connecting these things

and an 'identity' arrow $\mathrm{id} : A \to A$ for each thing A

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Arrows are composable *i.e.*

$$\frac{f : A \to B \qquad g : B \to C}{g \cdot f : A \to C}$$

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(composition is associative and has id as neutral element)

Examples

- Sets and functions (**Set**)
- Posets and monotone functions (**Pos**)
- Vector spaces and linear functions (**Vec**)
- Natural numbers and matrices (**Mat**)
- Reals numbers with infinity and $\leq$ relation (**R**)

Finality

Definition

An obj. X is final if for every obj. A there exists a unique arrow

$$! : A \to X$$

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In **Set**, **Pos**, **Vec** it is singleton set (represents 'discarding data')

In **Mat** it is 0 (discarding analogy still applies)

In **R** it is infinity (the greatest element)

Categorical Products

Definition

An obj. X called product of objs. A and B if

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[d, dotted, "\langle f, g \rangle"] \arrow[dr, "g"] & \\ A & X \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd}$$

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In **Set**, **Pos**, **Vec** given by Cartesian product

In **Mat** given by addition (leads to vertical stacking of matrices)

In **R** given by minimum (*i.e.* the greatest lower bound)

Categorical Exponentials

Definition

An obj. X called exponential of objs. A and B if

$$\begin{tikzcd} A \times B \arrow[d, dotted, "(\lambda f) \times \mathrm{id}"'] \arrow[dr, "f"] & \\ X \times B \arrow[r, "\mathrm{app}"'] & C \end{tikzcd}$$

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Other examples will be analysed later on

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Cartesian-closed Categories

We will now vastly generalise our semantics of λ-calculus . . .

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... via the so-called Cartesian-closed categories

Cartesian-closed Categories

We will now vastly generalise our semantics of λ-calculus ...

... via the so-called Cartesian-closed categories

Definition

A category is called Cartesian-closed if it has a final object, binary products, and exponentials

Denotational Semantics in a Cartesian-closed Category

Types $\mathbb{A}$ interpreted as objects $[\![\mathbb{A}]\!]$

$$[\![1]\!] = 1$$
$$[\![\mathbb{A} \times \mathbb{B}]\!] = [\![\mathbb{A}]\!] \times [\![\mathbb{B}]\!]$$
$$[\![\mathbb{A} \to \mathbb{B}]\!] = [\![\mathbb{B}]\!]^{[\![\mathbb{A}]\!]}$$

Typing contexts Γ interpreted as products

$$[\![\Gamma]\!] = [\![x_1 : \mathbb{A}_1, \ldots, x_n : \mathbb{A}_n]\!] = [\![\mathbb{A}_1]\!] \times \cdots \times [\![\mathbb{A}_n]\!]$$

λ-terms $\Gamma \vdash t : \mathbb{A}$ interpreted as arrows

$$[\![\Gamma \vdash t : \mathbb{A}]\!] : [\![\Gamma]\!] \longrightarrow [\![\mathbb{A}]\!]$$

Denotational Semantics in a Cartesian-closed Category

$$\frac{x_i : \mathbb{A} \in \Gamma}{[\![\Gamma \vdash x_i : \mathbb{A}]\!] = \pi_i} \qquad \frac{}{[\![\Gamma \vdash * : 1]\!] = !} \qquad \frac{[\![\Gamma \vdash t : \mathbb{A} \times \mathbb{B}]\!] = f}{[\![\Gamma \vdash \pi_1 t : \mathbb{A}]\!] = \pi_1 \cdot f}$$

$$\frac{[\![\Gamma \vdash t : \mathbb{A}]\!] = f \qquad [\![\Gamma \vdash s : \mathbb{B}]\!] = g}{[\![\Gamma \vdash \langle t, s \rangle : \mathbb{A} \times \mathbb{B}]\!] = \langle f, g \rangle} \qquad \frac{[\![\Gamma, x : \mathbb{A} \vdash t : \mathbb{B}]\!] = f}{[\![\Gamma \vdash \lambda x : \mathbb{A}.\, t : \mathbb{A} \to \mathbb{B}]\!] = \lambda f}$$

$$\frac{[\![\Gamma \vdash t : \mathbb{A} \to \mathbb{B}]\!] = f \qquad [\![\Gamma \vdash s : \mathbb{A}]\!] = g}{[\![\Gamma \vdash t\, s : \mathbb{B}]\!] = \mathrm{app} \cdot \langle f, g \rangle}$$

Case-study: The Boolean Category

$$\begin{tikzcd} 0 \arrow[loop left, "\leq"] \arrow[rr, "\leq"] & & 1 \arrow[loop right, "\leq"] \end{tikzcd}$$

Show that it has final object and binary products

Show that it has exponentials
(hint: use negation and disjunction)

Show that $b \to c \leq (a \to b) \to (a \to c)$

Case-study: Programming to the Rescue !

Previous problem proved via the program

$$f : \mathbb{B} \to \mathbb{C} \vdash \lambda g.\, \lambda x.\, f(g\,(x)) : (\mathbb{A} \to \mathbb{B}) \to (\mathbb{A} \to \mathbb{C})$$

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Much simpler than before !!!

But wait ... is then λ-calculus really a programming language ?

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Monoidal Categories

A monoidal category allows us

- to pair two objs. A and B into a single one $A \otimes B$
- and analogously for morphisms, *i.e.*

$$\frac{f : A \to A' \qquad g : B \to B'}{f \otimes g : A \otimes B \to A' \otimes B'}$$

- It has an obj. I such that $i : I \otimes A \xrightarrow{\cong} A$ for every obj. A

A number of coherence laws hold . . .

Also assume the existence of a symmetry map $\mathrm{sw} : A \otimes B \xrightarrow{\cong} B \otimes A$

Examples

- Every category with binary products and final object is monoidal
- **Vec** with the tensor product (the mathematical basis of entanglement) is monoidal
- **R** with addition is monoidal

Categorical Exponentials Revisited

Definition

An obj. X called exponential of objs. A and B if

$$\begin{tikzcd}
A \otimes B \arrow[d, dotted, "(\lambda f) \otimes \mathrm{id}"'] \arrow[dr, "f"] & \\
X \otimes B \arrow[r, "\mathrm{app}"'] & C
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In **Set**, **Pos** given by the set of (monotone) functions $A \to B$

In **Vec** given by linear maps

Monoidal-closed Categories

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Definition

A category called monoidal-closed if it is monoidal and has exponentials

Denotational Semantics in a Monoidal-closed Category

Types $\mathbb{A}$ interpreted as objects $[\![\mathbb{A}]\!]$

$$[\![\mathbb{I}]\!] = I$$
$$[\![\mathbb{A} \otimes \mathbb{B}]\!] = [\![\mathbb{A}]\!] \otimes [\![\mathbb{B}]\!]$$
$$[\![\mathbb{A} \multimap \mathbb{B}]\!] = [\![\mathbb{B}]\!]^{[\![\mathbb{A}]\!]}$$

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Denotational Semantics in a Monoidal-closed Category

$$\frac{}{[\![x : \mathbb{A} \vdash x : \mathbb{A}]\!] = \mathsf{id}} \qquad \frac{}{[\![(-) \vdash * : \mathbb{I}]\!] = \mathsf{id}}$$

$$\frac{[\![\Gamma \vdash t : \mathbb{I}]\!] = f \qquad [\![\Delta \vdash s : \mathbb{A}]\!] = g}{[\![\Gamma, \Delta \vdash t \ \mathsf{to} \ * . \ s : \mathbb{A}]\!] = i \cdot (f \otimes g)} \qquad \frac{[\![\Gamma \vdash t : \mathbb{A}]\!] = f \qquad [\![\Delta \vdash s : \mathbb{B}]\!] = g}{[\![\Gamma, \Delta \vdash t \otimes s : \mathbb{A} \otimes \mathbb{B}]\!] = f \otimes g}$$

$$\frac{[\![\Gamma \vdash t : \mathbb{A} \otimes \mathbb{B}]\!] = f \qquad [\![\Delta, x : \mathbb{A}, y : \mathbb{B} \vdash s : \mathbb{C}]\!] = g}{[\![\Gamma, \Delta \vdash \mathsf{pm} \ t \ \mathsf{to} \ x \otimes y . s : \mathbb{C}]\!] = g \cdot \mathrm{sw} \cdot (f \otimes \mathsf{id})}$$

$$\frac{[\![\Gamma, x : \mathbb{A} \vdash t : \mathbb{B}]\!] = f}{[\![\Gamma \vdash \lambda x : \mathbb{A}. \ t : \mathbb{A} \multimap \mathbb{B}]\!] = \lambda f} \qquad \frac{[\![\Gamma \vdash t : \mathbb{A} \multimap \mathbb{B}]\!] = f \quad [\![\Delta \vdash s : \mathbb{A}]\!] = g}{[\![\Gamma, \Delta \vdash t \, s : \mathbb{B}]\!] = \mathrm{app} \cdot (f \otimes g)}$$

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(Semantics slightly oversimplified)

Case-study: The Category R

$$\ldots \xrightarrow{\leq} 0 \xrightarrow{\leq} 1 \xrightarrow{\leq} \ldots \xrightarrow{\leq} \infty$$

Show that it is monoidal via addition

Show that it has exponentials
(hint: consider subtraction)

Show that $a + b \leq c \Rightarrow 0 \leq c - b - a$

Case-study: Programming (again) to the Rescue !

Previous problem proved via the program

$$(-) \vdash \lambda y_1.\ \lambda y_2.\ t[y_1 \otimes y_2/x] : \mathbb{A} \multimap (\mathbb{B} \multimap \mathbb{C})$$

(where $x : \mathbb{A} \otimes \mathbb{B} \vdash t : \mathbb{C}$ serves as a witness to $a + b \leq c$)

Case-study: Pointed Sets and Strict Functions

Definition

A set X is called pointed if it has a distinguished element 0_X

Definition

A function of pointed sets $f : X \to Y$ is strict if $f(0_X) = 0_Y$

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PSet denotes the category of pointed sets and strict maps

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An interesting category to interpret strict evaluation

Case-study: Pointed Sets and Strict Functions

PSet is monoidal. $A \otimes B$ is set of pairs $a \otimes b$ with equations

$$0_A \otimes b = b \otimes 0_B = 0_{A \otimes B}$$

Fundamental for strict evaluation

Case-study: Pointed Sets and Strict Functions

Show that it has exponentials ...

... and projections $\pi_1 : A \otimes B \to A$ and $\pi_2 : A \otimes B \to B$

Case-study: Pointed Sets and Strict Functions

Show that it has exponentials ...

... and projections $\pi_1 : A \otimes B \to A$ and $\pi_2 : A \otimes B \to B$

Show that if $[\![\mathrm{div}]\!] = x \mapsto 0$ then we have

$$\pi_2 \; (\mathrm{div} \otimes t) \;=\; \mathrm{div}$$